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Quantum Field Theory and the Coloring Problem of Graphs

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Abstract. The ϕ^k theory is compared with the multilinear theory of scalar fields $\phi_1, \phi_2, \ldots, \phi_k$ having the same mass as that of ϕ. In particular, it is shown that Feynman integrals encountered in the ϕ^3 theory are not necessarily present also in the $\phi_1\phi_2\phi_3$ theory, but they are if they correspond to planar Feynman graphs having no tadpole part. Furthermore, a necessary and sufficient condition for the presence of a ϕ^3 Feynman integral in the $\phi_1\phi_2^2$ theory is found. Those considerations are applications of graph theory, especially of the coloring problem of graphs, to Feynman graphs.

1. Introduction and Results

The main question which we discuss in the present paper is as follows: Let $\phi_1, \phi_2, \ldots, \phi_k$ be k scalar quantized fields having an identical mass, which is also equal to the mass of another scalar field ϕ; then, to what extent the multilinear theory of $\phi_1, \phi_2, \ldots, \phi_k$ (its interaction Lagrangian density is $\mathscr{L}_I^{(k)} = \lambda\,\phi_1\phi_2 \ldots \phi_k$) is different from the ϕ^k theory ($\mathscr{L}_I = (\lambda/k!)\colon\!\phi^k\colon$)? It seems to be generally believed indistinctly that every Feynman integral of the ϕ^k theory would be present also in the $\phi_1\phi_2\ldots\phi_k$ theory, that is, every Feynman graph appearing in the ϕ^k theory[1] could be renamed as a Feynman graph of the $\phi_1\phi_2\ldots\phi_k$ theory. The existence of counterexamples to this conjecture was pointed out explicitly by the present author [1] (see Fig. 1) in connection with the proof of divergence

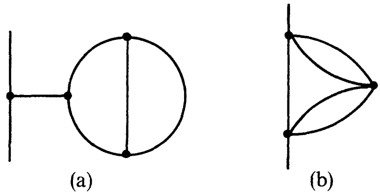

Fig. 1. Examples of ϕ^k Feynman graphs which are not realizable in the $\phi_1\phi_2 \ldots \phi_k$ theory [(a) for $k=3$ and (b) for $k=4$; for $k \geqq 5$, add $k-4$ internal lines and $k-4$ external lines to (b)]

[1] Hereafter, we abbreviate a Feynman graph appearing in the ϕ^k theory as a ϕ^k Feynman graph.

of the perturbation series. Since it is easy to construct non-tadpole counterexamples for $k \geqq 4$, in the present paper we concentrate our attention to the $k=3$ case. In this case, we obtain the following results: (1) Any ϕ^3 Feynman graph involving at least one tadpole part cannot be renamed as a $\phi_1 \phi_2 \phi_3$ Feynman graph. (2) Any planar ϕ^3 Feynman graph involving no tadpole part can be renamed as a $\phi_1 \phi_2 \phi_3$ Feynman graph, provided that the celebrated "four-color conjecture of planar maps" is valid. (3) There exist some non-planar Feynman graphs involving no tadpole part which cannot be renamed as $\phi_1 \phi_2 \phi_3$ Feynman graphs.

As a related topic to the above problem, we investigate what kind of ϕ^3 Feynman graphs can be realized also in the $\phi_1 \phi_2^2$ theory. Our result is as follows: A ϕ^3 Feynman graph G can be renamed as a $\phi_1 \phi_2^2$ Feynman graph if and only if G contains no vertex with which two or three tadpole parts are incident.

The problems stated above are rather academic. It is of more practical importance to find how many times an arbitrary Feynman integral of the ϕ^k theory appears in the $\phi_1 \phi_2 \dots \phi_k$ theory. The relative weight factor is closely related to a "coloring polynomial", which is well known in graph theory (for details, see Section 4). We establish a theorem which states that this factor is expressed in terms of a line-coloring polynomial. Line-coloring polynomials are calculated for some particular Feynman graphs, and some physical implication of the result is pointed out.

In the present paper, we have confined ourselves to the scalar-field case. If our considerations are extended to the Dirac-field case, it may be closely related to the colored quarks [2] or quarks of parastatistics [3].

In Section 2, notation and terminology are explained. In Section 3, we discuss what ϕ^3 Feynman graphs are realizable in the $\phi_1 \phi_2 \phi_3$ theory. In Section 4, we investigate the relationship between the weight factor of a Feynman integral in a multilinear theory and the line-coloring polynomial of the corresponding Feynman graph. Section 5 is devoted to the consideration on the ϕ^3 and $\phi_1 \phi_2^2$ theories.

2. Preliminaries

We extensively employ the terminology and notation used in the present author's book [4]. To be self-contained, however, we briefly review them below (for complete definitions, see Ref. [4] and [5]).

A *graph* G consists of *lines* and *vertices*; every line is *incident* with two vertices. If two vertices with which a line l is incident are identical, l is called a *loop line*. Two or more lines which are incident with two common distinct vertices are called *parallel lines*. In graph theory [5], a graph should contain neither loop lines nor parallel lines; an object which may contain them is called a pseudograph in graph theory. Since we call a pseudograph a graph, a graph-theoretical graph is called a *genuine graph* in the following.

The number of lines which are incident with a vertex a is called the *degree* of a, where each loop line incident with a should be counted twice. If all vertices of a graph G have the same degree, G is called a *homogeneous graph;* if their degree is k, G is a *degree-k homogeneous graph.*

For a graph G, the set of its lines and that of its vertices are denoted by $|G|$ and by $v(G)$, respectively. The number of the lines and that of the vertices are denoted by $N(G)$ and by $M(G)$, respectively. If G is a degree-k homogeneous graph, we of course have

$$kM(G) = 2N(G)\,. \tag{2.1}$$

A *circuit* C is a set of lines in G such that it topologically makes a circle. If a set, I, of lines contains a line incident with a vertex a, we say that I *passes through* a. The number of lines contained in I is denoted by $N(I)$. If C is a circuit, C passes through $N(C)$ vertices.

For $l \in |G|$, $G-l$ stands for the graph which is obtained from G by deleting a line l. Likewise for $I \subset |G|$, $G-I$ stands for the graph obtained by deleting all lines of I. A *reduced graph* G/l is the graph which is obtained from G by contracting l; G/I is similarly defined.

A *connected graph* is a graph which is topologically connected. Given a connected graph G (for a disconnected graph, take its connected component), a line l is called a *cut-line* of G if $G-l$ is disconnected. A line l is a cut-line if and only if there is no circuit containing l. If a connected graph G contains no cut-lines, G is called *strongly connected.*

A *planar graph* is a graph which can be embedded in a plane. A *tree graph* is a connected graph which contains no circuit.

A *Feynman graph* has two kinds of lines called *internal lines* and *external lines.* An external line is incident with only one vertex. Apart from external lines, a Feynman graph is analogous to a graph. For a Feynman graph G, $|G|$ stands for the set of all *internal* lines and $N(G)$ is their number. The number of the external lines of G is denoted by $n(G)$. For a ϕ^k Feynman graph, (2.1) is modified as

$$kM(G) = 2N(G) + n(G)\,. \tag{2.2}$$

As usual, Feynman graphs with $n(G)=0$, $n(G)=1$, $n(G)=2$, and $n(G)=3$ are called *vacuum-polarization graphs, tadpole graphs, self-energy graphs* and *vertex graphs,* respectively. Vacuum-polarization graphs are nothing but graphs. For sub-Feynman graphs, "graphs" in their names should be replaced by *parts.*

To *open* an internal line l incident with two vertices a and b means to replace l by two external lines incident with a and with b. To *link* an external line to another is the operation converse to the above.

A Feynman graph G is *planar* if the graph G' is planar, where G' is obtained from G by introducing a new vertex with which all external lines of G are incident.

Now, as we shall see below, our problem is essentially the *coloring problem* of graphs [6]. Everybody who is interested in mathematics knows the *four-color conjecture of planar maps:* In order to color any planar map in such a way that any country has a color different from the colors of its adjacent countries, it will be sufficient to use only four different colors. This proposition remains unproven since about one hundred years ago, but its validity was checked for all planar maps with up to 39 countries. In graph theory, it is customary to consider a dual graph [7] of a planar map; then the coloring of countries becomes to give colors (or symbols) to all vertices of any planar graph in such a way that no pair of adjacent vertices have the same color (or symbol). In general, if all vertices of a graph G can be colored with m different colors in such a way that no pair of adjacent vertices have the same color, G is said to be *m-colorable.* If G is m-colorable but not $(m-1)$-colorable, then m is called the *chromatic number* of G.

Quite analogously to the vertex coloring, we can define the *line-coloring problem.* If we can give m colors to all lines of a graph G in such a way that no adjacent lines have the same color[2], where *adjacent lines* are lines which are incident with a common vertex, then G is said to be *m-line-colorable.* The minimum number of m is called the *line-chromatic number* of G, which is denoted by $\chi(G)$. The concept of the line-coloring is straightforwardly extended to Feynman graphs. It is equivalent to the line-coloring problem to see whether or not a given ϕ^k Feynman graph can be renamed as a Feynman graph of a multilinear theory, because if the suffices $1, 2, \ldots, k$ of ϕ are regarded as colors the multilinear Lagrangian exactly represents the condition of the line-coloring. If we consider a Feynman graph having a loop line, it is evidently impossible to color the lines of G. Hence, except in Section 5, we forbid the presence of loop lines; in other words, we take a normal product in the interaction Lagrangian of the ϕ^k theory.

Theorems established in graph theory usually refer to genuine graphs. In order to apply them to Feynman graphs, we have to make two modifications, namely, to permit parallel lines and to include external lines. The most convenient way of reducing a Feynman graph G to a graph is to consider G^2, where G^2 is obtained from two G's by linking the $n(G)$ external lines of the first G to the corresponding ones of the second G.

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[2] Hereafter we always impose this condition in the line-coloring, so that we will not write it explicitly.

Theorem 2.1. *Given a Feynman graph G, the graph G^2 has the following properties:*

(1) $\chi(G^2)=\chi(G)$.

(2) *G^2 is a degree-k homogeneous graph if and only if G is a ϕ^k Feynman graph.*

(3) *G^2 is planar if and only if G is planar.*

Thus the line-coloring problem of a Feynman graph G reduces to that of a graph G^2.

Theorem 2.2. [8]. *If G is a genuine graph and if d is the maximum degree of vertices in G, then we have*

$$d \leqq \chi(G) \leqq d+1\,. \tag{2.3}$$

We have to note, however, that if parallel lines are present, there is no upper bound on $\chi(G)-d$; hence Theorem 2.2 is not very much useful for our purpose. As for the lower bound, it is evident that for any ϕ^k Feynman graph G we have

$$\chi(G) \geqq k\,. \tag{2.4}$$

Theorem 2.3. *Given a (possibly disconnected) Feynman graph G, let $G_1, G_2, \ldots, G_s$ be connected components, which are strongly connected, of the Feynman graph which is obtained from G by opening all cut-lines of G. Then:*

(1) *G is k-line-colorable if and only if all $G_1, G_2, \ldots, G_s$ are k-line-colorable.*

(2) *G is a ϕ^k Feynman graph if and only if all $G_1, G_2, \ldots, G_s$ are ϕ^k Feynman graphs.*

(3) *G is planar if and only if all $G_1, G_2, \ldots, G_s$ are planar.*

Owing to Theorem 2.3, we may confine ourselves to strongly connected Feynman graphs, provided that we discuss the whole set of ϕ^k Feynman graphs. We cannot, however, omit the possible existence of cut-lines if we discuss the graphs G^2 instead of Feynman graphs G.

3. Line-Coloring Problem of ϕ^3 Feynman Graphs

Since, as shown in Fig. 1, we can easily construct non-tadpole examples of ϕ^k Feynman graphs with $\chi(G)>k$ for $k\geqq 4$, in this section we concentrate our attention to the line-coloring problem of ϕ^3 Feynman graphs.

In order to translate the properties concerning genuine graphs into those concerning Feynman graphs without loop lines, we have to examine the effect of the presence of parallel lines in addition to that of external

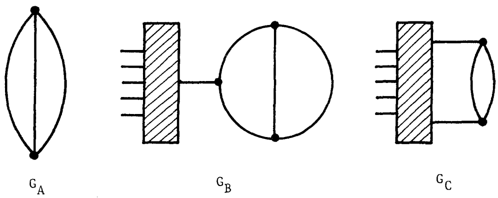

Fig. 2. All possibilities of parallel lines in the ϕ^3 theory

lines. Fortunately, in the ϕ^3 theory, there are only three types of possibilities of parallel lines as shown Fig. 2. It is evident that $\chi(G_A)=3$, and G_B reduces to a simpler Feynman graph and the simplest tadpole graph G_B^0 with $\chi(G_B^0)=4$ because of Theorem 2.3. As for G_C, the relevant part is called the *second-order self-energy part* as usual.

Theorem 3.1. *Given a ϕ^3 Feynman graph G, let $\tilde{G}$ be the ϕ^3 Feynman graph which is obtained from G by contracting every second-order self-energy part of G together with one of its two legs. Then we have $\chi(G)=3$ if and only if $\chi(\tilde{G})=3$, and G is planar if and only if $\tilde{G}$ is planar.*

Owing to Theorems 3.1 and 2.1, *the line-coloring problem of ϕ^3 Feynman graphs reduces to that of degree-3 homogeneous, genuine graphs.* Hereafter in this section we use the above consideration tacitly whenever necessary.

Theorem 3.2. *Let G be a degree-3 homogeneous graph. A necessary and sufficient condition for $\chi(G)=3$ is that there are disjoint circuits $C_1, C_2, \ldots, C_s$ such that $N(C_j)$ is even for every j and $\bigcup_{j=1}^{s} C_j$ passes through all vertices of G.*

Proof. If G is colored with three colors 1, 2 and 3, delete all lines of color 3. The resultant graph H is a degree-2 homogeneous graph with $v(H)=v(G)$, that is, $|H|$ is a union of disjoint circuits passing through all vertices of G. Furthermore, each circuit is of even length because H is 2-line-colorable. The proof of the converse proceeds exactly conversely. q.e.d.

Theorem 3.3. *If G is a ϕ^3 tadpole graph, we always have $\chi(G)=4$.*

Proof. From (2.2) we have

$$3M(G)=2N(G)+1\,, \tag{3.1}$$

whence $M(G)$ is odd. Applying Theorem 3.2 to G^2, therefore, we find $\chi(G)\neq 3$ because it is impossible to construct $C_1, C_2, \ldots, C_s$ as requested. Hence we obtain $\chi(G)=4$ because of Theorems 2.1 and 2.2. q.e.d.

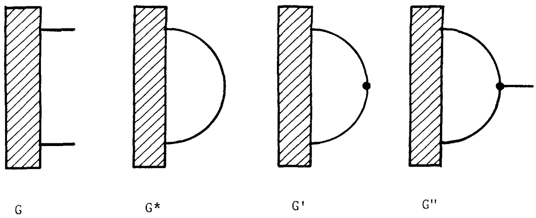

Fig. 3. Given a ϕ^3 self-energy graph G, auxiliary graphs G^*, G', and G'' are constructed in connection with Theorem 3.4

Theorem 3.4. *Let G be a ϕ^3 self-energy graph; then we have $\chi(G)=3$ if and only if $\chi(G^*)=3$, where G^* is the degree-3 homogeneous graph which is obtained from G by linking the two external lines of G (see Fig. 3). Furthermore, G is planar if and only if G^* is planar.*

Proof. If $\chi(G^*)=3$, it is evident that $\chi(G) \leqq 3$. Hence we have $\chi(G)=3$ because of (2.4). Conversely, suppose that $\chi(G)=3$. Then the lines of G can be colored with three colors 1, 2 and 3. If the colors of the two external lines are different, a graph G' is 3-line-colorable, where G' is obtained from G by introducing a new vertex e with which the two external lines of G are incident (see Fig. 3). Therefore, a ϕ^3 tadpole graph G'', which is obtained from G' by adding an external line incident with e, must be 3-line-colorable in contradiction with Theorem 3.3. Therefore, the two external lines of G must have the same color. Hence G^* is 3-line-colorable, but $\chi(G^*)<3$ is impossible; hence $\chi(G^*)=3$. q.e.d.

If the above consideration is applied to a self-energy part of a ϕ^3 Feynman graph, we obtain a generalization of Theorem 3.1.

Theorem 3.5. *Let G be a ϕ^3 vertex graph; then we have $\chi(G)=3$ if and only if $\chi(G')=3$, where G' is the degree-3 homogeneous graph which is obtained from G by introducing a new vertex with which the three external lines of G are incident (that is, G' is the graph considered for defining the planarity of G).*

Proof. It is evident that $\chi(G)=3$ follows from $\chi(G')=3$. Suppose that $\chi(G)=3$. In a line-coloring of G with three colors, if there are two external lines of the same color, a ϕ^3 tadpole graph, which is obtained by linking them, must be 3-line-colorable in contradiction with Theorem 3.3. Therefore the color of the three external lines are different from each other. Hence $\chi(G')=3$. q.e.d.

The next theorem is a well-known theorem in graph theory.

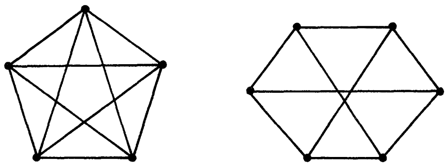

Fig. 4. Kuratowski graphs

Theorem 3.6. [9] [3]. *The four-color conjecture of planar maps is equivalent to the following proposition: "$\chi(G)=3$ for any planar, degree-3 homogeneous graph G having no cut-lines."*

This theorem can be translated into a statement concerning planar ϕ^3 Feynman graphs.

Theorem 3.7. *We have $\chi(G)=3$ for any planar ϕ^3 Feynman graph G having no tadpole part, provided that the four-color conjecture is valid.*

Proof. Since G^2 is a planar, degree-3 homogeneous graph because of Theorem 2.1, our theorem follows from Theorem 3.6 if we can show that G^2 has no cut-lines. Let R be the reduced graph which is obtained from G^2 by contracting all non-cut-lines of G^2; R is obviously a tree graph. Hence if R is non-trivial, it has a degree-1 vertex a [10]. The part of G^2 which has been contracted to a in R is nothing but a tadpole part of G^2, which is also a tadpole part of G in contradiction with the assumption. Therefore R must be trivial, that is, G^2 has no cut-lines. q.e.d.

Thus it is in all probabilities impossible to construct a planar ϕ^3 Feynman graph G having no tadpole part such that $\chi(G)=4$. In order to construct examples of $\chi(G)=4$, we have to consider non-planar ϕ^3 Feynman graphs. According to the celebrated Kuratowski's theorem [11], a graph G is non-planar if and only if at least one of the Kuratowski graphs depicted in Fig. 4 is embedded in G. Some examples of non-planar ϕ^3 Feynman graphs having no tadpole part which cannot be renamed as $\phi_1\phi_2\phi_3$ Feynman graphs are shown in Fig. 5. The vacuum-polarization graph Fig. 5 (a) is known as the Petersen graph [12]. The self-energy graph Fig. 5 (b) reduces to Fig. 5 (a) if we consider G^* (see Theorem 3.4). The vertex graph Fig. 5 (c) again reduces to Fig. 5 (a) if we consider G' (see Theorem 3.5).

[3] The key to the proof of Theorem 3.6 is to represent the four colors of a map by four elements of an abelian group $\{(0,0),(1,0),(0,1),(1,1)\text{: mod. } 2\}$. The border line of two countries shall be colored by the group-theoretical sum of their colors. Then we obtain a line-coloring with three colors $(1,0)$, $(0,1)$ and $(1,1)$. [Cut-lines are colored with $(0,0)$ by the above rule, whence they must be excluded.]

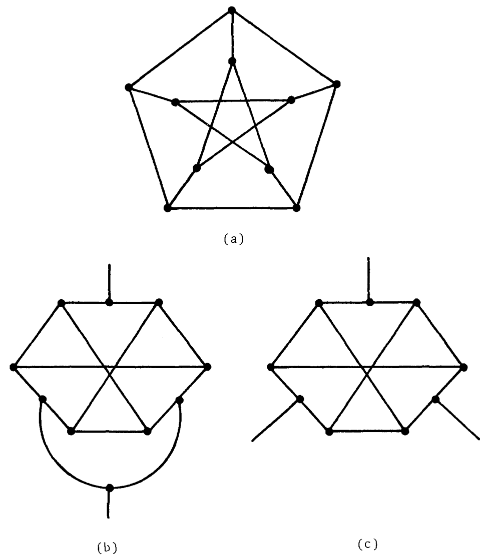

Fig. 5. No-tadpole ϕ^3 Feynman graphs which are not realizable in the $\phi_1\phi_2\phi_3$ theory

4. Line-Coloring Polynomials

The difference between the ϕ^k theory and the $\phi_1\phi_2\ldots\phi_k$ theory is not merely the fact that some Feynman integrals of the former do not appear in the latter, but a more physically significant point is that a ϕ^k Feynman graph generally corresponds to *several* $\phi_1\phi_2\ldots\phi_k$ Feynman graphs. In this section, we investigate this problem by somewhat generalizing it in the following way.

Let $\phi_1, \phi_2, \ldots, \phi_m$ be scalar fields having the same mass as that of ϕ. We compare the theory in which the interaction Lagrangian density is given by

$$\mathscr{L}_I^{(m)} \equiv \lambda \sum_{(j_1,\cdots,j_k)} \phi_{j_1}\phi_{j_2}\cdots\phi_{j_k}, \tag{4.1}$$

where the summation goes over all possible combinations of k distinct integers among $(1, 2, \ldots, m)$, with the ϕ^k theory described by

$$\mathscr{L}_I \equiv (\lambda/k!):\phi^k:. \tag{4.2}$$

Given a ϕ^k Feynman graph G, we denote by $Q_G(m)$ the number of ways of coloring the lines of G, where in counting this number not only external lines but also vertices and internal lines of G are supposed to be *distinguishable.* We call $Q_G(m)$ a *line-coloring polynomial,* since, as we shall see below, it is a polynomial in m. From the definition, we immediately have

$$\begin{aligned} Q_G(m) &> 0 \quad \text{for} \quad m \geqq \chi(G), \\ Q_G(m) &= 0 \quad \text{for} \quad m < \chi(G). \end{aligned} \tag{4.3}$$

Theorem 4.1. *Given a ϕ^k Feynman graph G, let I_G be the Feynman integral corresponding to G. Let w_G and $w_G^{(m)}$ be the weights of I_G in the S-matrix of the ϕ^k theory and in that of the $\mathscr{L}_I^{(m)}$ theory, respectively. Then*

$$w_G^{(m)}/w_G = Q_G(m). \tag{4.4}$$

Proof. In the ϕ^k theory, I_G is characterized by the external lines, which are distinguishable, and the topological structure of G, that is, internal lines and vertices are *indistinguishable* from each other if their topological positions relative to the remainder are the same. When we give every vertex of G a specific name, that is, when the vertices of G are regarded as distinguishable, we denote G by G^v for definiteness. Now, according to the Feynman rules, we know

$$w_G = 1/s_G p_G, \tag{4.5}$$

where $1/s_G$ is the statistical factor and $1/p_G$ is the permutation symmetry factor of G. More precisely, s_G is given by

$$s_G = \prod_{h=2}^{k} (h!)^{N_h}, \tag{4.6}$$

where N_h stands for the number of h parallel lines involved in G, while p_G is the number of the vertex permutations of G^v which leave G^v invariant.

In the $\mathscr{L}_I^{(m)}$ theory, there are several Feynman graphs G_j which correspond to I_G. As before, we denote the G_j having distinguishable vertices by $(G_j)^v$. The statistical factor of G_j is unity because parallel lines are always colored with different colors in the "colored" Feynman graph G_j. Let p_j be the number of the vertex permutations of $(G_j)^v$ which leave $(G_j)^v$ invariant. Then we have

$$w_G^{(m)} = \sum_j 1/p_j, \tag{4.7}$$

where the summation goes over all G_j's which are obtainable by coloring G.

It is convenient to replace the summation over G_j's in (4.7) by that over the colored Feynman graphs, $(G^v)_i$, which are what are obtained by coloring the *vertex-distinguishable* Feynman graph G^v. We define a mapping $f: \{(G^v)_i\} \to \{G_j\}$ by the operation of abolishing the distinguishability of vertices. The mapping f is of course onto but not one-to-one. The inverse image, $f^{-1}(G_j)$, of a particular G_j is generated from its one element $(G^v)_i$ by all possible vertex permutations which leave G^v invariant. Since the vertex permutations which leave $(G_j)^v$ invariant give no change to any element, the number of the elements of $f^{-1}(G_j)$ equals p_G/p_j because of the theorem concerning the order of a factor group. Therefore (4.7) reduces to

$$w_G^{(m)} = C_G^{(m)}/p_G\,, \tag{4.8}$$

where $C_G^{(m)}$ denotes the total number of $(G^v)_i$'s. Since in each $(G^v)_i$ only the internal lines are mutually indistinguishable, we have

$$C_G^{(m)} = Q_G(m)/s_G\,. \tag{4.9}$$

Thus we obtain (4.4) by collecting (4.5), (4.8) and (4.9). q.e.d.

The *coloring polynomial* (or chromatic polynomial) $P_G(m)$, which is well known in graph theory [13], concerns vertex colorings, that is, $P_G(m)$ is the number of ways of coloring the (distinguishable) vertices of a graph G. We can, however, express $Q_G(m)$ in terms of the (vertex-) coloring polynomial; indeed,

$$Q_G(m) = P_{L[G]}(m)\,, \tag{4.10}$$

where $L[G]$ is the *line graph* [14] of a Feynman graph G, which is constructed in the following way: $v(L[G])$ consists of vertices which are in one-to-one correspondence with the internal and external lines of G; if we denote the vertex corresponding to a line l of G by a_l, $L[G]$ has a line connecting a_l and $a_{l'}$ if and only if l and l' are incident with a common vertex in G, where if l and l' are parallel in G, $L[G]$ should have two parallel lines incident with a_l and $a_{l'}$. By definition, we have

$$M(L[G]) = N(G) + n(G), \quad n(L[G]) = 0\,, \tag{4.11}$$

and if G is a ϕ^k Feynman graph then

$$N(L[G]) = \tfrac{1}{2}k(k-1)\,M(G)\,. \tag{4.12}$$

Because of (4.10), various properties of $P_H(m)$ [13] are transcribed into those of $Q_G(m)$. In particular, $Q_G(m)$ is a polynomial in m of degree $N \equiv N(G) + n(G)$, and the coefficient of m^N is $+1$.

When we have a line-coloring of G made with exactly s colors, we obtain

$$m(m-1)\dots(m-s+1) \tag{4.13}$$

destinct line-colorings by permuting the available m colors. We call two line-colorings which cannot be obtained by a color permutation *inequivalent*. Let c_s be the number of inequivalent line-colorings of G made with exactly s colors; c_s is of course independent of m. From (4.3) and (4.13), we obtain

$$Q_G(m) = \sum_{s=\chi(G)}^{N} c_s m(m-1) \dots (m-s+1), \tag{4.14}$$

where $c_N = 1$. In (4.14), we have $c_s > 0$, because we have $c_{\chi(G)} > 0$ from (4.3) and if $c_s > 0$ and if $s < \min(N, m)$ then $c_{s+1} > 0$ since an $(s+1)$-coloring is obtained from an s-coloring by changing a color of a line such that there exists another line having the same color as its, into an $(s+1)$-th color.

Example 1. If G is a ϕ^k tree Feynman graph for which $n(G) = (k-2) \cdot M(G) + 2$, we have

$$Q_G(m) = m \prod_{j=0}^{k-2} (m-j-1)^{M(G)}. \tag{4.15}$$

Example 2. If G is a ϕ^k single-loop Feynman graph (i.e. $|G|$ is a circuit), for which $n(G) = (k-2)\, M(G)$, we have

$$Q_G(m) = [(m-1)^{M(G)} + (-1)^{M(G)}(m-1)] \prod_{j=0}^{k-3} (m-j-2)^{M(G)}. \tag{4.16}$$

Proof. It is sufficient to consider the $k=2$ case. Suppose r lines which are in a row. Fix the color of one end line. Let α_r be the number of line-colorings such that two end lines are colored with different colors. Then the number of line-colorings such that the two end lines are colored with the same color is just α_{r-1}. Hence we have a recurrence formula

$$\alpha_r = (m-2)\alpha_{r-1} + (m-1)\alpha_{r-2}. \tag{4.17}$$

Solving the difference equation (4.17) under $\alpha_1 = 0$ and $\alpha_2 = m-1$, we find

$$\alpha_r = [(m-1)^r + (-1)^r (m-1)]/m. \tag{4.18}$$

The number of line-colorings of a single loop (circuit) is equal to $m\alpha_{M(G)}$. q.e.d.

As seen above, $Q_G(m)$ increases exponentially when $n(G)$ increases. This fact physically implies that the multilinear theory predicts that multiple-production cross sections are very much larger than those in the ϕ^k theory.

5. ϕ^3 Theory and the $\phi_1\phi_2^2$ Theory

As a symmetric theory of ϕ_1, ϕ_2 and ϕ_3 which should be compared with the ϕ^3 theory, in addition to the multilinear theory it is also reasonable to consider a theory having an interaction Lagrangian density

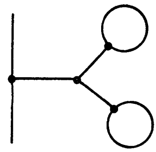

Fig. 6. A ϕ^3 Feynman graph which is not realizable in the $\phi_1\phi_2^2$ theory

proportional to $(\phi_1+\phi_2+\phi_3)^3$. By expanding it, we have three types of terms: ϕ_i^3, $\phi_i\phi_j^2$ $(i\neq j)$ and $\phi_1\phi_2\phi_3$. Hence, in this section, we investigate the relationship between the ϕ^3 theory $(\mathscr{L}_I=(\lambda/6)\phi^3)$ and the $\phi_1\phi_2^2$ theory $(\mathscr{L}_I'=(\lambda/2)\phi_1\phi_2^2)$. Throughout this section, for generality we do not necessarily take a normal product in both $\mathscr{L}_I$ and $\mathscr{L}_I'$, that is, we admit the presence of loop lines in Feynman graphs. The problem which we discuss below is to find what kind of ϕ^3 Feynman graphs can be renamed as $\phi_1\phi_2^2$ Feynman graphs. Of course, any $\phi_1\phi_2\phi_3$ Feynman graph is realizable as a $\phi_1\phi_2^2$ Feynman graph by identifying ϕ_2 and ϕ_3, but the converse is not true.

Theorem 5.1. *If a ϕ^3 Feynman graph G can be renamed as a $\phi_1\phi_2^2$ Feynman graph, then the "tail" line of each tadpole part must correspond to ϕ_1.*

Proof. If the tail line is of ϕ_2, we meet a contradiction by tracing adjacent ϕ_2 lines in the tadpole part successively. q.e.d.

From Theorem 5.1, we see that if a ϕ^3 Feynman graph G contains a vertex with which two or three tadpole-part tails are incident, then G cannot be renamed as a $\phi_1\phi_2^2$ Feynman graph (see Fig. 6).

Next, we quote Petersen's theorem. As discussed in Section 3, it is easy to modify it so as to permit the presence of parallel lines and loop lines.

Theorem 5.2. [15]. *Let G be a degree-3 homogeneous graph having no cut-lines. Then there exist a degree-1 homogeneous graph G_1 and a degree-2 homogeneous graph G_2 such that*

$$v(G_1)=v(G_2)=v(G),$$
$$|G_1|\cup|G_2|=|G| \tag{5.1}$$

and $|G_1|$ and $|G_2|$ are disjoint.

We translate this theorem into that for Feynman graphs.

Theorem 5.3. *In order for a ϕ^3 Feynman graph G to be realizable in the $\phi_1\phi_2^2$ theory, it is necessary and sufficient that G has no vertex with which two or three tadpole-part tails are incident.*

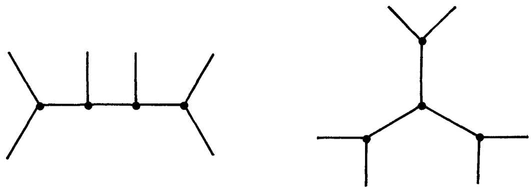

Fig. 7. Some tree Feynman graphs of $n(G)=6$ having different values of Q'_G

Proof. As remarked above, the necessity follows from Theorem 5.1. In the following, we prove the sufficiency. By the same reasoning as in the proof of Theorem 3.7, we see from Theorem 5.2 that if G has no tadpole part it is realizable in the $\phi_1\phi_2^2$ theory. Hence we consider the case in which G has tadpole parts. Owing to Theorem 5.1, we can discuss the lines in each tadpole part independently of the remainder. Let T be a tadpole part which contains no proper sub-tadpole part. Let l_1 be the tail of T and l_2 and l'_2 be the two lines adjacent to l_1 *inside* T (if l_2 is a loop line, l'_2 is not distinct from l_2). Since from Theorem 5.1 l_1 is a ϕ_1 line, l_2 and l'_2 must be ϕ_2 lines. Therefore T is realizable in the $\phi_1\phi_2^2$ theory if and only if $(T-l_1)/l_2$ is so. Since $(T-l_1)/l_2$ is, however, a ϕ^3 Feynman graph having no tadpole part, it is realizable in the $\phi_1\phi_2^2$ theory. Thus our problem concerning G reduces to that concerning G/I, where $I=|T|\cup\{l\}$ and l is one of the two lines adjacent to l_1 *outside* T. Hence mathematical induction establishes the theorem. q.e.d.

Theorem 5.3 is a complete solution to the problem posed at the beginning of this section. Our next problem is of course to find the ratio Q'_G of the weight of a Feynman integral I_G in the $\phi_1\phi_2^2$ theory to that in the ϕ^3 theory. Unfortunately, we have no general results on this problem; hence we here present some examples.

Example 1. If G is a ϕ^3 multiperipheral graph, we have[4]

$$Q'_G=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n(G)+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n(G)+1}\right]. \tag{5.2}$$

Tree Feynman graphs having the same number of external lines do not necessarily have the same value of Q'_G. For example, $Q'_G=13$ for Fig. 7 (a) as seen from (5.2), but $Q'_G=12$ for Fig. 7 (b).

Example 2. If G is a ϕ^3 single-loop graph,

$$Q'_G=\left(\frac{1+\sqrt{5}}{2}\right)^{n(G)}+\left(\frac{1-\sqrt{5}}{2}\right)^{n(G)}. \tag{5.3}$$

[4] (5.2) is Binet's formula for the Fibonacci sequence.

The method of the proofs of (5.2) and (5.3) is analogous to that of Example 2 in Section 4. We again see that Q'_G increases exponentially as $n(G)$ increases.

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13. Ref. [5], pp. 145—148.
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