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Analyticity Properties of the Ising Model in the Antiferromagnetic Phase

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Abstract. The partition function of the Ising antiferromagnet is proved to have no zeroes in an annulus around the origin in the complex z-plane. The intersection of this annulus with the positive real axis belongs to the antiferromagnetic region. The free energy and the correlation functions are analytic in the annulus.

Introduction

In the present article we study the cubic Ising model with repulsive nearest neighbour interaction in the antiferromagnetic region. For simplicity we take the two dimensional case, but the results remain true in higher dimensions.

All points x of the two dimensional lattice $\mathbb{Z}^2$ can have spin $\sigma_x = \pm 1$. To a finite volume Λ, a configuration $\boldsymbol{\sigma} = \{\sigma_x, x \in \Lambda\}$ in Λ and a boundary condition $\boldsymbol{\tau} = \{\tau_x, x \notin \Lambda\}$ we assign the energy

$$H_\Lambda(\boldsymbol{\sigma}|\boldsymbol{\tau}) = J \sum_{\substack{\langle x,y\rangle \\ x,y\in\Lambda}} \sigma_x\sigma_y - h \sum_{x\in\Lambda} \sigma_x + J \sum_{\substack{\langle x,y\rangle \\ x\in\Lambda, y\notin\Lambda}} \sigma_x\tau_y . \quad (1)$$

In this expression, $\langle x, y\rangle$ denotes the summation over pairs of nearest neighbours; repulsive interaction means that $J > 0$.

For low temperatures and small absolute values of the magnetic field h, the system is known to have at least two equilibrium states [3]. They can be obtained as limits of finite volume Gibbs distributions with the boundary conditions

$$\boldsymbol{\tau} = \pm \boldsymbol{\varepsilon} , \quad (2)$$

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respectively, where ε is given by

$$\begin{aligned} \varepsilon_x &= 1, \text{ if } x \text{ is even} \\ &-1, \text{ if } x \text{ is odd}. \end{aligned} \tag{3}$$

It seems of interest now to study the distribution of the zeroes of the partition function in the complex $z \equiv \exp(2\beta h)-$ plane, and the analyticity properties of the free energy $f(z)$. This is emphasized by the following. Let us add to the energy (1) a contribution due to a staggered field k, that is an extra term

$$-k \sum_{x \in \Lambda} \varepsilon_x \sigma_x .$$

Consider the free energy $f(z, w)$ as a function of the variables z and $w \equiv \exp(2\beta k)$. Take first $z=1$ ($h=0$). Then the model is equivalent to the attractive Ising model with homogeneous field k, so $f(1, w)$ has a singularity for $w=1$, and the zeroes of the partition function come arbitrary close to $w=1$ for large volumes. The present problem is, what happens for fixed $w=1$ in a neighbourhood of $z=1$.

From a lemma by Lee and Yang ([9], Section 5.1.1) it follows, that $f(z, w)$ is analytic in both variables in the regions

$$|w| > 1, |w|^{-1} < |z| < |w| ;$$
$$|w| < 1, |w| < |z| < |w|^{-1} .$$

The annulus of analyticity in z, found thus, shrinks to zero as w tends to 1. We shall show, that for $w=1$ and low temperatures there still is an annulus of the type

$$a^{-1} < |z| < a$$

where the partition function has no zeroes and the free energy is analytic.

Moreover, one knows from general low activity considerations ([9], Section 4.2.7), that in a region

$$|z| < b^{-1}, |z| > b ,$$

the partition function has no zeroes. These facts point to the conjecture [8], that below the critical temperature the zeroes lie on two curves around the origin, the intersections with the positive real axis giving the two critical fields. With an extra attractive next nearest neighbour potential this already appears for a volume as small as 4×6 [8]. Without these extra interactions, this volume is supposed to be too small to give an indication for the large volume behaviour [7, 8]. It seems very unlikely now, that the zeroes cluster on a whole interval of the positive real axis.

In §§ 1, 2 we describe contours and introduce our tool, the Minlos-Sinaĭ equations [1, 2] for outer contours. They were meant for the case of attractive interactions; for our purpose, a slight modification in the spirit of Dobrušin [3] has been made. In § 3 we study the zeroes of the partition function and the analyticity of the free energy, in § 4 the analyticity of the correlation functions.

In § 5 the hard core lattice gas is treated. The hamiltonian is formally the lattice gas analog of Eq.(1) (put $\sigma_x = 2n_x - 1$, $\mu = 2h + 8J$) with $J = +\infty$, μ finite. This means, that nearest neighbours cannot both be occupied. Then for large absolute values of $z \equiv \exp(\beta\mu)$ (i.e. in the region of two equilibrium states [3]) the partition function has no zeroes and the mentioned analyticity properties hold.

§ 1. Contours

All through this article, the volume Λ will consist of a finite number of simply connected pieces. Through §§ 1–3, the boundary condition $\boldsymbol{\tau} = +\boldsymbol{\varepsilon}$ is taken.

Given a configuration $\boldsymbol{\sigma}$ in Λ, draw a unit segment between two equal spins. One obtains a set of segments with the property that in each point of the dual lattice an even number of segments meet. Conversely to any such set of segments corresponds a unique configuration.

Given $\boldsymbol{\sigma}$, the *distance d* of a segment s to the boundary of Λ is the minimal number of segments crossed if s is connected to the border of Λ by a broken line through lattice points. A *contour* γ_i is now a maximal connected set of segments with fixed distance. Thus a set of segments is divided into a configuration of contours $\{\gamma_i\} = \gamma$, each contour γ_i having a certain distance $d(\gamma_i|\gamma)$ to the border of Λ. A contour of distance 0 is called *outer*. The name contour is justified by the fact, that it divides Λ into an inside and an outside part. Two contours at the same distance lie outside each other and do not touch. A contour with distance d is embraced by a contour of distance d–1 and may touch it in a finite number of points (see Fig. 1). There is a one to one correspondance between the configurations of contours γ with the mentioned properties and the configuration of spins $\boldsymbol{\sigma}$.

In terms of contours, the energy, Eq. (1) is:

$$U_\Lambda(\gamma) = \sum_i \left[2J|\gamma_i| - h(-1)^{d(\gamma_i|\gamma)}\eta(\gamma_i)\right] + H_\Lambda(\varepsilon, \varepsilon)\,, \tag{4}$$

where $|\gamma_i|$ is the length (number or segments) of γ_i and

$$\eta(\gamma_i) = 2 \quad [(\textit{no. of odd sites in } \gamma_i)\text{–}(\textit{no. of even sites in } \gamma_i)]\,. \tag{5}$$

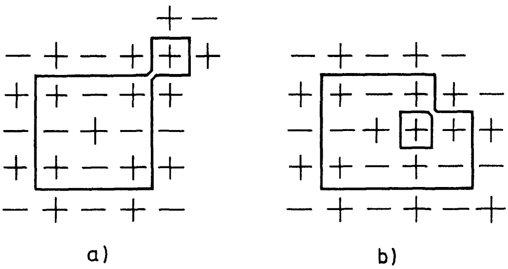

Fig. 1. (a) a contour; (b) a contour embracing another one

Formula (4) is most easily derived by building up γ contour by contour, from the empty configuration of contours $\emptyset$ (the ground state), starting with the outer contours and proceeding inward. Each step gives then the energy contribution as in Eq. (4).

If t_1 is any translation over one step, then

$$\eta(t_1\gamma_i) = -\eta(\gamma_i). \tag{6}$$

Since the inner of $t_1\gamma_i$ only differs from the inner of γ_i by two surface layers, this gives

$$|\eta(\gamma_i)| \leqq \tfrac{1}{2}|\gamma_i|. \tag{7}$$

We can then easily check that $\emptyset$ is the ground state for any $|h| \leqq 4J$ [Eqs. (4), (7)].

Given an allowed set of outer contours $\{\gamma_1, \ldots, \gamma_n\}$ we define the event:

$$\mathscr{B}_\Lambda(\gamma_1, \ldots, \gamma_n) = \{\gamma \quad \text{in} \quad \Lambda : \gamma_1, \ldots, \gamma_n \quad \text{are outer contours of } \gamma\} \tag{8}$$

with probability

$$\varrho_\Lambda(\gamma_1, \ldots, \gamma_n) = Q_\Lambda^{-1} \sum_{\gamma \in \mathscr{B}_\Lambda(\gamma_1, \ldots, \gamma_n)} e^{-\beta U_\Lambda(\gamma)} \tag{9}$$

where Q_Λ is the partition function

$$Q_\Lambda = \sum_\gamma e^{-\beta U_\Lambda(\gamma)}. \tag{10}$$

The quantities $\varrho_\Lambda(\gamma_1, \ldots, \gamma_n)$ are the correlation functions for outer contours.

Let γ have γ_1 as an outer contour. Then we define the configuration $T_{\gamma_1}(\gamma)$ by removing γ_1 from γ and shifting the contours inside γ_1 one step to the right. The Dobrušin-transformation T_{γ_1} [3] is one-to-one and from Eqs. (4) and (6) we have

$$U_\Lambda(\gamma) - U_\Lambda(T_{\gamma_1}(\gamma)) = 2J|\gamma_1| - h\eta(\gamma_1) \tag{11}$$

evidently independent of γ (as long as it has γ_1 as an outer contour).

Therefore, if we define the event

$$\mathscr{B}_{\Lambda,\gamma_1}(\gamma_2, \ldots, \gamma_n) = T_{\gamma_1}\mathscr{B}_\Lambda(\gamma_1, \ldots, \gamma_n) \tag{12}$$

we have

$$\varrho_\Lambda(\gamma_1, \ldots, \gamma_n) = v(\gamma_1) Pr[\mathscr{B}_{\Lambda,\gamma_1}(\gamma_2, \ldots, \gamma_n)], \tag{13}$$

where

$$v(\gamma_1) = e^{-2\beta J|\gamma_1| + \beta h\eta(\gamma_1)}. \tag{14}$$

The event $\mathscr{B}_{\Lambda,\gamma_1}$ will be the usual starting point for the derivation of correlation equations.

The use of correlation functions for outer contours instead of correlation functions for arbitrary contours is explained by the following. If γ_1 is an arbitrary contour, the second term on the r.h.s of Eq. (11) may have either sign, dependent on γ. So Eq. (14) is no longer valid and even an estimation for $|v(\gamma_1)|$ cannot be found for non real values of h.

It is also clear now why Λ must consist of simply connected pieces: if γ_1 enbraces a hole in Λ, it may not be possible to define T_{γ_1}.

§ 2. Correlation Equations for Outer Contours

In connection with a contour γ, the following sets of lattice points appear to be useful.

$\Theta(\gamma)$ = (points inside γ)–(points inside γ, adjacent[1] to γ)
$\Theta_1(\gamma)$ = (points inside γ)
$\Theta_2(\gamma)$ = (points inside γ) + (points outside γ, adjacent[1] to γ).

Let further $G(\gamma)$ be the set of contours δ embracing a point in $\Theta_2(\gamma) - \Theta(t\gamma)$, that is

$$\Theta_1(\delta) \cap [\Theta_2(\gamma) - \Theta(t\gamma)] \neq \emptyset \tag{15}$$

(the contour $t\gamma$ is obtained by shifting γ one place to the right).

After the Dobrušin transformation T_{γ_1} all the spins in $\Theta_2(\gamma_1) - \Theta(t\gamma_1)$ are fixed (see Fig. 2). That means, that a set of outer contours $(\gamma_2, \ldots, \gamma_n, \delta_1, \ldots, \delta_k)$ belongs to a configuration $\gamma \in \mathscr{B}_{\Lambda,\gamma_1}(\gamma_2, \ldots, \gamma_n)$ if $\delta_i \notin G(\gamma_1)$,

[1] A point x inside γ is called adjacent to γ, if the unit square around x has a segment in common with γ. A point x outside γ is called adjacent to γ, if the unit square around x has a point in common with γ. Cf. the definition of contour.

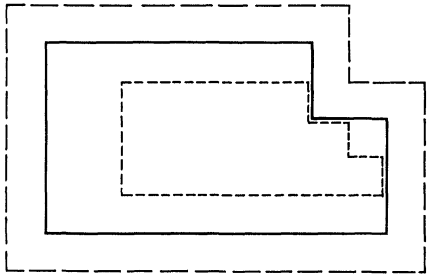

Fig. 2. —— Is a contour γ. The region in —— is $\Theta_1(\gamma)$, the region in — — is $\Theta_2(\gamma)$, the region in ... is $\Theta(t\gamma)$

$1 \leqq i \leqq k$. So the event $\mathscr{B}_{\Lambda,\gamma_1}(\gamma_2, \ldots, \gamma_n)$ admits the representation

$$\mathscr{B}_{\Lambda,\gamma_1}(\gamma_2, \ldots, \gamma_n) = \mathscr{B}_{\Lambda}(\gamma_2, \ldots, \gamma_n) - \bigcup_{\delta \in G(\gamma_1)} \mathscr{B}_{\Lambda}(\gamma_2, \ldots, \gamma_n, \delta). \tag{16}$$

Combining this with Eq. (13), we arrive at the equations

$$\varrho_{\Lambda}(\gamma_1, \ldots, \gamma_n) = v(\gamma_1) \sum_{k=0}^{\infty} (-1)^k \sum_{\substack{\{\delta_1, \ldots, \delta_k\} \\ \delta_i \in G(\gamma_1)}} \varrho_{\Lambda}(\gamma_2, \ldots, \gamma_n, \delta_1, \ldots, \delta_k). \tag{17}$$

It should be understood that $\varrho_{\Lambda}(\emptyset) = 1$.

The probabilistic derivation given here is valid as long as h is real. Then the quantities

$$\tilde{\varrho}_{\Lambda}(\gamma_1, \ldots, \gamma_n) = \sum_{\gamma \in \mathscr{B}_{\Lambda}(\gamma_1, \ldots, \gamma_n)} e^{-\beta U_{\Lambda}(\gamma)} \tag{18}$$

satisfy also Eq. (17), but with $\tilde{\varrho}_{\Lambda}(\emptyset) = Q_{\Lambda}$.

Note however that the quantities (18) with arbitrary complex h still satisfy Eq. (17), by analytic continuation.

At this point, we can look at the difference between the present equations and two resembling sets. In order to obtain $\mathscr{B}_{\Lambda,\gamma_1}$ from $\mathscr{B}_{\Lambda}$, Minlos and Sinaĭ [1, 2] only remove γ_1. Di Liberto's [4] $\mathscr{B}_{\Lambda,\gamma_1}(\gamma_2, \ldots, \gamma_n)$ consists of the configurations, which have $\gamma_2, \ldots, \gamma_n$ as outer contours and a layer of fixed spins immediately outside γ_1, not embraced by any other contour.

In both cases $v(\gamma_1)$, Eq. (14) becomes a more complicated quotient of partition functions, which seems difficult to estimate for non-real values of h.

§ 3. Zeroes of the Partition Function

In this section we shall determine a region in the complex $z \equiv \exp(2\beta h)$–plane, where the partition function $Q_{\Lambda}(\beta, z)$ has no zeroes.

Consider the space $\mathfrak{B}$ of infinite sequences of functions $\psi = \{\psi(\gamma_1), \psi(\gamma_1, \gamma_2), \ldots, \psi(\gamma_1, \ldots, \gamma_k), \ldots\}$, where the function $\psi(\gamma_1, \ldots, \gamma_k)$

is defined on the set of ordered sets of k outer contours on $\mathbb{Z}^2$. We give $\mathfrak{B}$ the structure of a Banach space by introducing the norm

$$\|\psi\| = \sup_{n \geqq 1} \sup_{\{\gamma_1, \dots, \gamma_n\}} |\psi(\gamma_1, \dots, \gamma_n)| \prod_{1 \leqq i \leqq n} (du)^{-|\gamma_i|}, \tag{19}$$

where $d > 1$ is a number to be determined later and

$$u = \exp(-2\beta J + \tfrac{1}{2}\beta|\mathrm{Re}h|). \tag{20}$$

Evidently ϱ_Λ and $\tilde{\varrho}_\Lambda$, Eqs. (9, 18), belong to $\mathfrak{B}$, because Λ is finite.

Definite on $\mathfrak{B}$ the linear operator K by

$$(K\psi)(\gamma_1, \dots, \gamma_n) = v(\gamma_1) \sum_{k \geqq 0} (-1)^k \sum_{\substack{\{\delta_1, \dots, \delta_k\} \\ \delta_i \in G(\gamma_1)}} \psi(\gamma_2, \dots, \gamma_n, \delta_1, \dots, \delta_k) \tag{21}$$

[for $n = 1$ we set $\psi(\emptyset) = 0$].
Moreover, let $\boldsymbol{\alpha}$ be the vector

$$\boldsymbol{\alpha} = \{v(\gamma_1), 0, 0, \dots\} \tag{22}$$

with norm d^{-4} [Eqs. (14), (19), (20)], and χ_Λ the operator with norm 1 defined by

$$\begin{gathered}(\chi_\Lambda\psi)(\gamma_1, \dots, \gamma_n) = \psi(\gamma_1, \dots, \gamma_n) \prod_{1 \leqq i \leqq n} \chi_\Lambda(\gamma_i), \\ \chi_\Lambda(\gamma) = 1 \quad \text{if} \quad \gamma \quad \text{in} \quad \Lambda, \\ \chi_\Lambda(\gamma) = 0 \quad \text{otherwise}. \end{gathered} \tag{23}$$

Then the correlation equations read

$$\varrho_\Lambda = \chi_\Lambda \boldsymbol{\alpha} + \chi_\Lambda K \varrho_\Lambda, \tag{24}$$

$$\tilde{\varrho}_\Lambda = Q_\Lambda \chi_\Lambda \boldsymbol{\alpha} + \chi_\Lambda K \tilde{\varrho}_\Lambda. \tag{25}$$

We now estimate the norm of the operator K. Let $\|\psi\| \leqq 1$. Then

$$\begin{aligned} &|(K\psi)(\gamma_1, \dots, \gamma_n)| \prod_{1 \leqq i \leqq n} (du)^{-|\gamma_i|} \\ &\leqq d^{-|\gamma_1|} \sum_{k \geqq 0} \sum_{\substack{\{\delta_1, \dots, \delta_k\} \\ \delta_i \in G(\gamma_1)}} \prod_{1 \leqq i \leqq k} (du)^{|\delta_i|} \\ &\leqq d^{-|\gamma_1|} \sum_{k \geqq 0} \left\{ \binom{2|\gamma_1| + 1}{k} \sum_{m\,\mathrm{even} \geqq 4} A_m (du)^m \right\}^k \\ &= d^{-|\gamma_1|} \left\{ 1 + \sum_{m\,\mathrm{even} \geqq 4} A_m (du)^m \right\}^{2|\gamma_1| + 1} \end{aligned} \tag{26}$$

Here, A_m is the number of contours of length m, embracing a given lattice point. Then Eq. (26) is derived by noting, that each δ_i must embrace a point in $\Theta_2(\gamma_1) - \Theta(t\gamma_1)$, and this set consists of at most $2|\gamma_1| + 1$ points.

We use the bound

$$A_m \leqq \tfrac{1}{4} m 3^{m-4}. \tag{27}$$

It is clear then, that for any $d > 1$ there is a number $u_0(d)$, such that $\|K\| < 1$ for $u < u_0(d)$. With Eq. (20) this condition becomes, with $u_0 = \exp(-2\beta_0 J)$,

$$|\mathrm{Re}\, h| < 4J(1 - \beta_0/\beta)$$

or

$$e^{-8(\beta-\beta_0)J} < |z| < e^{8(\beta-\beta_0)J}. \tag{28}$$

By choosing the best value for d, β_0 can be made as small as possible; we find

$$d = 1.1\,; \quad e^{2\beta_0 J} = 4.0\,.$$

Under the condition (28), Eqs. (24) and (25) have a unique solution in $\mathfrak{B}$, satisfying

$$|\varrho_\Lambda(\gamma_1, \dots, \gamma_n)| \leqq C \prod_{1 \leqq i \leqq n} (du)^{|\gamma_i|}, \tag{29}$$

$$|\tilde{\varrho}_\Lambda(\gamma_1, \dots, \gamma_n)| \leqq C |Q_\Lambda| \prod_{1 \leqq i \leqq n} (du)^{|\gamma_i|}, \tag{30}$$

where

$$C = d^{-4}(1 - \|K\|)^{-1}.$$

Because $\tilde{\varrho}_\Lambda$, as defined in Eq. (18), belongs to $\mathfrak{B}$, it coincides with the unique solution of Eq. (25). This implies, that $Q_\Lambda \neq 0$. Indeed, from $Q_\Lambda = 0$ it would follow, by Eq. (30), that $\tilde{\varrho}_\Lambda = 0$. However, $\tilde{\varrho}_\Lambda(\gamma_1, \dots, \gamma_n)$ is certainly unequal to zero if $\mathscr{B}_\Lambda(\gamma_1, \dots, \gamma_n)$ consists of only one configuration; take, e.g., unit squares around all lattice points with both coordinates even.

The unique solution of Eq. (24) coincides now with ϱ_Λ as defined in Eq. (9).

The reasoning of §§ 1–3 can be repeated word for word with the other boundary condition $\tau = -\varepsilon$. The respective boundary conditions $\tau = \pm\varepsilon$ will be indicated by the suffices $\pm$.

We may now summarize our results:

Theorem 1. *Let $\beta > \beta_0$, and let $A(\beta)$ be the annulus in the complex z-plane*

$$e^{-8(\beta-\beta_0)J} < |z| < e^{8(\beta-\beta_0)J}.$$

Then

a) *The partition functions satisfy*

$$Q_\Lambda^\pm(\beta, z) \neq 0$$

for $z \in A(\beta)$.

b) *The free energy*

$$f(\beta, z) = -\beta^{-1} \lim_{\Lambda \to \infty} |\Lambda|^{-1} \log Q_\Lambda^\pm(\beta, z) \tag{31}$$

extends from the interval $0 < z < \infty$ *to a function analytic in* $A(\beta)$.

Proof of b). Generally we have

$$Q_\Lambda^+(z) = Q_\Lambda^-(z^{-1}).$$

If Λ is a rectangle with an even side, moreover

$$Q_\Lambda^+(z) = Q_\Lambda^-(z)$$

by reflection symmetry. Then

$$Q_\Lambda^\pm(z) = Q_\Lambda^\pm(z^{-1}) \tag{32}$$

and the region

$$|z| \leqq e^{-8(\beta-\beta_0)J}$$

contains half of the zeroes, so that log $Q_\Lambda^\pm(z)$ is analytic in $A(\beta)$. The convergence of Eq. (31) for real z and the stability of the potential give then, by Vitali's theorem, the analyticity of $f(\beta, z)$ in $A(\beta)$ (cf. [9], p 111).

Let us conclude this section with a remark on inequality (29). If Λ consists of a number of simply connected sets, the correlation functions ϱ factorize into parts belonging to these different sets. With the help of this fact, one immediately sees that C in Eq. (29) can be replaced by 1:

$$|\varrho_\Lambda(\gamma_1, \ldots, \gamma_n)| \leqq \prod_{1 \leqq i \leqq n} (du)^{|\gamma_i|}. \tag{33}$$

§ 4. Analyticity of the Correlation Functions

In this section we investigate the analyticity properties of the usual spin correlation functions in the thermodynamic limit. For this purpose, it appears useful to introduce the following correlation functions.

$$\begin{aligned} \tau_\Lambda^\pm(X) &= Pr_\Lambda^\pm\{\sigma_x = \mp \varepsilon_x, x \in X\}, \\ \mu_\Lambda^\pm(X) &= Pr_\Lambda^\pm\{\sigma_x = \pm \varepsilon_x, x \in X\}, \end{aligned} \tag{34}$$

where the suffix $\pm$ stands for the boundary condition $\tau = \pm \varepsilon$, respectively. Near the ground state, the quantities $\pi_\Lambda^\pm(X)$ are small. By reversing all spins, we obtain the following relations between π and μ.

$$\begin{aligned}\pi_\Lambda^\pm(X) &= \sum_{Y\subset X} (-1)^{|Y|}\mu_\Lambda^\pm(Y),\\ \mu_\Lambda^\pm(X) &= \sum_{Y\subset X} (-1)^{|Y|}\pi_\Lambda^\pm(Y).\end{aligned} \tag{35}$$

From the F.K.G. inequalities [5, 6] it follows, that for $z>0$ when $\Lambda\to\infty$

$$\pi_\Lambda^\pm(X) \nearrow \pi^\pm(X). \tag{36}$$

We shall show, that for fixed β and X the quantities $\pi_\Lambda^\pm(X)$ are uniformly bounded in Λ and in $z\in A(\beta)$. By theorem 1a) the $\pi_\Lambda^\pm(X)$ are analytic in $A(\beta)$; Vitali's theorem implies then that the $\pi^\pm(X)$ are analytic functions of z in $A(\beta)$.

We start by expressing $\pi_\Lambda^\pm(X)$ in terms of the correlation functions for outer contours.

$$\pi_\Lambda^\pm(X) = \sum_{n\geqq 1} \sum_{\substack{\{X_1,\dots,X_n\}\\ \Sigma X_i = X}} \sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\\ X_i\subset\Theta_1(\gamma_i)}} \varrho_\Lambda^\pm(\gamma_1,\dots,\gamma_n) \prod_{1\leqq i\leqq n} \mu^\mp_{\Theta(\gamma_i)}(X_i). \tag{37}$$

The second sum runs over all partitions of the set X in n non-empty, non-intersecting subsets. The formula is derived by remarking, that a point $x\in X$ has a spin not fitting to the boundary condition only if it is embraced by a contour.

Wit Eq. (35), we find

$$\pi_\Lambda^\pm(X) = \alpha_\Lambda^\pm(X) + (-1)^{|X|} \sum_{\gamma:X\subset\Theta_1(\gamma)} \varrho_\Lambda^\pm(\gamma)\pi^\mp_{\Theta(\gamma)}(X), \tag{38}$$

where

$$\begin{aligned}\alpha_\Lambda^\pm(X) = &\sum_{\gamma:X\subset\Theta_1(\gamma)} \varrho_\Lambda^\pm(\gamma) \sum_{Y\subset X, Y\neq X} (-1)^{|Y|}\pi^\mp_{\Theta(\gamma)}(Y)\\ &+\sum_{n\geqq 2} \sum_{\substack{\{X_1,\dots,X_n\}\\ \Sigma X_i = X}} \sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\\ X_i\subset\Theta_1(\gamma_i)}} \varrho_\Lambda^\pm(\gamma_1,\dots,\gamma_n) \prod_{1\leqq i\leqq n} \mu^\mp_{\Theta(\gamma_1)}(X_i).\end{aligned} \tag{39}$$

Note, that $\alpha_\Lambda^\pm(X)$ is expressed in terms of correlation functions of order smaller then $|X|$ and in terms of correlation functions for outer contours. In particular, if X consists of one point x

$$\alpha_\Lambda^\pm(x) = \sum_{\gamma:x\in\Theta_1(\gamma)} \varrho_\Lambda^\pm(\gamma). \tag{40}$$

Using Eq. (33), one finds the useful estimation, independent of $\Lambda, X_1, \dots, X_n$

$$\sum_{\substack{\{\gamma_1,\dots,\gamma_n\}\\ X_i\subset\Theta_1(\gamma_i)}} |\varrho_\Lambda^\pm(\gamma_1,\dots,\gamma_n)| \leqq y^n, \tag{41}$$

where

$$y = \sum_{m \text{ even} \geqq 4} A_m (du)^m \tag{42}$$

is uniformly bounded in $z \in A(\beta)$. This leads to the following lemma.

Lemma. *The quantities* $\alpha_\Lambda^\pm(X)$ *and* $\pi_\Lambda^\pm(X)$ *are uniformly bounded in* Λ *and in* $z \in A(\beta)$ *by*

$$|\alpha_\Lambda^\pm(X)| \leqq \alpha(|X|, y) \tag{43}$$

$$|\pi_\Lambda^\pm(X)| \leqq \alpha(|X|, y) e^y . \tag{44}$$

For any $|X|$, $\alpha(|X|, y)$ *is a double polynomial in* y *and* e^y.

Proof. If Eq. (43) is given, Eq. (44) follows by iterating Eq. (38). Because Λ is finite, the series breaks off after a finite number of terms. The $(k+1)st$ term is estimated by

$$\alpha(|X|, y) y^k / k! .$$

Let now Eqs. (43) and (44) be given for any X, $|X| \leqq n$. Using them, together with Eq. (41), in Eq. (39), we find the bound (43) for $|X| = n+1$.

By Eq. (40), we can take

$$\alpha(1, y) = y .$$

This concludes the proof of the lemma by induction.

It should be noted, that the bounds $\alpha(|X|, y)$, found by the indicated procedure, are increasing very strongly with the number of points $|X|$.

The argument for the following theorem is now complete.

Theorem 2. *Let* ω^+ *and* ω^- *be the equilibrium states which are the limiting states for boundary conditions* $\tau = \varepsilon$ *and* $\tau = -\varepsilon$, *respectively, and let* A *be a local observable. Then the expectation values* $\omega^+(A)$ *and* $\omega^-(A)$ *extend from* $0 < z < \infty$ *to analytic functions in the annulus* $A(\beta)$, *Eq.* (28).

In particular, we have analyticity for the quantities $\omega(A)$, where ω is the translation invariant state

$$\omega = \tfrac{1}{2}(\omega^+ + \omega^-) .$$

In a region of the shape (28), with $z > 0$, ω is proved to be the only invariant equilibrium state [4].

§ 5. The Hard Core Lattice Gas

Another interesting model related to the one described is the hard core lattice gas, in which neighbouring sites cannot be occupied simultaneously.

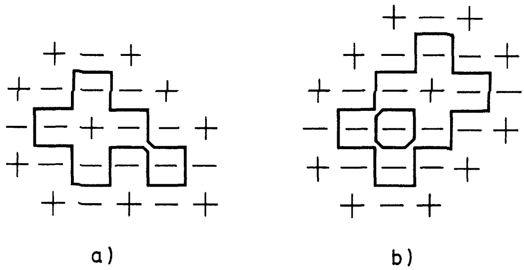

Fig. 3. (a) a contour; (b) a contour embracing another one

Given again the volume Λ and the alternating boundary condition ε, we define segments and contours as in §1. Generally, immediately outside a contour with even distance (in the sense of §1) there must be a layer of spins ε_x. In the hard core case, both spins adjacent to a segment must be -1. This means, that a contour can only occur and have even distance, if all the lattice points immediately outside it are odd. For contours with odd distance the converse statement holds. In particular, all contours must be "stair-shaped": if in a point two segments meet, then these are perpendicular (see Fig. 3).

With these extra restrictions on allowed configurations of contours, we find for the energy

$$U_\Lambda(\gamma) = \frac{1}{4}\mu \sum_i |\gamma_i| + H_\Lambda(\varepsilon|\varepsilon),$$

where μ is the chemical potential.

Using the Dobrušin transformation, we then derive the correlation equations for outer contours, Eq. (17), where it is understood now that

$$\nu(\gamma_1) = \exp(-\tfrac{1}{4}\beta\mu|\gamma_1|) = z^{-|\gamma_1|/4}.$$

Following the lines of §§ 3–4, we arrive at

Theorem 3. *Let* $|z| > z_0$. *Then:*

a) *The grand partition functions satisfy*

$$Z_\Lambda^\pm(z) \neq 0.$$

b) *If* $p(z)$ *denotes the limiting pressure*

$$p(z) = \lim_{\Lambda\to\infty} |\Lambda|^{-1} \log Z_\Lambda^\pm(z),$$

then $p(z) - \frac{1}{2}\log z$ *is analytic in* z.

c) *If* ω^+ *and* ω^- *are the limiting states, then* $\omega^+(A)$ *and* $\omega^-(A)$ *are analytic in* z *for any local observable* A.

d) *Let ϱ be the density, i.e. the one point correlation function in the state $\omega = \frac{1}{2}(\omega^+ + \omega^-)$, or also*

$$\varrho = z\frac{d}{dz}p(z).$$

Then the high density expansion

$$p(\varrho) = -\tfrac{1}{2}\log\left(\tfrac{1}{2} - \varrho\right) + \sum_{n \geqq 0} b_n\left(\tfrac{1}{2} - \varrho\right)^n,$$

has a finite radius of convergence.

Point d) does not need further comment.

Conclusion

We considered properties of analyticity in the variable z. It is evident however, that the free energy and the correlation functions are analytic in both z and β in the region

$$\exp\left[-8J(\mathrm{Re}\beta - \beta_0)\right] < |z| < \exp\left[8J(\mathrm{Re}\beta - \beta_0)\right].$$

Our results in §§ 3–4 are valid below a temperature given by $\exp(2\beta_0 J) = 4.0$. This is not too far from the critical temperature, $\exp(2\beta_c J) = 1 + \sqrt{2}$. The value for β_0 may be slightly improved with a better bound for A_m, Eq. (27). For the hard core case the situation is worse. The critical activity is expected to be about $z_c = 3.8$, our z_0 is of the order of 75.

The addition of a small translation invariant interaction to the energy (1) does not affect the existence of two different equilibrium states, because the energy change by T_{γ_1}, Eq. (11), would remain of the order of $|\gamma_1|$. However, it would no longer be independent of the rest of the configuration γ. This can be repaired by considering contours with distance smaller than the range of the interaction as one large contour. In such way analyticity of the free energy in the extra interaction can be proved, but in a region which shrinks to zero as the range of the interaction grows. So all this is not sufficient for another proof of Di Liberto's result [4], that there is a unique invariant equilibrium state.

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