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The Classical Limit of n-Vector Spin Models

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Abstract. It is proved that the free energy of a system of n-dimensional spins with Kac type potential is equal, in the infinite range zero strength limit, to the free energy of the corresponding Curie-Weiss system in which every spin interacts equally with every other spin.

1. Introduction

In 1966 Lebowitz and Penrose [1] proved that the free energy of a classical system of particles in ν-dimensions with pair potential $v(\boldsymbol{r})$ of Kac type,

$$v(\boldsymbol{r}) = q(\boldsymbol{r}) + \gamma^\nu \varrho(\gamma \boldsymbol{r}) \tag{1.1}$$

approaches the van der Waals free energy with Maxwell construction in the limit $\gamma \to 0+$ (after the thermodynamic limit) provided the short range repulsive (hard core) part of the potential $q(\boldsymbol{r})$ and the long range attractive part of the potential $\gamma^\nu \varrho(\gamma \boldsymbol{r})$ satisfied certain conditions (stated in [1]).

It is not difficult, as suggested by Lebowitz and Penrose, to extend the analysis to Ising ferromagnets (or equivalently, attractive lattice gases) and show that the classical Curie-Weiss theory of magnetism can be obtained from a $\gamma \to 0+$ limit [2].

Here we consider the n-vector model, first introduced by Stanley [3], composed of a set of N, n-dimensional spins

$$\boldsymbol{S}_i = (S_{i1}, S_{i2}, \ldots, S_{in}), \qquad i = 1, 2, \ldots, N \tag{1.2}$$

occupying the vertices of a ν-dimensional lattice, with norm

$$\|\boldsymbol{S}_i\| = \left(\sum_{k=1}^{n} S_{ik}^2 \right)^{1/2} = n^{1/2} \tag{1.3}$$

and with interaction energy

$$E = - \sum_{1 \le i < j \le N} \varrho_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j - \boldsymbol{H} \cdot \sum_{i=1}^{N} \boldsymbol{S}_i\,, \tag{1.4}$$

where ϱ_{ij} is the coupling constant between the ith and jth spins and $\boldsymbol{H}$ is the external magnetic field.

The main interest in this class of models stems from the fact that as special cases of (1.4) one has the Ising model ($n=1$), the planar classical Heisenberg model ($n=2$), the classical Heisenberg model ($n=3$) and the spherical model ($n\to\infty$) [4, 5].

Our concern here is with the $\gamma\to 0+$ limit (v and n fixed) of (1.4) for a potential of Kac type

$$\varrho_{ij}=\gamma^{v}\varrho(\gamma|\boldsymbol{r}_i-\boldsymbol{r}_j|), \tag{1.5}$$

where $\boldsymbol{r}_i$ is the position vector of the ith lattice site. We will assume throughout (in order to guarantee the existence of the thermodynamic limit) that

$$g(0,\gamma)=\gamma^{v}\sum_{\boldsymbol{l}}\varrho(\gamma|\boldsymbol{l}|), \tag{1.6}$$

where the sum extends over the infinite lattice, exists for all $\gamma>0$. In addition, we assume that $\varrho_{ij}\geqq 0$, that

$$g(0)=\lim_{\gamma\to 0+} g(0,\gamma)=\int\varrho(|\boldsymbol{r}|)\,d\boldsymbol{r} \tag{1.7}$$

exists (as a Riemann integral) and that $\varrho(\vec{r})$ is everywhere bounded.

The normalized partition function is defined by

$$Q_N(\beta,\gamma)=[Z_N(0,\gamma)]^{-1}\,Z_N(\beta,\gamma) \tag{1.8}$$

where $\beta=(kT)^{-1}$,

$$Z_N(\beta,\gamma)=\int_{\|\boldsymbol{S}_i\|=n^{1/2}}\cdots\int \exp(-\beta E)\,d\boldsymbol{S}_1\ldots d\boldsymbol{S}_N\,, \tag{1.9}$$

and

$$Z_N(0,\gamma)=[2\pi^{n/2}n^{(n-1)/2}/\Gamma(n/2)]^N\,. \tag{1.10}$$

The limiting free energy per spin $\psi(\beta,\gamma)$ is defined by

$$-\beta\psi(\beta,\gamma)=\lim_{N\to\infty}N^{-1}\log Q_N(\beta,\gamma)\,, \tag{1.11}$$

and our aim here is to prove the following

Theorem. *For a system of n-dimensional spins with interaction energy (1.4) and with free energy* $\psi(\beta,\gamma)$ *defined by (1.11)*

$$\lim_{\gamma\to 0+}\psi(\beta,\gamma)=ng(0)\,\eta^2/2-\beta^{-1}\log\left[\frac{\Gamma(n/2)\,I_{n/2-1}(n\beta g(0)\,\eta+n^{1/2}\beta H)}{(n\beta g(0)\,\eta/2+n^{1/2}\beta H/2)^{n/2-1}}\right] \tag{1.12}$$

where $I_\mu(x)$ *is the modified Bessel function of the first kind of order* μ, η *is the solution of*

$$\eta=I_{n/2}(n\beta g(0)\eta+n^{1/2}\beta H)/I_{n/2-1}(n\beta g(0)\,\eta+n^{1/2}\beta H) \tag{1.13}$$

which minimizes the right hand side of (1.12), and the potential $\varrho_{ij}(\geqq 0)$ *(1.5) satisfies the conditions (1.6) and (1.7).*

For the special case $n=1$, (1.12) reduces to the classical Curie-Weiss free energy [2] (since $I_{1/2}(x)=(\pi x/2)^{-1/2}\sinh x$ and $I_{-1/2}(x)=(\pi x/2)^{-1/2}\cosh x$). For $n>1$, Silver *et al.* [6] have shown that the limiting free energy per spin for a Curie-Weiss system of N, n-dimensional spins (1.2) and (1.3) with interaction energy

$$E'=-\frac{g(0)}{N}\sum_{1\leq i<j\leq N}\boldsymbol{S}_i\cdot\boldsymbol{S}_j-\boldsymbol{H}\cdot\sum_{i=1}^{N}\boldsymbol{S}_i \tag{1.14}$$

is given by (1.12) and (1.13).

A complete discussion of the thermodynamics and critical behavior of (1.12) (which is the same as for the ordinary, $n=1$, Curie-Weiss theory) can be found in [6].

To prove the theorem we obtain upper and lower bounds on the free energy $\psi(\beta,\gamma)$ (1.11) and show that the two bounds coalesce to give the stated result in the limit $\gamma\to 0^+$.

2. Upper Bound on the Free Energy

For simplicity we impose periodic (Born Von Karman) boundary conditions on the potential (1.5) so that

$$\sum_{j=1}^{N}\varrho_{ij}=g_N(0,\gamma) \tag{2.1}$$

for all $i=1,2,\ldots,N$ [in the limit $N\to\infty$, $g_N(0,\gamma)$ approaches $g(0,\gamma)$ (1.6)].

We write the interaction energy (1.4) as ($\varrho_{ii}=0$)

$$\begin{aligned}E=&-1/2\sum_{i,j=1}^{N}\varrho_{ij}(\boldsymbol{S}_i-m\hat{\boldsymbol{H}})\cdot(\boldsymbol{S}_j-m\hat{\boldsymbol{H}})-1/2m\hat{\boldsymbol{H}}\cdot\sum_{i,j=1}^{N}\varrho_{ij}(\boldsymbol{S}_i+\boldsymbol{S}_j)\\&+m^2/2\sum_{i,j=1}^{N}\varrho_{ij}-\boldsymbol{H}\cdot\sum_{i=1}^{N}\boldsymbol{S}_i\end{aligned} \tag{2.2}$$

where $\hat{\boldsymbol{H}}$ is the unit vector in the direction of $\boldsymbol{H}$ and m will be fixed in a moment to give (1.12) as an upper bound on $\lim_{\gamma\to 0+}\psi(\beta,\gamma)$.

Using (2.1) and (2.2) the normalized partition function (1.8) can be written as

$$\begin{aligned}Q_N(\beta,\gamma)=&[Z_N^C(\beta,\gamma,m)/Z_N(0,\gamma)]\\&\cdot\int_{\|\boldsymbol{S}_i\|=n^{1/2}}\cdots\int\exp\left[\beta/2\sum_{i,j=1}^{N}\varrho_{ij}(\boldsymbol{S}_i-m\hat{\boldsymbol{H}})\cdot(\boldsymbol{S}_j-m\hat{\boldsymbol{H}})\right]\\&\left\{\exp[-\beta m^2Ng_N(0,\gamma)/2+(\beta mg_N(0,\gamma)+H)\hat{\boldsymbol{H}}\cdot\sum_{i=1}^{N}\boldsymbol{S}_i]/Z_N^C(\beta,\gamma,m)\right\}\prod_{i=1}^{N}d\boldsymbol{S}_i\\&=Q_N^C(\beta,\gamma,m)\left\langle\exp\left[\beta/2\sum_{i,j=1}^{N}\varrho_{ij}(\boldsymbol{S}_i-m\hat{\boldsymbol{H}})\cdot(\boldsymbol{S}_j-m\hat{\boldsymbol{H}})\right]\right\rangle_C\end{aligned} \tag{2.3}$$

where $H = \|\boldsymbol{H}\|$,

$$Q_N^C(\beta, \gamma, m) = Z_N^C(\beta, \gamma, m)/Z_N(0, \gamma)\,, \tag{2.4}$$

$$\begin{aligned} Z_N^C(\beta, \gamma, m) = & \int_{\|\boldsymbol{S}_i\| = n^{1/2}} \cdots \int \\ & \exp\left[-\beta m^2 N g_N(0, \gamma)/2 + (\beta m g_N(0, \gamma) + \beta H)\,\hat{H} \cdot \sum_{i=1}^{N} \boldsymbol{S}_i\right] \prod_{i=1}^{N} d\boldsymbol{S}_i \\ & = \exp[-\beta m^2 N g_N(0, \gamma)/2] \\ & \cdot \left(\int_{\|\boldsymbol{S}\| = n^{1/2}} \exp[(\beta m g_N(0, \gamma) + H)\,\hat{H} \cdot \boldsymbol{S}]\, d\boldsymbol{S} \right)^N \end{aligned} \tag{2.5}$$

and $\langle \cdots \rangle_C$ denotes an average with respect to the distribution function

$$P_N^C(\boldsymbol{S}_1, \ldots, \boldsymbol{S}_N) = \prod_{i=1}^{N} p^C(\boldsymbol{S}_i)\,, \tag{2.6}$$

$$\begin{aligned} p^C(\boldsymbol{S}) = \exp[(\beta m g_N(0, \gamma) + \beta H)\,\hat{H} \cdot \boldsymbol{S}] / & \int_{\|\boldsymbol{S}\| = n^{1/2}} \\ & \cdot \exp[(\beta m g_N(0, \gamma) + \beta H)\,\hat{H} \cdot \boldsymbol{S}]\, d\boldsymbol{S}. \end{aligned} \tag{2.7}$$

Making use of Jensen's inequality ($\langle \exp X \rangle \geqq \exp \langle X \rangle$) and the fact that the spins occur independently in P_N^C (2.6), (2.3) gives

$$Q_N(\beta, \gamma) \geqq Q_N^C(\beta, \gamma, m) \exp\left[\beta/2 \sum_{i,j=1}^{N} \varrho_{ij} (\langle \boldsymbol{S}_i \rangle_C - m\hat{H}) \cdot (\langle \boldsymbol{S}_j \rangle_C - m\hat{H})\right]. \tag{2.8}$$

To obtain the desired lower bound on $Q_N(\beta, \gamma)$ we choose, since $\langle \boldsymbol{S}_i \rangle_C = \langle \boldsymbol{S} \rangle_C$ is independent of i, and from (2.7) and (2.12) below, in the direction of $\boldsymbol{H}$,

$$m\hat{H} = \langle \boldsymbol{S} \rangle_C\,, \tag{2.9}$$

so that

$$Q_N(\beta, \gamma) \geqq Q_N^C(\beta, \gamma, m)\,. \tag{2.10}$$

To evaluate $\langle \boldsymbol{S} \rangle_C$ and $Q_N^C(\beta, \gamma, m)$ we need the following results:

$$\int_{\|\boldsymbol{S}\| = n^{1/2}} \exp(\boldsymbol{\alpha} \cdot \boldsymbol{S})\, d\boldsymbol{S} = 2\pi^{n/2} n^{(n-1)/2} I_{n/2-1}(n^{1/2} \|\boldsymbol{\alpha}\|)/(n^{1/2} \|\boldsymbol{\alpha}\|/2)^{n/2-1} \tag{2.11}$$

and

$$\int_{\|\boldsymbol{S}\| = n^{1/2}} \boldsymbol{S} \exp(\boldsymbol{\alpha} \cdot \boldsymbol{S})\, d\boldsymbol{S} = 2\pi^{n/2} n^{(n-1)/2} [I_{n/2}(n^{1/2} \|\boldsymbol{\alpha}\|)/(n^{1/2} \|\boldsymbol{\alpha}\|/2)^{n/2-1}]\, n^{1/2} \hat{\boldsymbol{\alpha}} \tag{2.12}$$

(2.11) can be found in Appendix A Silver *et al.* [6] and (2.12) follows from (2.11) and the fact that $\frac{d}{dx}(x^{-\alpha} I_\alpha(x)) = x^{-\alpha} I_{\alpha+1}(x)$.

From the definitions of $Q_N^C(\beta, \gamma, m)$ (2.4) and $\langle S \rangle_C$ (2.9) we obtain from (2.11) and (2.12) respectively,

$$Q_N^C(\beta, \gamma, m) = [\Gamma(n/2)]^N \exp[-\beta m^2 N g_N(0, \gamma)/2] \left[\frac{I_{n/2-1}(n^{1/2}(\beta m g_N(0,\gamma) + \beta H))}{(1/2 n^{1/2}(\beta m g_N(0,\gamma) + \beta H))^{n/2-1}}\right]^N \tag{2.13}$$

and

$$\begin{aligned} m &= \langle S \rangle_C \cdot \hat{H} \\ &= \left(\int_{\|S\| = n^{1/2}} S p^C(S)\, dS \right) \cdot \hat{H} \\ &= \frac{I_{n/2}(n^{1/2}(\beta m g_N(0,\gamma) + \beta H))\, n^{1/2}}{I_{n/2-1}(n^{1/2}(\beta m g_N(0,\gamma) + \beta H))} \,. \end{aligned} \tag{2.14}$$

Defining

$$\eta = m n^{-1/2}\,, \tag{2.15}$$

and allowing γ to approach zero after N approaches infinity, η becomes a solution of (1.13) and from (1.11), (2.10) and (2.13)

$$\begin{aligned} \lim_{\gamma \to 0^+} \psi(\beta, \gamma) &= \lim_{\gamma \to 0^+} \lim_{N \to \infty} (-\beta^{-1} N^{-1} \log Q_N(\beta, \gamma)) \\ &\leqq \psi^C(\beta) \end{aligned} \tag{2.16}$$

where $\psi^C(\beta)$ is the right hand side of (1.12).

This completes the derivation of the upper bound.

3. Lower Bound on the Free Energy

We begin by writting

$$\begin{aligned} Q_N(\beta, \gamma) = {} & [Z_N(0, \gamma)]^{-1} \exp(-N n \gamma^\nu \varrho(0)\, \beta/2)\,. \\ & \cdot \int \cdots \int_{\|S_i\| = n^{1/2}} \exp\left[\beta/2 \sum_{i,j=1}^{N} \gamma^\nu \varrho(\gamma |r_i - r_j|)\, S_i \cdot S_j + \beta H \cdot \sum_{i=1}^{N} S_i\right] \prod_{i=1}^{N} dS_i \end{aligned} \tag{3.1}$$

where a diagonal term ($i = j$) has been added and subtracted from the quadratic term, with $\varrho(0)$ chosen (sufficiently large) to make $\sum_{i,j=1}^{N} \gamma^\nu \varrho(\gamma |r_i - r_j|)\, S_i \cdot S_j$ positive definite.

We can then use the following elementary generalization of a well known identity [7],

$$\begin{aligned} \exp\left(\beta/2 \sum_{i,j=1}^{N} \varrho_{ij} S_i \cdot S_j\right) = {} & (2\pi)^{-Nn/2} (\operatorname{Det} \varrho)^{-n/2} \\ & \cdot \int_{-\infty}^{\infty} \cdots \int \exp\left(-1/2 \sum_{i,j=1}^{N} (\varrho^{-1})_{ij} X_i \cdot X_j + \beta^{1/2} \sum_{i=1}^{N} S_i \cdot X_i\right) \prod_{i=1}^{N} dX_i \end{aligned} \tag{3.2}$$

which is valid for any positive definite symmetric matrix $\varrho=(\varrho_{ij})$ and real n-dimensional vectors S_i, to write

$$Q_N(\beta,\gamma)=[Z_N(0,\gamma)]^{-1}\exp(-Nn\gamma^\nu\varrho(0)\beta/2)(2\pi)^{-Nn/2}(\operatorname{Det}\varrho)^{-n/2} \cdot \int_{\|S_i\|=n^{1/2}}\cdots\int\prod_{i=1}^N dS_i\int_{-\infty}^{\infty}\cdots\int\prod_{i=1}^N dX_i \cdot\exp\left(-1/2\sum_{i,j=1}^N(\varrho^{-1})_{ij}X_i\cdot X_j+\sum_{i=1}^N S_i\cdot(\beta^{1/2}X_i+\beta H)\right). \quad (3.3)$$

Interchanging orders of integration we then obtain

$$Q_N(\beta,\gamma)=[\Gamma(n/2)]^N\exp(-Nn\gamma^\nu\varrho(0)\beta/2)(2\pi)^{-Nn/2}(\operatorname{Det}\varrho)^{-n/2} \cdot\int_{-\infty}^{\infty}\int\prod_{i=1}^N dX_i\exp\left[-1/2\sum_{i,j=1}^N(\varrho^{-1})_{ij}X_i\cdot X_j+1/2z\sum_{i=1}^N\|X_i\|^2\right] \cdot\prod_{i=1}^N\frac{\exp(-\|X_i\|^2/2z)I_{n/2-1}(n^{1/2}\|\beta^{1/2}X_i+\beta H\|)}{(n^{1/2}\|\beta^{1/2}X_i+\beta H\|/2)^{n/2-1}} \quad (3.4)$$

where use has been made of (2.11) and in anticipation of the next step we have added and subtracted a term $\left(\sum_{i=1}^N\|X_i\|^2/2z\right)$ in the exponent.

To obtain an upper bound for $Q_N(\beta,\gamma)$ we first increase the right hand side of (3.4) by replacing $\|\beta^{1/2}X_i+\beta H\|$ by the larger quantity $\beta^{1/2}\|X_i\|+\beta H$ (this follows from the fact that $I_\mu(|\alpha|)/|\alpha|^\mu$ is an increasing function of $|\alpha|$) and then maximize each term in the resulting product in (3.4) separately for each i. The maximum occurs for $\|X_i\|=X$ a solution of

$$X/z=(\beta n)^{1/2}I_{n/2}(n^{1/2}(\beta^{1/2}X+\beta H))/I_{n/2-1}(n^{1/2}(\beta^{1/2}X+\beta H)). \quad (3.5)$$

The remaining integral in (3.4) can then be performed immediately to give

$$Q_N(\beta,\gamma)\leqq[\Gamma(n/2)]^N\exp(-Nn\gamma^\nu\varrho(0)\beta/2)[\operatorname{Det}(I-\varrho/z)]^{-n/2} \quad (3.6)$$
$$\cdot[I_{n/2-1}(n\beta z\bar\eta+n^{1/2}\beta H)\exp(-\bar\eta^2\beta nz/2)/(n\beta z\bar\eta/2+n^{1/2}\beta H/2)^{n/2-1}]$$

where $\bar\eta$ is defined by

$$\bar\eta=Xz^{-1}(\beta n)^{-1/2} =I_{n/2}(n\beta z\bar\eta+n^{1/2}\beta H)/I_{n/2-1}(n\beta z\bar\eta+n^{1/2}\beta H). \quad (3.7)$$

The manipulation leading to (3.6) obviously requires the matrix $I-\varrho/z$ to be positive definite, which will certainly be the case if z is greater than the maximum eigenvalue of ϱ. For $N=m^\nu$ spins located on the vertices of a regular ν-dimensional hypercubic lattice the eigenvalues of ϱ are given by

$$\lambda(\boldsymbol{k}) = \gamma^\nu \sum_{\boldsymbol{l}} \varrho(\gamma \|\boldsymbol{l}\|) \exp(2\pi i \boldsymbol{k} \cdot \boldsymbol{l}/m) \tag{3.8}$$

where the sum extends over all lattice vectors $\boldsymbol{l}$, including $\boldsymbol{l} = \boldsymbol{0}$. Since the $\boldsymbol{l} = \boldsymbol{0}$ term is immaterial for sufficiently small γ ($\varrho(0)$, of order unity, was chosen to make all $\lambda(\boldsymbol{k}) > 0$) and the maximum eigenvalue is $\lambda(\boldsymbol{0})$ (since we are assuming all $\varrho(X) \geqq 0$), the results (3.6) and (3.7) are valid as long as, from (2.1),

$$z > \gamma^\nu \sum_{\boldsymbol{l} \neq \boldsymbol{0}} \varrho(\gamma \|\boldsymbol{l}\|) = g_N(0, \gamma). \tag{3.9}$$

Now since ϱ is a Toeplitz matrix, Szegö's theorem [8] gives

$$\begin{aligned} f_\nu(z, \gamma) &= \lim_{N \to \infty} N^{-1} \log \operatorname{Det}(I - \varrho/z) \\ &= (2\pi)^{-\nu} \int_0^{2\pi} \cdots \int \log(1 - g(\boldsymbol{\theta}, \gamma)/z)\, d^\nu \theta \end{aligned} \tag{3.10}$$

where, noting (3.8),

$$g(\boldsymbol{\theta}, \gamma) = \sum_{\boldsymbol{l}} \gamma^\nu \varrho(\gamma \|\boldsymbol{l}\|)\, e^{i \boldsymbol{l} \cdot \boldsymbol{\theta}}. \tag{3.11}$$

It follows then from (3.6) that

$$\begin{aligned} \psi(\beta, \gamma) &= -\lim_{N \to \infty} (N\beta)^{-1} \log Q_N(\beta, \gamma) \\ &\geqq nz\bar{\eta}^2/2 - \beta^{-1} \log \left[\frac{\Gamma(n/2)\, I_{n/2-1}(n\beta z\bar{\eta} + n^{1/2}\beta H)}{(n\beta z\bar{\eta}/2 + n^{1/2}\beta H/2)^{n/2-1}} \right] \\ &\quad + n\gamma^\nu \varrho(0)/2 + n(2\beta)^{-1} f_\nu(z, \gamma) \end{aligned} \tag{3.12}$$

for all

$$z > g(0, \gamma). \tag{3.13}$$

Taking the limit $z \to g(0, \gamma)+$ in (3.7) and (3.12), $\bar{\eta}$ becomes η given by (1.13) in the limit $\gamma \to 0+$, the first two terms in (3.12) become $\psi^C(\beta)$ (2.16) [the right hand side of (1.12)] and since $\varrho(0)$ is of order unity

$$\lim_{\gamma \to 0+} \psi(\beta, \gamma) \geqq \psi^C(\beta) + \lim_{\gamma \to 0+} n(2\beta)^{-1} f_\nu(g(0, \gamma), \gamma). \tag{3.14}$$

In view of the upper bound (2.16), the theorem will be proved once we have shown that the second term in (3.14) is zero.

Consider first the case $\nu = 1$. From (3.11) we have

$$g(\theta, \gamma) = 2\gamma \sum_{l=1}^{\infty} \varrho(\gamma l) \cos l\theta \tag{3.15}$$

which can be approximated arbitrarily closely for small γ by

$$G(\theta, \gamma) = 2 \int_0^\infty \varrho(X) \cos(\theta X/\gamma)\, dX. \tag{3.16}$$

Since we are assuming that $\varrho(X)$ is bounded for $0 \leqq X < \infty$ and that $\int_0^\infty \varrho(X)\,dX$ exists (as a Riemann integral), $G(\theta,\gamma)$ and hence $g(\theta,\gamma)$ approach zero as $\gamma \to 0+$ by the Riemann-Lebesgue lemma, for all $\varepsilon \leqq \theta < 2\pi$ and (arbitrarily small) $\varepsilon > 0$. It follows almost immediately from (3.10) that $f_1(g(0,\gamma),\gamma)$ also approaches zero as $\gamma \to 0+$.

The case of arbitrary v is a straightforward generalization of the above argument.

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