Commun. math. Phys. 33, 23—42 (1973)

Investigation of the Critical Point in Models of the Type of Dyson's Hierarchical Models

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Received March 27, 1973

Abstract. We consider the classical spin models where the Hamiltonians are small modifications of the Hamiltonians of Dyson's hierarchical models. Under some assumptions we investigate rigorously the neighbourhood of the critical point and find the critical indices. It follows that in the cases under consideration phenomenological Landau's theory of phase transitions is valid.

Introduction

In classical lattice ferromagnets the critical temperature T_{cr} separates the domains with zero and non-zero spontaneous magnetization. The behaviour of different thermodynamical parameters near T_{cr} was considered rigorously for the two-dimensional Ising model using Onsager's exact solution (see [1, 2]), and some other models (see [3]) also using the exact formula for the free energy.

Recently Dyson introduced so-called hierarchical models immitating in many respects the lattice systems with pairwise long-range power interaction. Under several natural assumptions Dyson proved that the spontaneous magnetization in his models is non-zero for sufficiently large β ([4–6]).

We consider in this paper a class of models slightly generalizing hierarchical models and find rigorously under some conditions critical indices for them. Recently some non-rigorous results in this direction were obtained by Baker [7]. Our case corresponds to the "gaussian case" of his paper. Closely related results were presented also in [8] (see also [9]).

Now we want to describe briefly our main results. We impose certain conditions on the distribution of the mean spin in a finite volume of fixed size for an interval of temperatures. Under these conditions we prove the existence of a critical temperature T_{cr} inside this interval and for $T = T_{cr}$ we establish the limit distribution for the mean spin which is a gaussian distribution with a non-usual normalization. This permits us to find the asymptotic of binary correlation functions for $T = T_{cr}$.

For $T > T_{cr}$ we obtain the asymptotic expression for the susceptibility and binary correlation functions. For $T < T_{cr}$ we find an asymptotic expression for the spontaneous magnetization. The critical indices are just as predicted by Landau's theory ([10, 9]).

The case $T=T_{cr}$ was considered by both authors. The cases $T<T_{cr}$, $T>T_{cr}$ were considered by the first author.

1. Hierarchical Models and Asymptotically-Hierarchical Models

We begin with the description of general hierarchical models. Assume to be given a sequence of positive integers $r_n, r_n \geqq 2$, $n=1, \ldots, q_n = \prod_{i=1}^{n} r_i$, and for any integer $r \geqq 2$ some positive quadratic form $Q_r(t_1, \ldots, t_r)$. In general hierarchical models one considers the sequence of volumes $V_n = V_{n,0}$ in which each volume $V_{n,0}$ consists of q_n points. The volume $V_{n,0}$ is decomposed into r_n subvolumes $V_{n-1,i}$, $i=1, \ldots, r_n$, each of which consists of q_{n-1} points; the subvolume $V_{n-1,i}$ is decomposed into r_{n-1} subvolumes $V_{n-2,i}$ each of which consists of q_{n-2} points, and so on. For any $k \geqq 0$ the volume $V_{n,0}$ is decomposed into $r_n \cdot \ldots \cdot r_{n-k}$ subvolumes $V_{n-k-1,i}$ each of which consists of q_{n-k-1} points. The Hamiltonian is defined by a sequence of positive numbers b_p and has the form

$$H_n = -\sum_{p=1}^{n} b_p \sum_{i=0,\ \bigcup_{k=1}^{r_p} V_{p-1,i_k} = V_{p,i}}^{q_n/q_p - 1} Q_{r_p}(s_{i_1}, s_{i_2}, \ldots, s_{i_{r_p}}).$$

Here s_{i_l} is the mean spin in the volume V_{p-1,i_l}. We assume that the spin s_x, $x \in V_{n,0}$ takes values ± 1.

Dyson ([4–6]) considered the case $r_n \equiv 2$ and $Q_2(t_1, t_2) = \left(\frac{t_1 + t_2}{2}\right)^2$. Baker considered the cases $r_n \equiv 2, 4, 8$ ([7]).

We shall consider below slightly more general models which we shall call asymptotically-hierarchical models (a.h.m.). The Hamiltonian in these models is defined by an integer n_0 and can be written for $n > n_0$ in the following form

$$H_n = -\sum_{i=1}^{q_n/q_{n_0}} H_{n_0}(S_{n_0,i}) - \sum_{p=n_0+1}^{n} b_p \sum_{i,\ \bigcup_{k=1}^{r_p} V_{p-1,i_k} = V_{p,i}} Q_{r_p}(s_{i_1}, \ldots, s_{i_{r_p}}).$$

Here $S_{n_0,i}$ is the configuration of spins in $V_{n_0,i}$ and H_{n_0} is an arbitrary Hamiltonian in the volume of q_{n_0} points, satisfying the symmetry condition: $H_{n_0}(S_{n_0,i}) = H_{n_0}(-S_{n_0,i})$.

In the sequel only the case $r_n \equiv r$ and $b_p = c^p$ will be discussed. Moreover we shall assume that the quadratic form $Q_r(t_1, \ldots, t_r)$ is symmetric in its arguments and has the form

$$Q_r(t_1, \ldots, t_r) = g_r\left(\frac{t_1 + \cdots + t_r}{r}\right)^2 + h_r\left(\frac{t_1^2 + \cdots + t_r^2}{r}\right);$$

we shall assume that $g_r = g$, $h_r = h$, $h + g > 0$, $rg + ch > 0$. The extension of our results to other cases will be done in another paper.

For a.h.m. the limit free energy per particle χ exists and is a convex function of its arguments if $c < r$. An extension of Dyson's arguments shows that spontaneous magnetization for sufficiently large β is non-zero if $c > 1$.

2. Formulation of Results

For a.h.m. let us denote $g_k(s;\beta) = \mathrm{Prob}_k(s_{k,0} = s \mid \beta)$, where $\mathrm{Prob}_k(\cdot \mid \beta)$ means the Gibbs probability distribution with parameter β and zero magnetic field in the volume $V_{k,0}$.

For $k > n_0$ one can write the following recurrence relations:

$$g_k(s;\beta) = \frac{\Xi_{k-1}^r(\beta)}{\Xi_k(\beta)} \cdot \sum_{\frac{s_1 + \cdots + s_r}{r} = s} \exp(\beta c^{k-1} Q(s_1, s_2, \ldots, s_r))\, g_{k-1}(s_1;\beta) \ldots g_{k-1}(s_r;\beta) \tag{1}$$

where $\Xi_{k(\beta)}$ is the grand partition function in the volume $V_{k,0}$. For all k the probabilities $g_k(s;\beta)$ are even functions of s.

Let us introduce some normalization for spins $s_{k,0}$ putting $c^{\frac{k}{2}} s_{k,0} = z_k$. It is evident that z_k takes the values $\frac{-r^k + 2m}{r^k} \cdot c^{\frac{k}{2}}$, $0 \leqq m \leqq r^k$. Therefore for $\Delta_k = \frac{c^{\frac{k}{2}}}{r^k}$ and $f_k(z;\beta) = g_k(zc^{-\frac{k}{2}};\beta)$ we have instead of (1)

$$f_k(z;\beta) = L_k \sum_{\frac{z_1 + \cdots + z_r}{r} = \frac{z}{\sqrt{c}}} \exp(\beta Q(z_1, \ldots, z_r)) \cdot f_{k-1}(z_1;\beta) \cdot \ldots \cdot f_{k-1}(z_r;\beta)\, \Delta_{k-1}^{r-1}. \tag{2}$$

Here L_k is a normalization constant.

It is natural to assume that for large k the behaviour of the functions f_k is the same as the behaviour of the iterates of the non-linear integral transformation

$$Q_\beta(f) = \frac{\int_{\sum_{i=1}^r z_i = \frac{r}{\sqrt{c}} z} \exp(\beta Q(z_1, \ldots, z_r))\, f(z_1) \cdot \ldots \cdot f(z_r) \prod_i dz_i}{\int \exp(\beta Q(z_1, \ldots, z_r)\, f(z_1) \cdot \ldots \cdot f(z_r) \prod_i dz_i}.$$

The investigation of such transformations begins with the search for the fixed points or the eigenvectors, i.e. the functions f which satisfy the equality

$$f(z)=\lambda \int_{\sum z_i = \frac{r}{\sqrt{c}} z} \exp(\beta Q(z_1, \ldots, z_r))\, f(z_1) \cdot \ldots \cdot f(z_r) \prod_i dz_i = \tilde{Q}_\beta(f) \qquad (3)$$

for some constant λ. Having such a fixed point it is necessary to investigate its stability properties. Only fixed points with some stability properties can appear as limits of our difference transformations (2).

It is easy to verify that the function $\exp(-a_0(\beta)z^2)$ satisfies the Eq. (3) with $a_0(\beta)=\frac{g+h}{r-c}\beta$, i.e. it is an eigenfunction of the transformation $\tilde{Q}_\beta$. Thus the function $\sqrt{\frac{a_0}{\pi}}\exp(-a_0(\beta)z^2)=e_\beta(z)$ is a fixed point for the transformation Q_β.

To investigate the stability properties of $e_\beta(z)$ let us consider now the differential of the transformation Q_β near the point $e_\beta(z)$. It is easier to consider the differential of the transformation $\tilde{\tilde{Q}}_\beta$ near the point $e_\beta(z)$. Taking $e(z)=e_\beta(z)+\varepsilon h(z)$ one finds easily

$$\begin{aligned}
\mathscr{D}\tilde{Q}_\beta h &= \frac{d}{d\varepsilon}\tilde{Q}_\beta(e_\beta+\varepsilon h)|_{\varepsilon=0}\\
&= \frac{d}{d\varepsilon}\left(\frac{a_0}{\pi}\right)^{\frac{r}{2}} \int_{\sum z_i=\frac{r}{\sqrt{c}}z} \exp(\beta Q(z_1,\ldots,z_r))\,(e_\beta(z_1)+\varepsilon h(z_1))\\
&\quad \cdot \ldots \cdot (e_\beta(z_r)+\varepsilon h(z_r)) \prod_i dz_i\\
&= \exp(-a_0 z^2)\, r\left(\frac{\zeta(\beta)}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} \exp\left(\zeta(\beta)\left(\frac{z}{\sqrt{c}}-z_1\right)^2 + a_0(\beta)z_1^2\right)\\
&\quad \cdot h(z_1)\,dz_1
\end{aligned}$$

where

$$\zeta(\beta)=\frac{h\cdot\beta-\mathrm{a}_0(\beta)\,r}{r-1},$$

which is the well-known integral transformation with Gaussian kernel (see [11]). This operator is a compact selfadjoint operator in the Hilbert space $\mathscr{H}(\gamma_0)=L^2_{\mathrm{ev}}(\mathbb{R}^1; e^{-\gamma_0 z^2})$ of real valued even square-integrable functions with the weight $e^{-\gamma_0 z^2}$, where $\gamma_0=\gamma_0(\beta)=\zeta(\beta)\left(1-\frac{1}{c}\right)+2a_0(\beta)$; the spectrum of this integral operator in the space $\mathscr{H}(\gamma_0)$ consists of the infinite sequence of numbers $r, \frac{r}{c}, \frac{r}{c^2}, \frac{r}{c^3}, \ldots$. The corresponding eigenfunctions are $\{e^{-a_0(\beta)z^2} G_{2i}(z;\gamma)\}_{i=0}^{\infty}$,

$$\gamma = \gamma(\beta) = \zeta(\beta)\left(1 - \frac{1}{c}\right),$$

where

$$G_{2i}(z;\gamma) = \sqrt[4]{\frac{\gamma}{\pi}}\,\frac{\sqrt{(2i)!}}{2^i}\sum_{j=0}^{i}\frac{(-1)^{i-j}(2\sqrt{\gamma}\,z)^{2j}}{(2j)!\,(i-j)!}$$

is the $2i$-th Hermite polynomial.

The first eigenvalue r is determined by the fact that our transformation is r-linear. The second eigenvalue is always more than one. If $c > \sqrt{r}$ then all other eigenvalues are less than one. We shall show that in an a.h.m. the convergence of $f_k(z;\beta)$ to a gaussian distribution for $\beta = \beta_{cr}$ is in general possible if $c > \sqrt{r}$. Thus we impose the first important condition.

Condition 1. $c > \sqrt{r}$.

Now we are going to discuss our next condition. As the reader will see its exact formulation is not too short. Therefore we want to explain its meaning. One can hope that $f_n(z;\beta)$ for $\beta = \beta_{cr}$ will tend to $e_{\beta_{cr}}(z)$ if the functions $f_{n_0}(z;\beta_{cr})$ are sufficiently close to $e_{\beta_{cr}}(z)$ in an appropriate sense for some sufficiently large $n = n_0$. Here we give the exact formulation of what we mean by "sufficiently close" for our problem. The number $n_0 = n_0(c, Q)$ depends only on the constant c and the quadratic form Q. Its exact value is defined by some number (near ten) of inequalities appearing during the proof. Therefore we shall not give its explicit expression here.

Furthermore, if we are given a family of probability distributions $f_{n_0}(z;\beta)$ for some interval of temperatures $\boldsymbol{\beta} = [\beta^-, \beta^+]$ we don't know the value of the critical temperature without solving the whole sequence of recurrence Eqs. (2). However it is possible to formulate conditions which guarantee that the critical temperature will lie inside the interval $\boldsymbol{\beta}$.

Now we proceed to the exact formulation.

Condition 2. Let us choose three numbers $0 < \varrho, q, \xi < 1$ depending only on c in such a way that

$$\frac{r}{c^2} < \varrho < 1\,;$$

$$\frac{r^{\frac{1}{2}}}{c^{1\,3/4}},\ \frac{r^2}{c^4} < q < \frac{r}{c^2}\,; \tag{4}$$

$$c^{-\frac{1}{3}},\ q^{\frac{1}{4}},\ \frac{qc^2}{r} < \xi < 1\,.$$

There exist a number $n_0(c)$, an interval of inverse temperatures $\boldsymbol{\beta}_{n_0} = [\beta_{n_0}^-, \beta_{n_0}^+]$, and a differentiable function $b_{n_0}(\beta) = b(\beta)$, defined on this interval, the number d satisfying the inequalities

$$0<d,\ r+1<d_1^2 \quad \text{for} \quad d_1=\frac{d(\sqrt{c}-1)}{2\sqrt{c}},$$

for which

a$_1$) $b(\beta_{n_0}^-)=-\left(\frac{c}{2}\varrho\right)^{n_0}$, $b(\beta_{n_0}^+)=\left(\frac{c}{2}\varrho\right)^{n_0}$, $|b(\beta)|<\left(\frac{c}{2}\varrho\right)^{n_0}$ for $\beta\in(\beta_{n_0}^-,\beta_{n_0}^+)$.

a$_2$) for each $\beta\in\boldsymbol{\beta}_{n_0}$ the function $f_{n_0}(z;\beta)$ can be represented in the following form

$$f_{n_0}(z;\beta)=L_{n_0}(\beta)\exp\left(-\left(a_0(\beta)+\left(\frac{2}{c}\right)^{n_0}b(\beta)\right)z^2\right)(1+q_{n_0}(z;\beta))$$

with $L_{n_0}(\beta)$ being a constant factor (with respect to z) and for the "small perturbation" $q_{n_0}(z;\beta)$ satisfying

a$_2'$) for $|z|\leqq\frac{d\sqrt{n_0}}{\sqrt{a_0}}$

$$q_{n_0}(z;\beta)=\delta_{n_0}(\beta)\,G_4(z;\gamma)+R_{n_0}(z;\beta)$$

with $\gamma=\gamma(\beta)=\zeta(\beta)\left(1-\frac{1}{c}\right)$ and $\delta_{n_0}(\beta)$, $R_{n_0}(z;\beta)$ are differentiable functions of $\beta\in\boldsymbol{\beta}_{n_0}$,

$$|R_{n_0}(z;\beta)|\leqq q^{n_0}\leqq-\delta_{n_0}\xi^{n_0},$$

$$-\delta_{n_0}\leqq q^{\frac{n_0}{2}}\xi^{n_0}.$$

a$_2''$) for $|z|>\frac{d\sqrt{n_0}}{\sqrt{a_0}}$

$$0\leqq 1+q_{n_0}(z;\beta)\leqq\exp(-v_{n_0}z^4),$$

$$0<v_{n_0}<-0,1\cdot\delta_{n_0}.$$

Remark. The proof of Theorem 1 below shows that the Condition 2 is stable in the sense that it is fulfilled for $n>n_0$ if it is fulfilled for $n=n_0$.

Now we can give the exact formulation of our theorems.

Theorem 1. *If the Conditions 1, 2 hold, then there exists one and only one* $\beta_{cr}\in\boldsymbol{\beta}_{n_0}$ *for which*

$$\lim_{n\to\infty}\mathrm{Prob}_n\{t_1<s_{n,0}c^{\frac{n}{2}}<t_2;\beta_{cr}\}=\frac{1}{\sqrt{2\pi\sigma}}\int_{t_1}^{t_2}e^{-\frac{u^2}{2\sigma}}du$$

for some positive σ *and arbitrary* t_1, t_2.

Theorem 1 shows that for $\beta=\beta_{cr}$ the mean spin $s_{n,0}$ has gaussian distribution with non-usual normalization.

Corollary 1. *For $\beta=\beta_{cr}$ the binary correlation function $\langle s_x, s_y\rangle_{n,\beta_{cr}}$, $x, y \in V_{n,0}$ satisfies the inequalities*

$$C_1 \cdot c^{-d(x,y)} \leqq \langle s_x, s_y\rangle_{n,\beta_{cr}} \leqq C_2\, c^{-d(x,y)}$$

with some constants C_1, C_2, where $d(x, y)$ is the least number k such that $x, y \in V_{k,i}$ for some i.

Theorem 2. *Under the conditions of Theorem 1 let $\beta_{n0}^{+}<\beta<\beta_{cr}$. Then for the mean spin $s_{n,0}$*

$$\lim_{n\to\infty} \operatorname{Prob}_n\{t_1 < s_{n,0} r^{\frac{n}{2}} < t_2\,;\beta\} = \frac{1}{\sqrt{2\pi\sigma_1(\beta)}} \int_{t_1}^{t_2} e^{-\frac{u^2}{2\sigma_1(\beta)}}\,du$$

where $\sigma_1(\beta) \sim$ const $(\beta_{cr}-\beta)^{-1}$.

Corollary 2. *Under the conditions of Theorem 2 the binary correlation functions $\langle s_x, s_y\rangle_{n,\beta}$, $x, y \in V_{n,0}$ satisfy for all n the inequalities*

$$\frac{C_1'}{(\beta_{cr}-\beta)^2}\left(\frac{r^2}{c}\right)^{-d(x,y)} \leqq \langle s_x, s_y\rangle_{n,\beta} \leqq \frac{C_2'}{(\beta_{cr}-\beta)^2}\left(\frac{r^2}{c}\right)^{-d(x,y)}$$

with some constants C_1', C_2'.

Theorem 3. *Under the conditions of Theorem 1 let $\beta_{cr}<\beta<\beta_{n0}^{-}$. Then there exist positive functions $m(\beta)$, $\sigma_2(\beta)$, $m(\beta)\sim$* const $(\beta-\beta_{cr})^{\frac{1}{2}}$, *$\sigma_2(\beta)\sim$* const $(\beta-\beta_{cr})$, *such that for a sequence $m_n(\beta)$, $\lim_{n\to\infty} m_n(\beta)=m(\beta)$,*

$$\lim_{n\to\infty} \operatorname{Prob}_n\{t_1<(s_{n,0}-m_n(\beta))r^{\frac{n}{2}}<t_2;\beta\} = \frac{1}{2}\,\frac{1}{\sqrt{2\pi\sigma_2(\beta)}} \int_{t_1}^{t_2} e^{-\frac{u^2}{2\sigma_2(\beta)}}\,du\,,$$

$$\lim_{n\to\infty} \operatorname{Prob}_n\{t_1<(s_{n,0}+m_n(\beta))r^{\frac{n}{2}}<t_2;\beta\} = \frac{1}{2}\,\frac{1}{\sqrt{2\pi\sigma_2(\sigma)}} \int_{t_1}^{t_2} e^{-\frac{u^2}{2\sigma_2(\beta)}}\,du\,.$$

3. Thermodynamical Limit of Hierarchical and Asymptotically-Hierarchical Models

Let V be a countable set and r_n be a sequence of integers, $r_n>1$. Following Vershik let us say that a hierarchical $\{r_n\}$-structure is defined on V if there is defined a decreasing sequence $\xi_1 \geqq \xi_2 \geqq \cdots$ of partitions of V with the following properties:

a$_1$) $\xi_1=\varepsilon$ where ε is as usual the partition of V into separate points;

a$_2$) any element of ξ_i consists of r_i elements of ξ_{i-1};

a$_3$) for any two points $x, y \in V$ there exists a number $d(x, y)$ such that x, y belong to the same element of any partition ξ_i, $i \geqq d(x, y)$.

In the following $d(x, y)$ will be the least number with this property. The number $\prod_{i=1}^{d(x,y)} r_i$ plays the role of a distance between x, y. All spaces with $\{r_n\}$-structure are naturally isomorphic.

If V is the space with $\{r_n\}$-structure then $V^{(k)} = V \mid \xi_k$ is the space with $\{r'_n\}$-structure where $r'_n = r_{n+k}$.

Let us denote by $G = G(V)$ the group of all finite permutations of V leaving each ξ_i invariant, and let $\Omega(V)\,(\Omega_0(V))$ be the space of all real-valued (± 1-valued) functions on V. We can define in the usual way the probability distributions on $\Omega(V)$ and $\Omega_0(V)$, gaussian probability distributions on $\Omega(V)$, the distributions invariant under the group G, and so on.

Now let us return to the definition of Dyson's hierarchical models. One can consider the volumes $V_{n,0}$ as an increasing sequence of subsets of the infinite space V with $\{r_n\}$-structure such that $\bigcup_{n=1}^{\infty} V_{n,0} = V$. The sequence of probability distributions $\mathrm{Prob}_n(\cdot \mid \beta)$ can be considered as a sequence of probability distributions on $\Omega_0(V)$ defined on an increasing sequence of corresponding σ-algebras. Dyson in [4] proved in fact the following theorem.

Theorem 4. *For any $\beta > 0$ the sequence of probability distributions $\mathrm{Prob}_n(\cdot \mid \beta)$ converges in a natural sense to a limit Gibbsian distribution $P(\cdot \mid \beta)$ defined on the σ-algebra of measurable subsets of $\Omega_0(V)$ and invariant under the group G.*

For the asymptotically-hierarchical models the same considerations lead to the following theorem.

Theorem 5. *Under the conditions of Theorem 1 the probability distributions $\mathrm{Prob}_n(\cdot \mid \beta)$ converge in a natural way to a limit Gibbsian distribution $P(\cdot \mid \beta)$. The distribution on $\Omega(V^{(n_0)}) = \Omega(V \mid \xi_{n_0})$ induced by a map*

$$\pi_{n_0} : \Omega_0(V) \to \Omega(V_{n_0}) \quad \textit{where} \quad \pi_{n_0}(f)(x) = \frac{1}{\prod_{i=1}^{n_0} r_i} \sum_{y \in x} f(y)$$

is invariant under the corresponding group G.

Now we shall formulate the results concerning the limit distributions $P(\cdot \mid \beta)$ which are equivalent to the Theorems 1, 2, 3. Let be $s_n = \frac{1}{r^n} \sum_{x \in C_n} s(x)$ where C_n is an arbitrary element of the partition ξ_n entering into the definition of a hierarchical structure on V. We shall consider the distribution of s_n which depends only on n but not on C_n.

Theorem 1′. *Under the conditions of Theorem 1 for $\beta=\beta_{cr}$ and arbitrary fixed t_1, t_2*

$$\lim_{n\to\infty} \mathrm{P}\{t_1 < s_n \cdot c^{\frac{n}{2}} < t_2 ; \beta_{cr}\} = \frac{1}{\sqrt{2\pi\sigma'}} \int_{t_1}^{t_2} e^{-\frac{u^2}{2\sigma'}} du$$

for some positive constant σ'.

Corollary 1′. *There exist constants C_1, C_2 such that for $\beta=\beta_{cr}$*

$$C_1 c^{-d(x,y)} \leqq \langle s_x, s_y \rangle_{\beta_{cr}} \leqq C_2 c^{-d(x,y)}.$$

Theorem 2′. *Under the conditions of Theorems 1 and 2 for arbitrary fixed t_1, t_2*

$$\lim_{n\to\infty} \mathrm{P}\{t_1 < s_n r^n < t_2 ; \beta\} = \frac{1}{\sqrt{2\pi\sigma_1'(\beta)}} \int_{t_1}^{t_2} e^{-\frac{u^2}{2\sigma_1'(\beta)}} du$$

where $\sigma_1'(\beta) \sim \mathrm{const}(\beta_{cr}-\beta)^{-1}$.

Corollary 2′. *Under the conditions of Theorem 2 the binary correlations functions $\langle s_x, s_y\rangle_\beta$, $x, y \in V$ satisfy the inequalities*

$$\frac{C_1'}{\beta_{cr}-\beta} r^{-d(x,y)} \leqq \langle s_x, s_y \rangle_\beta \leqq \frac{C_2'}{\beta_{cr}-\beta} r^{-d(x,y)}$$

with some constants C_1', C_2'.

Theorem 3′. *In the notation of Theorem 3 under the conditions of the Theorem 1 for $\beta_{cr} < \beta < \beta_{n_0}^-$ and arbitrary t_1, t_2*

$$\lim_{n\to\infty} \mathrm{P}\{t_1 < (s_n - m_n(\beta)) r^{\frac{n}{2}} < t_2\} = \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma_2'(\beta)}} \int_{t_1}^{t_2} e^{-\frac{u^2}{2\sigma_2'(\beta)}} du$$

where $\sigma_2'(\beta) \sim \mathrm{const}(\beta-\beta_{cr})^{-1}$.

4. Proof of Theorem 1

We shall consider only the case $r=2$ and $Q(z_1, z_2) = c\left(\frac{z_1+z_2}{2}\right)^2$. The generalization to other cases is straight-forward. We shall construct a sequence of imbedded segments $\boldsymbol{\beta}_n = [\beta_n^-, \beta_n^+]$ and a sequence of differentiable functions $b_n(\beta)$ defined on $\boldsymbol{\beta}_n$ such that for all $\beta \in \boldsymbol{\beta}_n$ the functions $f_n(z;\beta)$ can be represented in the following form

$$f_n(z;\beta) = L_n(\beta) \exp\left[-\left(a_0(\beta) + \left(\frac{2}{c}\right)^n b_n(\beta)\right) z^2\right] (1 + q_n(z;\beta)) \qquad (5)$$

where $L_n(\beta)$, $b_n(\beta)$, $q_n(z;\beta)$ satisfy some estimates listed below. From these estimates it follows that

$$\left(\frac{2}{c}\right)^n b_n(\beta)\xrightarrow[n\to\infty]{}0;\ 0\leqq 1+q_n(z;\beta)\leqq 2 \quad \text{for} \quad \beta\in\boldsymbol{\beta}_n$$

and $-\infty<z<\infty$;

$$q_n(z;\beta)\xrightarrow[n\to\infty]{}0 \quad \text{uniformly for} \quad |z|<\frac{d\sqrt{n}}{\sqrt{a_0}},$$

$\beta\in\boldsymbol{\beta}_n$, d is a constant depending only on c, $L_n(\beta)\to L=\sqrt{\dfrac{a_0(\beta)}{\pi}}$. Thus we get the assertion of Theorem 1 for $\beta_{cr}=\bigcap\limits_n \boldsymbol{\beta}_n$.

The representation (5) is non-unique because one can change b_n or L_n and include the difference in q_n without changing f_n. The most crucial part of the proof is the special choice of L_n and b_n. This will be described precisely during the proof.

The variable z in (5) varies from $-c^{\frac{n}{2}}$ to $c^{\frac{n}{2}}$ with the step $\Delta_n=2c^{\frac{n}{2}}\cdot 2^{-n}$. The set of values of z will be denoted as M_n.

The substitution of (5) into the formula (2) gives $\left(\text{one should remember that } r=2,\ Q(z_1,z_2)=c\left(\dfrac{z_1+z_2}{2}\right)^2\right)$

$$f_{n+1}(z;\beta)=\tilde{L}_{n+1}\exp(-\tilde{\lambda}_{n+1}z^2)\sqrt{\frac{2\lambda_n}{\pi}}\sum_u e^{-2\lambda_n u^2}\left(1+q_n\left(\frac{z}{\sqrt{c}}-u;\beta\right)\right)$$
$$\cdot\left(1+q_n\left(\frac{z}{\sqrt{c}}+u;\beta\right)\right)\Delta_n, \tag{6}$$

where

$$\tilde{\lambda}_{n+1}=\tilde{\lambda}_{n+1}(\beta)=a_0(\beta)+\left(\frac{2}{c}\right)^{n+1}b_n(\beta),$$

$$\lambda_n=\lambda_n(\beta)=a_0(\beta)+\left(\frac{2}{c}\right)^n b_n(\beta),\ \Delta_n=2c^{\frac{n}{2}}2^{-n}.$$

The summation goes over u for which $\dfrac{z}{\sqrt{c}}-u\in M_n$, $\dfrac{z}{\sqrt{c}}+u\in M_n$ and $z\in M_{n+1}$. Let us put

$$1+\tilde{q}_{n+1}(z;\beta)=\sqrt{\frac{2\lambda_n}{\pi}}\sum_u e^{-2\lambda_n u^2}\left(1+q_n\left(\frac{z}{\sqrt{c}}-u;\beta\right)\right)$$
$$\cdot\left(1+q_n\left(\frac{z}{\sqrt{c}}+u;\beta\right)\right)\Delta_n. \tag{7}$$

If we introduce the linear integral operator

$$\mathscr{A}_n q = 2\sqrt{\frac{2\lambda_n}{\pi}} \int_{-\infty}^{+\infty} e^{-2\lambda_n u^2} q\left(\frac{z}{\sqrt{c}} - u\right) du \tag{8}$$

and assume, that the function $q_n(z;\beta)$ is extended to the whole line $\mathbb{R}^1$ as a nice function of $z \in \mathbb{R}^1$ then we can rewrite (7) in the following way

$$\tilde{q}_{n+1} = \mathscr{A}_n q_n + S_n(q_n)$$

where $S_n(q_n)$ will be treated as a remainder term. In fact $S_n(q_n)$ is the sum of two terms. The first one appears from the non-linear part of the formula (7):

$$S_n^{(1)}(q_n) = \sqrt{\frac{2\lambda_n}{\pi}} \sum_u e^{-2\lambda_n u^2} q_n\left(\frac{z}{\sqrt{c}} - u; \beta\right) q_n\left(\frac{z}{\sqrt{c}} + u; \beta\right) \Delta_n. \tag{9}$$

The second term $S_n^{(2)}(q_n)$ has the form

$$S_n^{(2)}(q_n) = (\tilde{\mathscr{A}}_n - \mathscr{A}_n)\,(q_n + \tfrac{1}{2}), \tag{9'}$$

where

$$\tilde{\mathscr{A}}_n q = 2\sqrt{\frac{2\lambda_n}{\pi}} \sum_u e^{-2\lambda_n u^2} q_n\left(\frac{z}{\sqrt{c}} - u; \beta\right) \Delta_n. \tag{10}$$

These properties of $\mathscr{A}_n$ can be easily verified:

1°. $\mathscr{A}_n$ is a self-adjoint compact operator in the Hilbert space $\mathscr{H}(\gamma_n) = L^2_{\mathrm{ev}}(\mathbb{R}^1; e^{-\gamma_n z^2})$ of real-valued even square-integrable functions with the weight $e^{-\gamma_n z^2}$, $\gamma_n = 2\lambda_n\left(1 - \frac{1}{c}\right)$.

2°. $\mathscr{A}_n \mathbb{1} = 2 \cdot \mathbb{1}$.

3°. The spectrum of $\mathscr{A}_n$ in the space $\mathscr{H}(\gamma_n)$ consist of the numbers $\left\{\frac{2}{c^j}\right\}_{j=0}^{\infty}$; the corresponding eigenfunctions are even Hermite polynomials $G_{2j}(z;\gamma_n)$, $j = 0, 1, \ldots$.

4°. $\|\mathscr{A}_n q\|_C \leqq 2\|q\|_C$

$\|\mathscr{A}_n q\|_{C^1} \leqq K\|q\|_C$

where K is some constant.

From 3° it follows that two eigenvalues $2, 2\cdot c^{-1}$ are always greater than 1; the others are smaller than one because of the condition $c > \sqrt{2}$. The main idea is to choose $b_n(\beta)$ and $L_n(\beta)$ in such a way that the projections of $q_n(z;\beta)$ on the expanding subspaces $\mathscr{H}_{\exp}(\gamma_n)$ generated by eigenvectors $G_0(z;\gamma_n)$, $G_2(z;\gamma_n)$ are equal to zero or at least very small. The possibility of doing so for $n = n_0$ follows easily from the Condition 2.

Let us fix $\omega = \omega(c) > 1$ so that $\omega - 1$ is sufficiently small and consider the sequence $n_i = [n_0 \omega^i]$, $i = 0, 1, \ldots$. We shall prove that for $n = n_0, n_1, n_2, \ldots$, the representations

$$f_n(z;\beta) = L_n(\beta) \exp\left(-\left(a_0(\beta) + \left(\frac{2}{c}\right)^n b_n(\beta)\right) z^2\right)(1 + q_n(z;\beta))$$

can be chosen in such a way that

1) $L_n(\beta)$, $b_n(\beta)$ are differentiable functions of $\beta \in \boldsymbol{\beta}_n$,

$$b_n(\beta_n^-) = -\left(\frac{c}{2}\varrho\right)^n, \quad b_n(\beta_n^+) = \left(\frac{c}{2}\varrho\right)^n, \quad |b_n(\beta)| \leqq \left(\frac{c}{2}\varrho\right)^n, \quad \beta \in \boldsymbol{\beta}_n.$$

2) $q_n(z;\beta)$, $z \in M_n$, $|z| < \dfrac{d\sqrt{n}}{\sqrt{a_0}}$, can be represented in the form

$$q_n(z;\beta) = \delta_n G_4(z;\gamma_n) + R_n(z;\beta) \tag{11}$$

where $\delta_n = \delta_n(\beta)$ is a differentiable function of $\beta \in \boldsymbol{\beta}_n$ and

$$K_n^{(1)}\left(\frac{2}{c^2}\right)^{n-n_0} < -\delta_n < K_n^{(2)}\left(\frac{2}{c^2}\right)^{n-n_0}$$

where $K_n^{(i)} = \delta_{n_0}\left(\prod_{j=n_0}^{n}(1+\alpha^j)\right)^{(-1)^i}$, $i = 1, 2$, $0 < \alpha = \alpha(c) < 1$, $R_n(z;\beta)$ is a differentiable function of $\beta \in \boldsymbol{\beta}_n$,

$$|R_n(z;\beta)| < q^n. \tag{12}$$

3) for $|z| > \dfrac{d\sqrt{n}}{\sqrt{a_0}}$

$$0 \leqq 1 + q_n(z;\beta) \leqq \exp(-v_n z^4)$$

with

$$v_n = v_{n_0}\left(\frac{2}{c^2}\right)^{n-n_0} \prod_{j=n_0}^{n}(1+\alpha^j)^{-1}.$$

In order to use the contractive properties of $\mathscr{A}_n$ we shall employ a more detailed representation of $q_n(z;\beta)$ for $n_i < n \leqq n_{i+1}$. Namely, let us write

2′) for $|z| < \dfrac{d\sqrt{n}}{\sqrt{a_0}}$

$$q_n(z;\beta) = \sum_{j=2}^{N} \delta_n^{(j)} G_{2j}(z;\gamma_n) + H_n(z;\beta) + T_n(z;\beta)$$

where $N = N(c)$ is so big that

$$\xi^N < \frac{1}{2}, \left(\frac{2}{c_0^N}\right)^{\omega-1} e^{d^2} < \frac{q}{2}, \qquad c_0 = \xi c > 1\,;$$

$$K_n^{(1)}\left(\frac{2}{c^2}\right)^n < -\delta_n^{(2)} < K_n^{(2)}\left(\frac{2}{c^2}\right)^n,$$

$$|\delta_n^{(j)}| < 2q^{n_1}\left(\frac{2}{c_0^j}\right)^{n-n_1-1}, \qquad j = 3, \ldots, N.$$

The function $H_n(z;\beta)$ is extended to the whole segment

$$D_n = \left[-\frac{d\sqrt{n}}{\sqrt{a_0}}, \frac{d\sqrt{n}}{\sqrt{a_0}}\right]$$

as a C^1-function and $H_n(z;\beta) = 0$ for $z \notin D_n$ and

$$\left\| \sum_{j=2}^{N} \delta_n^{(j)} G_{2j}(z;\gamma_n) + H_n(z;\beta) \right\|_{C^m(D_n)} \leqq K_m 3^{n-n_1} q^{n_1},$$

$m = 0, 1$, $K_m = K_m(c)$ is a constant,

$$(H_n, G_{2j})_{\mathscr{H}(\gamma_n)} = \int_{z\in D_n} H_n(z;\beta)\, G_{2j}(z;\gamma_n) e^{-\gamma_n z^2}\, dz = 0, \qquad j = 0, 1, \ldots, N,$$

$$\|H_n\|_{\mathscr{H}(\gamma_n)} \leqq 2q^{n_i}\left(\frac{2}{c^{N+1}}\right)^{n-n_i-1}.$$

All the functions $\delta_n^{(j)} = \delta_n^{(j)}(\beta)$, $2 \leqq j \leqq N$, $H_n(z;\beta)$, $T_n(z;\beta)$ are differentiable functions of $\beta \in \boldsymbol{\beta}_n$ and

$$|T_n(z;\beta)| < (\xi q)^{n_1+1} 3^{n-n_1+1}.$$

It is very important that in 2), 2′) the main role is played by the projection of q_n on the third eigenspace generated by $G_4(z;\gamma_n)$ and that this projection is negative. From this fact it follows that $1 + q_n(z;\beta)$ has the form drawn in Fig. 1:

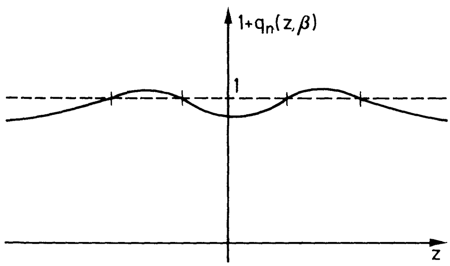

Fig. 1

Let us denote by $(\mathscr{U}_n)$ the properties 1), 2), 3) and by $(\mathscr{V}_n)$ the properties 1), 2′), 3). Now Theorem 1 will follow from the next three lemmas.

Lemma 1. *$(\mathscr{V}_n)$ implies $(\mathscr{U}_n)$ for $n=n_{i+1}$ if $N=N(c)$ is sufficiently large.*

Lemma 2. *For $n_i<n\leqq n_{i+1}$, q_{n+1} can be chosen in such a way that $(\mathscr{V}_n)$ implies $(\mathscr{V}_{n+1})$.*

Lemma 3. *For $n=n_i$, q_{n+1} can be chosen in such a way that $(\mathscr{U}_n)$ implies $(\mathscr{V}_{n+1})$.*

Proof of Lemma 1. Only the implication 2′)⇒2) must be proved. From 2′) one has

$$\int_{z\in D_n} |H_n(z;\beta)|^2\,dz \leqq e^{d^2 n} \int_{z\in D_n} |H_n(z;\beta)|^2 e^{-\gamma_n z^2}\,dz\,; \tag{13}$$

$$\|H_n(z;\beta)\|_{C^1(D_n)} \leqq \left\| \sum_{j=2}^{N} \delta_n^{(j)} G_{2j}(z;\gamma_n) + H_n(z;\beta) \right\|_{C^1(D_n)}$$
$$+ \sum_{j=2}^{N} |\delta_n^{(j)}|\ \|G_{2j}(z;\gamma_n)\|_{C^1(D_n)} \leqq 1\,,$$

because $N=N(c)$ is fixed, $\|G_{2j}(z;\beta)\|_{C^1(D_n)}$ increase as some power of n, $|\delta_n^{(j)}|$ decrease as a geometric progression and n_0 can be assumed to be sufficiently large.

Let us introduce the function $\varphi(z)$ where

$$\varphi(z)=\begin{cases} 1+z & -1\leqq z\leqq 0 \\ 0 & \text{for other } z \end{cases}$$

From the inequality $\left|\dfrac{d}{dz} H_n(z;\beta)\right|<1$ one has

$$|H_n(z;\beta)| \geqq |H_n(z_0;\beta)\,\varphi(z-z_0)|$$

for any $z_0>0$, $z_0<\dfrac{d\sqrt{n}}{\sqrt{a_0}}$. This gives

$$\|H_n(z;\beta)\|_{L^2} \geqq |H_n(z_0;\beta)|\ \|\varphi(z-z_0)\|_{L^2} = \sqrt{\tfrac{1}{3}}\,|H_n(z_0;\beta)|\,.$$

Now the desired estimate for the $C(D_n)$-norm of $H_n(z;\beta)$ follows from the $\mathscr{H}(\gamma_n)$-norm of $H_n(z;\beta)$. Other terms in $R_n(z;\beta)$ have good C-norms, as can be seen directly from 2′). Thus we have the desired estimate for the C-norm of $R_n(z;\beta)$. Q.E.D.

Proof of Lemma 2. We shall use an important local property of the expression (7) for $1+\tilde{q}_{n+1}(z;\beta)$. Namely, the part of the sum in (7) for u, $|u|>\dfrac{d_1\sqrt{n}}{\sqrt{a_0}}$, with some constant $d_1=d_1(c)$ gives a value which is less then $\varepsilon = 5e^{-d_1^2 n}$ because from 2′) and 3) one has $0 \leqq 1 + q_n(z;\beta) \leqq 2$. Therefore $\tilde{q}_{n+1}(z_0;\beta)$ depends mainly on the values of $q_n(z;\beta)$ for $\left|z - \frac{z_0}{\sqrt{c}}\right| < \frac{d_1\sqrt{n}}{\sqrt{a_0}}$ with ε-error. In particular for

$$z \in D_{n+1} = \left[-\frac{d\sqrt{n+1}}{\sqrt{a_0}}, \frac{d\sqrt{n+1}}{\sqrt{a_0}}\right]$$

$\tilde{q}_{n+1}(z;\beta)$ is defined with ε-error by the values of $q_n(z;\beta)$ for $z \in D_n$ because it follows from the Condition 2 that

$$1 - \frac{1}{\sqrt{c}} > \frac{d_1}{d}.$$

Now we proceed to construct $q_{n+1}(z;\beta)$. First of all let us include the part of the sum in (7) with $|u| \geqq \frac{d_1\sqrt{n}}{\sqrt{a_0}}$ in $T_{n+1}(z;\beta)$. If d_1 is sufficiently large this term will satisfy the estimates for T_{n+1}.

Now we shall estimate the remainder term $S_n(q_n) = S_n^{(1)}(q_n) + S_n^{(2)}(q_n)$ [see (9)]. Due to the quadratic character of $S_n^{(1)}(q_n)$ one has

$$\|S_n^{(1)}(q_n)\|_{C(D_{n+1})} \leqq 1,1 \cdot \|q_n\|^2_{C(D_n)}.$$

This inequality shows that $S_n^{(1)}(q_n)$ can be also included in $T_{n+1}(z;\beta)$ if the estimate of $\|q_n\|^2_{C(D_n)}$ which follows from 2′) is better than the estimate of $T_{n+1}(z;\beta)$ in 2′). The last assertion is true when $\omega - 1$ is sufficiently small.

Let us denote

$$B_n q(z) = 2\sqrt{\frac{2\lambda_n}{\pi}} \int_{u \in D_n} e^{-2\lambda_n u^2} q\left(\frac{z}{\sqrt{c}} - u\right) du,$$

$$\tilde{B}_n q(z) = 2\sqrt{\frac{2\lambda_n}{\pi}} \sum_{u \in D_n} e^{-2\lambda_n u^2} q\left(\frac{z}{\sqrt{c}} - u\right) \Delta n.$$

It is easy to see that

$$\|\tilde{B}_n q\|_C \leqq 2,1 \, \|q\|_C.$$

So we can include the term $\tilde{B}_n T_n(z;\beta)$ into $T_{n+1}(z;\beta)$. Next we consider instead of the function $\tilde{B}_n\left(\frac{1}{2} + \sum_{j=2}^{N} \delta_n^{(j)} G_{2j}(z;\gamma_n) + H_n(z;\beta)\right)$ the function $\mathscr{A}_n\left(\frac{1}{2} + \sum_{j=2}^{N} \delta_n^{(j)} G_{2j}(z;\gamma_n) + H_n(z;\beta)\right)$. The error can be estimated using the C^1-estimate of 2′) of the function $\sum_{j=2}^{N} \delta_n^{(j)} G_{2j}(z;\gamma_n) + H_n(z;\beta)$ and the following inequalities:

$$\|(\tilde{B}_n - B_n)\, q\|_C \leqq \|q\|_{C^1}\, \Delta_n$$

(which follow from a simple interpolation formula) and

$$\|(\mathscr{A}_n - \mathscr{B}_n)\, q\|_C = \left\| 2 \sqrt{\frac{2\lambda_n}{\pi}} \int_{|u| \geqq \frac{d_1\sqrt{n}}{\sqrt{a_0}}} e^{-2\lambda_n u^2} q\left(\frac{z}{\sqrt{c}} - u\right) du \right\|_C$$
$$\leqq 2\|q\|_C\, e^{-d_1^2 n}.$$

Due to these inequalities we include the error into $T_{n+1}(z;\beta)$. From the property 3° of the operator $\mathscr{A}_n$ it follows that $\mathscr{A}_n(\frac{1}{2}) = 1$ and

$$\mathscr{A}_n\left(\sum_{j=2}^{N} \delta_n^{(j)} G_{2j}(z;\gamma_n) + H_n(z;\beta)\right) = \sum_{j=2}^{N} \delta_n^{(j)} \frac{2}{c^j} G_{2j}(z;\gamma_n)$$
$$+ \mathscr{A}_n H_n(z;\beta), \qquad H_n \perp G_{2j} \quad \text{in} \quad \mathscr{H}(\gamma_n), \qquad j = 0, 1, \ldots, N,$$

and consequently

$$\|\mathscr{A}_n H_n\|_{\mathscr{H}(\gamma_n)} \leqq \frac{2}{c^{N+1}} \|H_n\|_{\mathscr{H}(\gamma_n)}.$$

It follows from the last estimate that

$$\mathscr{A}_n H_n(z;\beta) = \sum_{j=0}^{N} \tilde{\delta}_{n+1}^{(j)} G_{2j}(z;\gamma_n) + \tilde{H}_{n+1}(z;\beta)$$

where $\tilde{H}_{n+1}(z;\beta)$ is a smooth function of $z \in D_{n+1}$, $\tilde{H}_{n+1}(z;\beta) = 0$ for

$$z \notin D_{n+1}, (\tilde{H}_{n+1}, G_{2j})_{\mathscr{H}(\gamma_n)} = 0, j = 0, 1, \ldots, N, |\tilde{\delta}_{n+1}^{(j)}|,$$

$$\|\tilde{H}_{n+1}\|_{\mathscr{H}(\gamma_n)} \leqq \frac{2}{c^N} \|H_n\|_{\mathscr{H}(\gamma_n)}.$$

Therefore denoting $\tilde{\tilde{\delta}}_{n+1}^{(j)} = \frac{2}{c^j} \delta_n^{(j)} + \tilde{\delta}_{n+1}^{(j)}$ we have

$$1 + \tilde{q}_{n+1}(z;\beta) = 1 + \sum_{j=0}^{1} \tilde{\delta}_{n+1}^{(j)} G_{2j}(z;\gamma_n) + \sum_{j=2}^{N} \tilde{\tilde{\delta}}_{n+1}^{(j)} G_{2j}(z;\gamma_n)$$
$$+ \tilde{H}_{n+1}(z;\beta) + \tilde{T}_{n+1}(z;\beta),$$

where $\tilde{\tilde{\delta}}_{n+1}^{(j)}, \tilde{H}_{n+1}, \tilde{T}_{n+1}$ satisfy 2′). Let us put

$$\gamma_{n+1} = 2\left(a_0(\beta) + \left(\frac{2}{c}\right)^{n+1} b_n(\beta)\right)\left(1 - \frac{1}{c}\right)$$

and rewrite the last equality putting γ_{n+1} instead of γ_n

$$1 + \tilde{q}_{n+1}(z;\beta) = 1 + \sum_{j=0}^{1} \delta_{n+1}^{(j)} G_{2j}(z;\gamma_{n+1}) + \sum_{j=2}^{N} \delta_{n+1}^{(j)} G_{2j}(z;\gamma_{n+1})$$
$$+ H_{n+1}(z;\beta) + T_{n+1}(z;\beta),$$

where $\delta_{n+1}^{(j)}, j=2, \ldots, N, H_{n+1}, T_{n+1}$ also satisfy 2′) because

$$|\gamma_n - \gamma_{n+1}| < \left(\frac{2}{c}\right)^{n+1} |b_n(\beta)| < \frac{2}{c} \varrho^n, \quad |\delta_{n+1}^{(j)}| < |\gamma_n - \gamma_{n+1}| \, |\delta_n^{(2)}| \leqq \left(\frac{2\varrho}{c^2}\right)^n, \quad j=0,1.$$

For $z \in D_{n+1}$

$$1 + \sum_{j=0}^{1} \delta_{n+1}^{(j)} G_{2j}(z; \gamma_{n+1}) = e^{\sum_{j=0}^{1} \delta_{n+1}^{(j)} G_{2j}(z; \gamma_{n+1})} + T_{n+1}^{(1)}(z; \beta)$$

where $T_{n+1}^{(1)}$ satisfies the estimate of T_{n+1} in 2′). Now we can define q_{n+1} from the expression

$$1 + q_{n+1}(z; \beta) = e^{-\sum_{j=0}^{1} \delta_{n+1}^{(j)} G_{2j}(z; \gamma_{n+1})} (1 + \tilde{q}_{n+1}(z; \beta)).$$

It follows from the above estimates that with this definition of $q_{n+1}(z; \beta)$ 2′) is true.

The function $\sum_{j=0}^{1} \delta_{n+1}^{(j)} G_{2j}(z; \gamma_{n+1})$ is the even quadratic polynomial of z. Thus we have the representation (5) for $f_{n+1}(z; \beta)$ where the Gaussian multiplier is

$$\begin{aligned}\tilde{L}_{n+1}(\beta) \exp\left(-\left(a_0(\beta) + \left(\frac{2}{c}\right)^{n+1} b_n(\beta)\right) z^2 + \sum_{j=0}^{1} \delta_{n+1}^{(j)} G_{2j}(z; \gamma_{n+1})\right) \\ = L_{n+1}(\beta) \exp\left(-\left(a_0(\beta) + \left(\frac{2}{c}\right)^{n+1} b_{n+1}(\beta)\right) z^2\right).\end{aligned}$$

The last expression gives also the definition of the $b_{n+1}(\beta)$. It follows that

$$\left(\frac{2}{c}\right)^{n+1} |b_n(\beta) - b_{n+1}(\beta)| < K \left(\frac{2\varrho}{c^2}\right)^n, \quad K = K(c, \beta_{n0}^-, \beta_{n0}^+)$$

is a constant. Therefore

$$\left(\frac{2}{c}\right)^{n+1} b_{n+1}(\beta_n^+) > \left(\frac{2}{c}\right)^{n+1} b_n(\beta_n^+) - K \left(\frac{2\varrho}{c^2}\right)^n = \varrho^n \left(\frac{2}{c} - K \left(\frac{2}{c^2}\right)^n\right) > \varrho^n$$

and in a similar way $\left(\frac{2}{c}\right)^{n+1} b_{n+1}(\beta_n^-) < -\varrho^n$. Furthermore $b_{n+1}(\beta)$ is a differentiable function of β and we can find a segment $\beta_{n+1} = [\beta_{n+1}^-, \beta_{n+1}^+] \subset \boldsymbol{\beta}_n$ satisfying 1).

Next we consider the estimate 3) for $1 + q_{n+1}(z; \beta)$. It can be easily seen from 2) that the estimate 3) is true not only for $|z| > \frac{d\sqrt{n}}{\sqrt{a}}$ but also for $|z|>d_2$ where d_2 is some constant not depending on n. Therefore $\{z\colon 1+q_{n+1}(z;\beta)>1\}\subset D=[-d_2,d_2]$ and we can write for all z

$$0\leqq 1+q_n(z;\beta)\leqq e^{-\nu_n z^4}(1+\mu_n\chi_D(z))$$

where $\nu_n=\nu_{n_0}\left(\frac{2}{c^2}\right)^{n-n_0}\prod_{j=n_0}^{n}(1+\alpha^j)^{-1}$, $\mu_n=2|\delta_n^{(2)}|$, and χ_D is the indicator of D. Substitution of the last inequality into (7) gives

$$1+\tilde{q}_{n+1}(z;\beta)<\sqrt{\frac{2\lambda_n}{\pi}}\sum_u e^{-2\lambda_n u^2}e^{-\nu_n\left(\frac{2}{c^2}z^4+\frac{12}{c}z^2u^2+2u^4\right)}$$

$$\cdot\left(1+\mu_n\chi_D\left(\frac{z}{\sqrt{c}}-u\right)\right)\left(1+\mu_n\chi_D\left(\frac{z}{\sqrt{c}}+u\right)\right)=a_1+a_2\mu_n+a_3\mu_n^2\,.$$

It is easy to see that for

$$0\leqq a_1<\exp\left(-\nu_n\frac{2}{c^2}z^4\right)\left(1-\frac{K_0}{n+1}\right)$$

where K_0 is a constant not depending on n,

$$0\leqq a_2<K_1\exp\left(-\nu_n\frac{2}{c^2}z^4\right),$$

$$a_3=0\quad\text{because}\quad \chi_D\left(\frac{z}{\sqrt{c}}-u\right)\chi_D\left(\frac{z}{\sqrt{c}}+u\right)=0\quad\text{for}\quad |z|>\frac{d_2\sqrt{n+1}}{\sqrt{a_0}}.$$

This gives

$$0\leqq 1+\tilde{q}_{n+1}(z;\beta)\leqq\exp\left(-\nu_n\frac{2}{c^2}z^4\right)\left(1-\frac{K_0}{n+1}+K_1\mu_n\right)$$

$$\leqq\exp\left(-\frac{2}{c^2}\nu_n z^4\right).$$

Now 3) for $1+q_{n+1}(z;\beta)$ is a simple consequence of the last inequality. Lemma 2 is proved.

Proof of Lemma 3. We rewrite the formula (7) in the following way

$$\tilde{q}_{n+1}(z;\beta)=\tilde{\mathcal{A}}_n q_n(z;\beta)+S_n^{(1)}(q_n)(z;\beta)$$

where

$$\tilde{\mathcal{A}}_n q(z)=2\sqrt{\frac{2\lambda_n}{\pi}}\sum_u e^{-2\lambda_n\left(\frac{z}{\sqrt{c}}-u\right)^2}q(u)\,\Delta_n \tag{14}$$

is the linear part of the transformation (7) and $S_n^{(1)}(q_n)$ defined in (9), is the nonlinear part of this transformation. One can easily verify the following properties of the operator $\tilde{\mathcal{A}}_n$:

1°. The formula (14) is meaningful for all $z \in \mathbb{R}^1$ even if $q_n(z;\beta)$ is defined only for $z \in M_n$ and

$$\|\tilde{\mathscr{A}}_n q\|_{C(\mathbb{R}^1)} < 3\,\|q\|_{C(M_n)}$$
$$\|\tilde{\mathscr{A}}_n q\|_{C^1(\mathbb{R}^1)} < K\,\|q\|_{C(M_n)} \qquad K \text{ is a constant}$$

not depending on q and n.

2°. For $|z| < \dfrac{d\sqrt{n+1}}{\sqrt{a_0}}$

$$\tilde{\mathscr{A}}_n G_4(z;\gamma_n) = \mathscr{A}_n G_4(z;\gamma_n) + (\tilde{\mathscr{A}}_n - \mathscr{A}_n)\, G_4(z;\gamma_n) = \frac{2}{c^2}\, G_4(z;\gamma_n) + T^{(1)}_{n+1}(z;\beta)$$

where $\mathscr{A}_n$ is the integral operator, defined in (8) and $T^{(1)}_{n+1}(z;\beta)$ satisfies the estimate of the $T_{n+1}(z;\beta)$ in 2′).

It follows from these properties of the operator $\tilde{\mathscr{A}}_n$ and from the estimate of $S^{(1)}_n(q_n)(z;\beta)$ during the proof of Lemma 2 that

$$\tilde{q}_{n+1}(z;\beta) = \frac{2}{c^2}\,\delta_n G_4(z;\gamma_n) + \tilde{\mathscr{A}}_n R_n(z;\beta) + T_{n+1}(z;\beta)$$

where the sum $\dfrac{2}{c^2}\,\delta_n G_4(z;\gamma_n) + \tilde{\mathscr{A}}_n R_n(z;\beta)$ satisfies the C^m-estimate, $m = 0, 1$, of 2′) because of property 1° of the operator $\tilde{\mathscr{A}}_n$ and $T_{n+1}(z;\beta)$ satisfies the corresponding estimate of 2′) too. Next we decompose the sum $\dfrac{2}{c^2}\,\delta_n G_4(z;\gamma_n) + \tilde{\mathscr{A}}_n R_n(z;\beta)$ (this decomposition is unique):

$$\frac{2}{c^2}\,\delta_n G_4(z;\gamma_n) + \tilde{\mathscr{A}}_n R_n(z;\beta) = \sum_{j=0}^{N} \delta^{(j)}_{n+1} G_{2j}(z;\gamma_{n+1}) + H_{n+1}(z;\beta) \qquad (15)$$

where $H_{n+1}(z;\beta) = 0$ for $|z| > \dfrac{d\sqrt{n+1}}{\sqrt{a_0}}$ and

$$(H_{n+1}, G_{2j})_{\mathscr{H}(\gamma_{n+1})} = \int_{z \in D_{n+1}} H_{n+1}(z;\beta)\, G_{2j}(z;\gamma_{n+1})\, e^{-\gamma_{n+1} z^2} dz = 0\,,$$
$$j = 0, 1, \ldots, N\,.$$

The desired estimates of $\delta^{(j)}_{n+1}, j = 2, \ldots, N$ and $H_{n+1}(z;\beta)$ of 2′) follow from the C-estimate of the $R_n(z;\beta)$ of 2) and from the property 1° of the operator $\tilde{\mathscr{A}}_n$. At last we annul the projections $\delta^{(j)}_{n+1} G_j(z;\gamma_{n+1}), j = 0, 1$, in the formula (15) on the expanding eigenvectors of the operator $\mathscr{A}_{n+1}$ in the same way as in the proof of the Lemma 2, i.e. changing a little the Gaussian multiplier in the representation (5). One can see that for $q_{n+1}(z;\beta)$ all the estimates of 2′) are true. The proof of the properties 1) and 3) of the function $q_{n+1}(z;\beta)$ is just the same as we used during the proof of Lemma 2. This completes the proof of Lemma 1 and also the proof of Theorem 1.

Proofs of Theorems 2, 3 will be published in another paper.

Acknowledgments. The authors express their gratitude to Professors R. L. Dobrushin, O. Lanford, I. M. Lifshitz, R. A. Minlos for valuable discussions. We thank very much Prof. Lanford for his help in preparing this text for publication.

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