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Mixing Properties, Differentiability of the Free Energy and the Central Limit Theorem for a Pure Phase in the Ising Model at Low Temperature

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Abstract. For the Ising model with nearest neighbour interaction it is shown that the spin correlations $\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle$ decrease exponentially as $d(A, B) \to \infty$ in a pure phase when the temperature is well below T_c. This is used to prove that the free energy $F(\beta, h)$ is infinitely differentiable in β and has one sided derivatives in h of all orders for $h = 0$. The bounds are also used to prove that the central limit theorem holds for several variables such as e.g. the total energy and the total magnetization of the system, the limit distribution being gaussian with variances determined by the second derivatives of $F(\beta, h)$.

Introduction

We consider the Ising model with nearest neighbour interaction in a finite box Λ on a ν-dimensional square lattice Z^ν. The spin at each point $p \in \Lambda$ takes the values $\sigma_p = \pm 1$, and the energy of a spin configuration is given by

$$-E_\Lambda(\sigma) = \tfrac{1}{2} \sum_{p,q \in \Lambda} J_{p,q} \sigma_p \sigma_q + \sum_{p \in \Lambda} \sigma_p \sum_{q \notin \Lambda} J_{p,q} + H \sum_{p \in \Lambda} \sigma_p \tag{1}$$

where $J_{p,q} = J > 0$ if p and q are neighbours and $J_{p,q} = 0$ otherwise, and H is the external magnetic field. We are only going to consider the situation where Λ is completely surrounded by $+$ spins, which give rise to the boundary term in the energy. The Boltzmannfactor is $e^{-\frac{E_\Lambda(\sigma)}{kT}}$. We put $\frac{2J}{kT} = \beta$ and $\frac{H}{kT} = h$ and denote the spin correlations $\left\langle \prod_{p \in A} \sigma_p \right\rangle$ by $\langle \sigma_A \rangle_{h,\Lambda}$ $A \subseteqq \Lambda$. The free energy (multiplied by $-kT$) is given by

$$F(\beta, h, \Lambda) = |\Lambda|^{-1} \log \sum_\sigma e^{-\frac{E_\Lambda(\sigma)}{kT}}. \tag{2}$$

When Λ increases to all of Z^ν (in the sense of van Hove) the $\langle \sigma_A \rangle_{h,\Lambda}$ decrease to limits $\langle \sigma_A \rangle_h$ which determine the state of an infinite system (i.e. all probabilities of events depending on a finite number of spins), and $F(\beta, h, \Lambda)$ converges to a limit $F(\beta, h)$. For $\beta < \beta_c$, the critical reciprocal temperature, the $\langle \sigma_A \rangle_h$ do not depend on the special boundary condition chosen and there is no phase transition in the system in the sense that $F(\beta, h)$ is differentiable in h for $h = 0$. For $\beta > \beta_c$ and $h = 0$ the limits of the correlation functions depend on the boundary conditions and the $\langle \sigma_A \rangle_0$ describe a pure phase with positive magnetization in the sense that

$$\langle \sigma_p \rangle_0 = m^* > 0 \quad \text{and} \quad \langle \sigma_A \sigma_B \rangle_0 - \langle \sigma_A \rangle_0 \langle \sigma_B \rangle_0 \to 0$$

when $d(A, B) \to \infty$. In this case the right and left derivatives of $F(\beta, h)$ with respect to h are different for $h = 0$ and are equal to $\pm m^*$ respectively. (These questions are discussed in [6].) When $h \neq 0$ the limits of the correlation functions do not depend on the boundary conditions and $F(\beta, h)$ is analytic in β, h. $F(\beta, h)$ is always independent of the boundary conditions chosen, and hence an even function of h.

In Section 1 we study the mixing properties of the state for β well above β_c. We show that

$$|\langle \sigma_A \sigma_B \rangle_{h,\Lambda} - \langle \sigma_A \rangle_{h,\Lambda} \langle \sigma_B \rangle_{h,\Lambda}| \leqq \text{const}\, e^{-(\beta - \beta')d(A,B)}$$

and also that

$$|\langle \sigma_A \rangle_{h,\Lambda} - \langle \sigma_A \rangle_h| \leqq \text{const}\, e^{-(\beta - \beta')d(A, \partial\Lambda)}$$

uniformly in Λ and $h \geqq 0$ when $\beta > \beta'$ for a certain $\beta' > \beta_c$ depending on the dimension.

In Section 2 we use these estimates to conclude that $F(\beta, h)$ is infinitely differentiable in h to the right and left for $h = 0$.

In particular the susceptibility is finite at $h = 0\pm$ and $m^*(\beta)$ is infinitely differentiable for $\beta > \beta'$. These results extend those recently obtained by Lebowitz [7] for $\beta < \beta_c$.

In Section 3 we show that the estimates of Section 1 can be used to prove the central limit theorem for several quantities such as the total magnetization and the total energy when $h \geqq 0$ and $\beta > \beta'$. The proof also works just as well for β small enough when the estimates of Fisher [2] show that the correlations also decay exponentially. To simplify the presentation we carry out the proofs in Section 1 for the two dimensional case, but it is seen that they work just as well in any number of dimensions.

1. Mixing Properties of the State

The proofs in this section will be based on arguments concerning the boundaries separating + and − spins. A spin configuration in Λ can uniquely be represented by drawing the family of contours separating + and − spins as indicated in Fig. 1 and described in detail in [8].

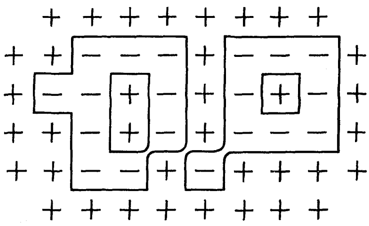

Fig. 1. A spin configuration and its associated contours

With the boundary condition chosen all contours are closed, and we consider them separated into simple closed contours as shown in Fig. 1. The outer contours are those not surrounded by any other ones. The following estimate of the probability that a contour γ is an outer contour of the configuration is basic in the following. It extends the well known bound used in Peierls' proof that a phase transition occurs for $h=0$ to the situation when $h \geqq 0$.

Lemma 1. *For all $h \geqq 0$ we have the bound*

$$P_{h,\Lambda}\,(\gamma \text{ is an outer contour}) \leqq e^{-\beta|\gamma|}. \tag{3}$$

Proof. Call the probability P. It can be written as

$$P = e^{-\beta|\gamma|} \frac{Z_-(\gamma)\, Z_0(\gamma)}{Z(\Lambda)} \tag{4}$$

where $Z_-(\gamma)$ denotes the partition function for all configurations inside γ which have all spins adjacent to γ equal to -1, $Z_0(\gamma)$ the partition function for all configurations outside γ with all spins adjacent to γ equal to $+1$ and without contours surrounding γ, and $Z(\Lambda)$ denotes the partition function for all configurations in Λ. If in $Z(\Lambda)$ we restrict the summation to all configurations having $+$ spins adjacent to γ both along the inside and outside and no contours surrounding γ we get something smaller. Hence $Z(\Lambda) \geqq Z_+(\gamma)\, Z_0(\gamma)$, where $Z_+(\gamma)$ is defined as $Z_-(\gamma)$ but with boundary spins $+1$ instead, and

$$P \leqq e^{-\beta|\gamma|} \frac{Z_-(\gamma)}{Z_+(\gamma)}. \tag{5}$$

When $h=0$ $Z_-(\gamma) = Z_+(\gamma)$, because to each term in one of the sums there is precisely one in the other obtained by reversing all the spins which has the same energy [remember that the energy is equal to const $+\beta$ (the length of the contours)]. To see that

$$\frac{Z_-(\gamma)}{Z_+(\gamma)} \leqq 1 \quad \text{for} \quad h \geqq 0$$

we show that
$$\frac{\partial}{\partial h}\log\frac{Z_-(\gamma)}{Z_+(\gamma)}\leqq 0\,.$$

In fact, this quantity is equal to
$$\sum_{p\,\text{inside}\,\gamma}\langle\sigma_p\rangle_{h,\gamma,-}-\langle\sigma_p\rangle_{h,\gamma,+}\,,$$
the averages being calculated in the two ensembles inside γ. But the F.G.K. inequalities [3] tell us that the $\langle\sigma_p\rangle_{h,\gamma}$ are increasing functions of any external fields acting inside γ. Hence the $-$ averages are $\leqq$ the $+$ averages since they can be obtained by letting an external field equal to $\mp\infty$ respectively act on the spins along γ.

Remark. The proof works just as well if the interactions $J_{p,q}$ and the external field are not uniform as long as they are $\geqq J>0$ and $\geqq 0$ respectively. This will be used in Section 3.

The estimate implies generally speaking that long contours are very unlikely. We know that the $\langle\sigma_A\rangle_{h,\Lambda}$ decrease as Λ increases, i.e. that $\langle\sigma_A\rangle_{h,\Lambda}\geqq\langle\sigma_A\rangle_{h,\Lambda'}$ when $\Lambda\subseteqq\Lambda'$ [8].

We next estimate how much they differ when A is far from the part of $\partial\Lambda$ which is not contained in $\partial\Lambda'$. ($\partial\Lambda$ is the set of points outside Λ which interact with points in Λ.)

Theorem 1. *Let $A\subseteqq\Lambda\subseteqq\Lambda'$ and $\Delta=\partial\Lambda\setminus(\partial\Lambda\cap\partial\Lambda')$, then*
$$0\leqq\langle\sigma_A\rangle_{h,\Lambda}-\langle\sigma_A\rangle_{h,\Lambda'}\leqq|A|\cdot 6(1-3e^{-\beta})^{-2}(3e^{-\beta})^{d(A,\Delta)} \tag{6}$$
when $\beta>\beta'=\log 3$ and $h\geqq 0$. Hence the same bounds are valid for $\langle\sigma_A\rangle_{h,\Lambda}-\langle\sigma_A\rangle_h$.

Proof. For each configuration in the larger box Λ' consider those contours that surround points of Δ. Since long contours are unlikely it is very unlikely that any of them also surrounds points of A if $d(A,\Delta)$ is large. For the same reason it is very unlikely that a contour in Λ' surrounds Λ. This is proved in the following lemma which we prove after Theorem 1:

Lemma 2. *For each configuration in Λ' let $\gamma_1,\ldots,\gamma_n$ be those outer contours that surround points of Δ (if there are any), and let E be the event: "one of $\gamma_1,\ldots,\gamma_n$ surrounds a point of A or some outer contour in Λ' surrounds Λ". Then*
$$P_{h,\Lambda'}(E)\leqq|A|\cdot 3(1-3e^{-\beta})^{-2}(3e^{-\beta})^{d(A,\Delta)} \tag{7}$$
when $\beta>\beta'$ and $h\geqq 0$.

Consider now $\langle\sigma_A\rangle_{h,\Lambda'}$ and split it according if E occurs or not:
$$\langle\sigma_A\rangle_{h,\Lambda'}=P_{h,\Lambda'}(E)\langle\sigma_A\rangle_{h,\Lambda',E}+(1-P_{h,\Lambda'}(E))\langle\sigma_A\rangle_{h,\Lambda',\bar{E}}\,. \tag{8}$$

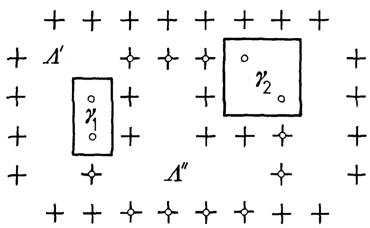

Fig. 2. Boundary condition for fixed $\gamma_1, \cdots, \gamma_n$. The points of $\partial \Lambda$ are marked by circles

For each family $\gamma_1, \dots, \gamma_n$ that can occur as a family defined above of a configuration in $\bar{E}$ consider the conditional average of σ_A given that the family was precisely $\gamma_1, \dots, \gamma_n$. That conditional average is however the average in the box $\Lambda'' \subseteqq \Lambda$ of points in Λ outside $\gamma_1, \dots, \gamma_n$ with the boundary condition + all along the outside of $\gamma_1, \dots, \gamma_n$ and on $\partial \Lambda$ as indicated in Fig. 2 because there is only nearest neighbour interaction. By the monotonicity of $\langle \sigma_A \rangle_{\Lambda''}$ that average is $\geqq \langle \sigma_A \rangle_{h,\Lambda}$ and summing over all possible $\gamma_1, \dots, \gamma_n$ we can conclude that $\langle \sigma_A \rangle_{h,\Lambda',\bar{E}} \geqq \langle \sigma_A \rangle_{h,\Lambda}$ also. From (8) we then see that

$$\langle \sigma_A \rangle_{h,\Lambda'} \geqq \langle \sigma_A \rangle_{h,\Lambda} - 2P_{h,\Lambda'}(E),$$

because $\sigma_A \geqq -1$ always and the theorem follows using the estimate of $P_{h,\Lambda'}(E)$ in Lemma 2.

Proof of Lemma 1. Consider a point $a \in A$ at distance d from Δ and let L be an infinite halfline on the lattice starting from a. Consider any γ surrounding a. It must separate two adjacent points on L; let $q = q(\gamma)$ be the last point on L not separated from a by γ. We have $d(a,q) + d(q,\Delta) \geqq d$, and because the segment (a,q) is surrounded by γ $|\gamma| \geqq 2d(a,q)$. Also $|\gamma| \geqq 2d(q,\Delta)$ if γ surrounds a point of Δ (see Appendix A). Hence we know that if $l = d(a,q)$ then $|\gamma| \geqq \max(2(d-l), 2l)$, so we get the estimate $P_{h,\Lambda'}$ (a surrounded by an outer contour surrounding a point of Δ)

$$\begin{aligned} &\leqq \sum_{q \in L} \sum_{\substack{q(\gamma)=q \\ \gamma \text{ surr. } a \text{ and} \\ \text{points of } \Delta}} e^{-\beta|\gamma|} \\ &\leqq \sum_{l \leqq \frac{d}{2}} \sum_{n \geqq 2(d-l)} (3e^{-\beta})^n + \sum_{l > \frac{d}{2}} \sum_{n \geqq 2l} (3e^{-\beta})^n \leqq 2(1-3e^{-\beta})^{-2} (3e^{-\beta})^d \end{aligned} \tag{9}$$

if $3e^{-\beta} < 1$ using the estimate 3^n for the number of contours of length n starting at a given segment. The probability that A is surrounded by some outer contour is estimated quite analogously by

$$\sum_{l \geqq d} \sum_{n \geqq 2l} (3e^{-\beta})^n \leqq (1-3e^{-\beta})^{-2} (3e^{-\beta})^d$$

since Λ can only be surrounded by a contour if $\Delta = \partial \Lambda$, and in this case $l \geqq d$. The lemma is hence proved by adding the above estimates for $a \in A$.

Theorem 1 can immediately be used to estimate the correlations between σ_A and σ_B when A and B are far apart:

Theorem 2.

$$0 \leqq \langle \sigma_A \sigma_B \rangle_{h,\Lambda} - \langle \sigma_A \rangle_{h,\Lambda} \langle \sigma_B \rangle_{h,\Lambda} \leqq 6(|A| + |B|)(1 - 3e^{-\beta})^{-2} (3e^{-\beta})^{\frac{d(A,B)}{2} - 1} \tag{10}$$

when $\beta > \beta'$ and $h \geqq 0$.

Proof. Let Λ' be the set of points in Λ whose distance from A is at most $\frac{d(A,B)}{2}$, $\Delta = \partial \Lambda' \setminus (\partial \Lambda' \cap \partial \Lambda)$ and $\Lambda'' = \Lambda \setminus (\Lambda' \cup \Delta)$. Then $d(A, \Delta) = \frac{d(A,B)}{2} + 1$ and $d(B, \Delta) \geqq \frac{d(A,B)}{2} - 1$.

If we constrain the spins on Δ to be all $+1$ Λ is split into the two independent boxes Λ' and Λ'' with $+$ boundary conditions. In this process $\langle \sigma_A \sigma_B \rangle_{h,\Lambda}$ can not decrease by G.K.S. inequalities [4, 5], and it is changed into $\langle \sigma_A \rangle_{h,\Lambda'} \langle \sigma_B \rangle_{h,\Lambda''}$.

These averages can be estimated by Theorem 1 and we get:

$$\begin{aligned} 0 &\leqq \langle \sigma_A \sigma_B \rangle_{h,\Lambda} - \langle \sigma_A \rangle_{h,\Lambda} \langle \sigma_B \rangle_{h,\Lambda} \leqq \langle \sigma_A \rangle_{h,\Lambda'} \langle \sigma_B \rangle_{h,\Lambda''} - \langle \sigma_A \rangle_{h,\Lambda} \langle \sigma_B \rangle_{h,\Lambda} \\ &= (\langle \sigma_A \rangle_{h,\Lambda'} - \langle \sigma_A \rangle_{h,\Lambda}) \langle \sigma_B \rangle_{h,\Lambda''} + \langle \sigma_A \rangle_{h,\Lambda} (\langle \sigma_B \rangle_{h,\Lambda''} - \langle \sigma_B \rangle_{h,\Lambda}) \\ &\leqq 6(|A| + |B|)(1 - 3e^{-\beta})^{-2} (3e^{-\beta})^{\frac{d(A,B)}{2} - 1}. \end{aligned} \tag{11}$$

The first inequality in (11) is the second G.K.S. inequality [4, 5].

Remark. Theorem 1 and 2 are also valid for the lattice gas occupation numbers $\varrho_p = \frac{1 + \sigma_p}{2}$ and their correlations $\langle \varrho_A \rangle_{h,\Lambda}$ since these have the same monotonicity properties as the $\langle \sigma_A \rangle_{h,\Lambda}$.

Although we are not going to need it in the following, we also show that a stronger mixing property holds for e.g. the ϱ_A, namely that the conditional average of ϱ_B given the configuration on a set A which does not surround B deviates very little from the unconditional average if $d(A, B)$ is big:

Theorem 3. *Let A be a rectangular region in Λ and α any configuration of spins in A. Then if $\langle \varrho_B \rangle_{h,\Lambda,\alpha}$ denotes the conditional average of ϱ_B given α we have:*

$$|\langle \varrho_B \rangle_{h,\Lambda,\alpha} - \langle \varrho_B \rangle_{h,\Lambda}| \leqq 6|B| (1 - 3e^{-\beta})^{-2} (3e^{-\beta})^{d(A,B)} \tag{12}$$

when $\beta > \beta'$ and $h \geqq 0$.

Proof. Let α_0 and α_1 be the configurations in A having all $\varrho_p = 0$ and 1 respectively. The F.G.K. inequalities [3] tell us that the conditional average for any given α is bounded below and above by the corresponding average given α_0 and α_1 respectively, so it is sufficient to prove the theorem for these two configurations. For α_1 it follows directly from Theorem 1 applied to ϱ_B, the smaller box being $\Lambda \backslash A$ and the bigger one Λ.

Consider now α_0, i.e. consider the box $\Lambda \backslash A$ with boundary condition $+1$ in $\partial \Lambda$ and -1 at the sites of A adjacent to ∂A. The contours in $\Lambda \backslash A$ again have to be closed and an odd number of them have to surround A. The argument of Lemma 1 can then be modified to give an analogous estimate:

Lemma 3. *For all* $h \geqq 0$ *we have the bound*

$$P_{h,\Lambda,\alpha_0}(\gamma \text{ is the outmost contour surrounding } A) \leqq e^{-\beta(|\gamma| - |\partial A|)} . \quad (13)$$

Proof. As in (4) we have

$$P = e^{-\beta|\gamma|} \frac{Z_-(\gamma)\, Z_0(\gamma)}{Z(\Lambda \backslash A)}, \qquad Z(\Lambda \backslash A) \geqq Z_+(\gamma)\, Z_0(\gamma) \quad \text{and} \quad \frac{Z_-(\gamma)}{Z_+(\gamma)}$$

is decreasing in h.

There is still a one to one correspondence between the terms in $Z_-(\gamma)$ and $Z_+(\gamma)$ obtained by reversing all spins inside γ (and outside of A). In this process all contours not adjacent to points in A are unchanged, and the segments adjacent to A not belonging to the contours become parts of them and vice versa. Hence for $h = 0$ the total energy of two corresponding configurations differ at most by $\pm |\partial A|$, so we have $\dfrac{Z_-(\gamma)}{Z_+(\gamma)} \leqq e^{\beta|\partial A|}$ for $h = 0$ and (10) is proved.

The argument of Lemma 2 then gives the following estimate:

Lemma 4. *Let* E *be the event: "the innermost contour surrounding* A *also surrounds a point of* B".

Then

$$P_{h,\Lambda,\alpha_0}(E) \leqq 2|B|\,(1 - 3e^{-\beta})^{-2}\,(3e^{-\beta})^{d(A,B)} \quad (14)$$

when $\beta > \beta'$ *and* $h \geqq 0$.

Proof. Let b be a point of B at distance d from A, γ a contour surrounding A and b, and let L and $q(\gamma)$ be defined as in the proof of Lemma 2. (L goes from b away from A.) We have $d(b,q) + d(q,A) \geqq d$ and (see Appendix A) $|\gamma| - |\partial A| \geqq 2d(b,q)$, $|\gamma| - |\partial A| \geqq 2d(q,A)$. Hence if $l = d(b,q)$ then $|\gamma| - |\partial A| \geqq \max(2(d-l), 2l)$, and (14) follows as in the proof of Lemma 2. (If the innermost contour surrounds b the outmost one also does.)

The proof of Theorem 3 for α_0 can now be completed using Lemma 4 and the argument of Theorem 1:

$$\langle \varrho_B \rangle_{h,\Lambda,\alpha_0} = P_{h,\Lambda,\alpha_0}(E) \langle \varrho_B \rangle_{h,\Lambda,E} + (1 - P_{h,\Lambda,\alpha_0}(E)) \langle \varrho_B \rangle_{h,\Lambda,\bar{E}} . \tag{15}$$

$\langle \varrho_B \rangle_{h,\Lambda,\bar{E}} \geqq \langle \varrho_B \rangle_{h,\Lambda}$, since for any contour γ surrounding A but not b the conditional average of ϱ_B given that the innermost contour was precisely γ is the average of ϱ_B in the region outside γ with the boundary condition $+1$ along the outside of γ. That average is however $\geqq \langle \varrho_B \rangle_{h,\Lambda}$ by F.G.K., and summing over all possible γ the inequality follows. From (15) we then get:

$$0 \geqq \langle \varrho_B \rangle_{h,\Lambda,\alpha_0} - \langle \varrho_B \rangle_{h,\Lambda} \geqq -P_{h,\Lambda,\alpha_0}(E) . \tag{16}$$

(The first inequality is true by F.G.K.), and Theorem 3 for α_0 follows using Lemma 4.

2. Differentiability of the Free Energy

For a finite system the derivatives of $F(\beta, h, \Lambda)$ are related to the correlations in the well known way which follows from (1) and (2):

$$\frac{\partial F(\beta, h, \Lambda)}{\partial \beta} = \frac{1}{2|\Lambda|} \sum_{\{p,q\} \subset \Lambda}{}' \langle \sigma_p \sigma_q \rangle_{h,\Lambda}$$

e.g., and similarly for other derivatives. In order to see the relations between the various derivatives and the correlations in general it is more perspicuous first to consider a system with an arbitrary many body interaction whose free energy is of the form

$$G = \log \sum_{\sigma} e^{\sum_A J_A \sigma_A} \tag{17}$$

and regard it as a function of all J_A, $A \subseteq \Lambda$. (We define $J_A = 0$ if $A \nsubseteq \Lambda$ in the following.)

From (17) we see that

$$\begin{aligned} \frac{\partial G}{\partial J_A} &= \langle \sigma_A \rangle \\ \frac{\partial \langle \sigma_A \rangle}{\partial J_B} &= \langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \end{aligned} \tag{18}$$

etc. and we can define the generalized Ursell functions by:

$$U_n(A_1, \cdots, A_n) = \frac{\partial^n G}{\partial J_{A_1} \cdots \partial J_{A_n}} . \tag{19}$$

As shown by Lebowitz [7] for the ordinary Ursell functions (having all $|A_i|=1$) if we have a bound

$$|U_2(A_1, A_2)| = |\langle\sigma_{A_1}\sigma_{A_2}\rangle - \langle\sigma_{A_1}\rangle\langle\sigma_{A_2}\rangle| \leqq \text{const}\, e^{-\varkappa d(A_1, A_2)}$$

we can get a bound

$$|U_n(A_1, \cdots, A_n)| \leqq C_n(a)\, e^{-\frac{\varkappa}{n} d(A_1 \cup \cdots \cup A_n)} \tag{20}$$

in terms of the diameter of $A_1 \cup \cdots \cup A_n$ if the diameters of $A_1, \cdots, A_n$ are all $\leqq a$ (see Appendix B). Hence by Theorem 2 we have such a bound uniformly when Λ arbitrary, $\beta \geqq \beta'' > \beta'$ and $h \geqq 0$ for any $\beta'' > \beta'$.

Since for the Ising model

$$J_A = \begin{cases} \beta/2 & \text{if } A \text{ is a pair of nearest neighbours in } \Lambda \\ h + a & \text{boundary term if } A \text{ is a one point set in } \Lambda \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

we have

$$\frac{\partial^{m+n} F(\beta, h, \Lambda)}{\partial\beta^m\, \partial h^n} = 2^{-m} |\Lambda|^{-1} \sum_{\substack{Q_1 \cdots Q_m \subseteq \Lambda \\ q_1 \cdots q_n \in \Lambda}}' U_{m+n}(Q_1, \cdots, Q_m, q_1, \cdots, q_n) \tag{22}$$

where the Q_i range over all pairs of neighbours in Λ and the q_i over all points in Λ.

Similarly

$$\frac{\partial^{m+n} \langle\sigma_A\rangle_{h,\Lambda}}{\partial\beta^m\, \partial h^n} = 2^{-m} |\Lambda|^{-1} \sum_{\substack{Q_1 \cdots Q_m \subseteq \Lambda \\ q_1 \cdots q_n \in \Lambda}}' U_{m+n+1}(A, Q_1, \cdots, Q_m, q_1, \cdots, q_n) \tag{23}$$

so that using the estimates above and Theorem 1 we can conclude that they converge as Λ increases:

Theorem 4. *For $\beta > \beta'$ the limiting free energy $F(\beta, h)$ and the $\langle\sigma_A\rangle_h$ are infinitely differentiable in β for $h = 0$. F, $\langle\sigma_A\rangle_h$ and all their β-derivatives have right and left derivatives of all orders with respect to h, and these are all the limits of the corresponding derivatives when $h \downarrow 0$. All derivatives for $h \geqq 0$ are the limits of the corresponding quantities in (22) and (23) when Λ increases to Z^ν in the sense of van Hove, and the limits are given by the corresponding expressions involving the limits $\langle\sigma_A\rangle_h$ in place of the $\langle\sigma_A\rangle_{h,\Lambda}$:*

$$\frac{\partial^{m+n} F(\beta, h)}{\partial\beta^m\, \partial h^n} = 2^{-m} \sum_{\substack{Q_1 \cdots Q_m \\ q_2 \cdots q_n}}' U_{m+n}(Q_1, \cdots, Q_m, q_1, \cdots, q_n) \tag{22'}$$

$$\frac{\partial^{m+n} \langle\sigma_A\rangle_h}{\partial\beta^m\, \partial h^n} = 2^{-m} \sum_{\substack{Q_1 \cdots Q_m \\ q_2 \cdots q_n}}' U_{m+n+1}(A, Q_1, \cdots, Q_m, q_1, \cdots, q_n)\,. \tag{23'}$$

(In these sums e.g. q_1 or Q_1 is fixed and the other arguments range over all of Z^ν.)

Proof. The convergence of (22) and (23) to (22′) and (23′) for all $\beta > \beta'$, $h \geqq 0$ follows from (20), Theorem 1 and the van Hove condition, which allow the boundary effects to be neglected. The fact that the convergence is uniform in any region $\beta \geqq \beta'' > \beta$, $h \geqq 0$ and the fact that $F(\beta, h, \Lambda)$ converges to $F(\beta, h)$ imply that $F(\beta, h)$ is differentiable as stated and that its derivatives are the limits of those of $F(\beta, h, \Lambda)$. The same argument applies to the $\langle \sigma_A \rangle_{h,\Lambda}$.

3. The Central Limit Theorem for Certain Random Variables

In this section we show that the central limit theorem holds for some quantities $X(\sigma)$ of the form

$$X(\sigma) = \sum_A X_A \sigma_A . \tag{24}$$

Generally speaking for a quantity as in (24) we expect $X - \langle X \rangle$ to have an approximatively gaussian distribution if in the sum there are many uniformly small terms which are sufficiently weakly dependent. This will turn out to be the case for e.g. the total magnetization and the total energy when Λ is large.

In the following we will only consider variables as in (24) with $X_A \neq 0$ only if A is a one point set or a nearest neighbour pair in Λ. For such variables the estimates previously derived can be used to get a rather straight-forward proof by expanding the generating function $\langle e^{tX} \rangle_{h,\Lambda}$ for $t \geqq 0$ to second order in t and estimating the remainder term suitably. The basic estimates needed can be collected in the following lemma:

Lemma 5. *If the X_A are restricted as just indicated and are all non negative, then*

$$\langle e^{tX} \rangle_{h,\Lambda} = \exp\left[t\langle X \rangle_{h,\Lambda} + \frac{t^2}{2}(\langle X^2 \rangle_{h,\Lambda} - \langle X \rangle_{h,\Lambda}^2) + O\left(t^3 \max_A X_A\right) \sum_A X_A^2 \right] \tag{25}$$

uniformly when $\beta \geqq \beta'' > \beta'$, $h \geqq 0$, $t \geqq 0$ and Λ arbitrary.

Proof. If we denote by $G(t)$ the free energy defined as in (17) for the interaction obtained from (21) by adding tX_A to J_A we get:

$$\begin{aligned} \langle e^{tX} \rangle_{h,\Lambda} &= \exp(G(t) - G(0)) \\ &= \exp\left(tG'(0) + \frac{t^2}{2} G''(0) + \frac{t^3}{6} G'''(\tilde{t}) \right) \end{aligned} \tag{26}$$

where $\tilde{t}$ is an intermediate point $0 \leqq \tilde{t} \leqq t$.

The first two terms in the Taylor expansion are equal to the corresponding ones in (25), and the third is equal to

$$R = \frac{t^3}{6}\left(\langle X^3\rangle - 3\langle X^2\rangle\langle X\rangle + 2\langle X\rangle^3\right),$$

the averages being evaluated with the interaction $J_A + \tilde{t} X_A$. Since $\tilde{t} X_A \geqq 0$ all our previous estimates are valid also for this interaction by the remark after Lemma 1, and we get from (20)

$$\begin{aligned} |R| &= \frac{t^3}{6}\left| \sum_{A_1, A_2, A_3} X_{A_1} X_{A_2} X_{A_3} U_3(A_1, A_2, A_3) \right| \\ &\leqq \frac{t^3}{6} C_3(2) \sum_{A_1, A_2, A_3} X_{A_1} X_{A_2} X_{A_3} e^{-\frac{\varkappa}{3} d(A_1 \cup A_2 \cup A_3)} \qquad (27) \\ &= O\left(t^2 \left(\max_A X_A\right) \sum_{A_1, A_2} X_{A_1} X_{A_2} e^{-\frac{\varkappa}{3} d(A_1 \cup A_2)}\right). \end{aligned}$$

The last sum can be estimated in terms of $\sum_A X_A^2$ as follows: Let B_1, B_2, B_3 be the three different one and two points sets with $J_B \neq 0$ that contain the origin e.g. Then the sum can be bounded by

$$\begin{aligned} &O\left(\sum_{i,j=1}^{3} \sum_{p,q} X_{B_i+p} X_{B_j+q} e^{-\frac{\varkappa}{3} d(p,q)}\right) \\ &\quad = O\left(\sum_r e^{-\frac{\varkappa |r|}{3}} \sum_{i,j=1}^{3} \sum_{p-q=r} X_{B_i+p} X_{B_j+q}\right) \qquad (28) \\ &\quad = O\left(\sum_B X_B^2\right). \end{aligned}$$

We can now prove the central limit theorem for the total energy and magnetization of the system:

Theorem 5. *When Λ increases in the sense of van Hove the distribution of the fluctuations in the total energy $E_0 = \sum'_{\{p,q\} \subset \Lambda} \sigma_p \sigma_q$ and the total magnetization*

$M = \sum_{p \in \Lambda} \sigma_p$, i.e. $e = |\Lambda|^{-1/2}(E_0 - \langle E_0\rangle_{h,\Lambda})$ and $m = |\Lambda|^{-1/2}(M - \langle M\rangle_{h,\Lambda})$

converges to a gaussian one with mean zero and second moments

$$\langle e^2\rangle = 4\frac{\partial^2 F(\beta, h)}{\partial \beta^2}, \qquad \langle em\rangle = 2\frac{\partial^2 F(\beta, h)}{\partial \beta\, \partial h}, \qquad \langle m^2\rangle = \frac{\partial^2 F(\beta, h)}{\partial h^2}$$

when $\beta > \beta'$ $h \geqq 0$.

Proof. Put $X_A = t_1 |\Lambda|^{-1/2}$ if A is a single point in Λ and $X_A = t_2 |\Lambda|^{-1/2}$ if A is a nearest neighbour pair in Λ and $X_A = 0$ otherwise and apply Lemma 5. The last term in (25) is $O(|\Lambda|^{-1/2})$ and hence goes to zero. The quadratic term converges to $\frac{1}{2}(t_1^2\langle m^2\rangle + 2t_1 t_2\langle em\rangle + t_2^2\langle e^2\rangle)$ by Theorem 4, so we see that the generating function of e, m converges to that of the gaussian defined by it:

$$\lim_{\Lambda} \langle e^{t_1 m + t_2 e}\rangle_{h,\Lambda} = \exp \tfrac{1}{2}(t_1^2\langle m^2\rangle + 2t_1 t_2\langle em\rangle + t_2^2\langle e^2\rangle) \tag{29}$$

for all $t_1, t_2 \geqq 0$. By the continuity theorem for generating functions proved in Appendix C this implies that the distribution of e and m converges to that gaussian distribution.

The gaussian distribution is the one prescribed by thermodynamic fluctuation theory, and it is completely determined by the limiting free energy $F(\beta, h)$. However it is too crude to approximate the averages $\langle E_0\rangle_{h,\Lambda}$ and $\langle M\rangle_{h,\Lambda}$ by $2|\Lambda| \dfrac{\partial F(\beta,h)}{\partial \beta}$ and $|\Lambda| \dfrac{\partial F(\beta,h)}{\partial h}$ because in general surface effects are not small compared to $|\Lambda|^{1/2}$. For example, using Theorem 1 it is easy to show that if Λ is a square with side L then

$$\langle M\rangle_{h,\Lambda} = L^2\langle\sigma_0\rangle_h + 4L\sum_{1}^{\infty}(\mu_d - \langle\sigma_0\rangle_h) + o(L) \tag{30}$$

where μ_d is the average magnetization at distance d from the boundary of an infinite half space with all spins $+1$ at the boundary. Hence to obtain the correct average of the gaussian distribution it is not enough to know $F(\beta, h)$; the "surface term" in $F(\beta, h, \Lambda)$ can in general not be neglected. This is even more pronounced in three dimensions, where the surface correction is proportional to $|\Lambda|^{2/3}$. In the proof of Theorem 5 we could more generally have considered the simultaneous distribution of e and m_0, m_1 the magnetizations of the two sublattices Λ_0 and Λ_1 of points whose coordinates have an even or odd sum. The covariances of the gaussian limit distribution are then $\langle e^2\rangle$, $\langle em_i\rangle = \frac{1}{2}\langle em\rangle$

$$\langle m_0^2\rangle = \langle m_1^2\rangle = \tfrac{1}{2}\sum_{p\in\Lambda_0}\langle\sigma_p\sigma_0\rangle_h - \langle\sigma_p\rangle_h\langle\sigma_0\rangle_h$$

and

$$\langle m_0 m_1\rangle = \tfrac{1}{2}\sum_{p\in\Lambda_1}\langle\sigma_p\sigma_0\rangle_h - \langle\sigma_p\rangle_h\langle\sigma_0\rangle_h\,.$$

Using this result and the transformation which reverses the spins on one sublattice and turns the system into an antiferromagnet with interaction $-\beta$ we get the central limit theorem for such a system:

Corrollary 5. *For an antiferromagnetic system with parameters* $-\beta$ *and* $h=0$ *the distribution of* e *and the total magnetization* m_a *converges* *to a gaussian one with zero mean and covariances*

and

$$\langle e^2\rangle, \langle em_a\rangle = \langle e(m_0 - m_1)\rangle = 0$$

$$\langle m_a^2\rangle = \langle (m_0 - m_1)^2\rangle = \langle m^2\rangle - 4\langle m_0 m_1\rangle .$$

(The averages involving e, m_0, m_1 *are those given above.)*

Since the estimate of Lemma 5 is uniform in Λ we can also use it to prove the central limit theorem for a sum over a large region of an infinite system:

Theorem 6. *Consider an infinite system and let a sequence of variables be defined as in* (24), *each by a finite sum. Then if*

$$\sum_A (X_A^{(n)})^2 \leqq \text{const}, \quad \max_A |X_A^{(n)}| \to 0$$

and

$$v^{(n)} \equiv \langle (X^{(n)})^2\rangle_h - \langle X^{(n)}\rangle_h^2 = \sum_{A,A_2} X_{A_1}^{(n)} X_{A_2}^{(n)} U_2(A_1, A_2) \to v$$

the distribution of $X^{(n)} - \langle X^{(n)}\rangle_h$ *converges to a gaussian one with zero mean and variance* v. *For example, if* $X^\varepsilon = \varepsilon^{\nu/2} \sum_p \varphi(\varepsilon p)\,\sigma_p$ *where* $\varphi(\cdot)$ *is any continuous function with compact support, then as* $\varepsilon \to 0$ *the distribution of* $X^\varepsilon - \langle X^\varepsilon\rangle_h$ *converges to a gaussian one with variance*

$$v = \left(\sum_p \langle \sigma_0 \sigma_p\rangle_h - \langle \sigma_0\rangle_h \langle \sigma_p\rangle_h\right) \int \varphi^2(x)\,dx = \frac{\partial^2 F(\beta, h)}{\partial h^2} \int \varphi^2(x)\,dx . \tag{31}$$

Proof. If $X_A^{(n)} \geqq 0$ the theorem follows immediately from (25) and the continuity theorem for generating functions. If $X_A^{(n)}$ are of both signs we split them into the positive and negative part:

$$X_A^{(n)} = X_{A,+}^{(n)} - X_{A,-}^{(n)} \quad \text{with} \quad X_{A,+}^{(n)} \cdot X_{A,-}^{(n)} = 0, \quad \text{and} \quad \sum_A (X_{A,\pm}^{(n)})^2 \leqq \text{const}.$$

Consider an arbitrary subsequence $\{n'\}$ of the given one. Since the quadratic form $\sum_{A_1,A_2} X_{A_1} X_{A_2} U_2(A_1, A_2)$ is bounded by $\text{const} \sum_A X_A^2$ (see (28)) we can always find a subsequence $\{n''\}$ of $\{n'\}$ such that the covariances of $X_+^{(n'')}$ and $X_-^{(n'')}$ converge. The proof for non-negative $X_A^{(n)}$ applies to any combination $t_+ X_+^{(n'')} + t_- X_-^{(n'')}$ with $t_+, t_- \geqq 0$, so by the continuity theorem the centered distribution of $(X_+^{(n'')}, X_-^{(n'')})$ converges to a gaussian one. From this follows that the centered distribution of $X^{(n'')} = X_+^{(n'')} - X_-^{(n'')}$ converges to a gaussian one with variance v. Since any subsequence of the given one contains a subsequence converging to the same gaussian distribution the sequence itself in fact converges to that distribution [1].

The limit theorem for "coarse grained" averages like X^ε implies more generally that if we consider any finite family $X_1^\varepsilon, \cdots, X_n^\varepsilon$ defined by

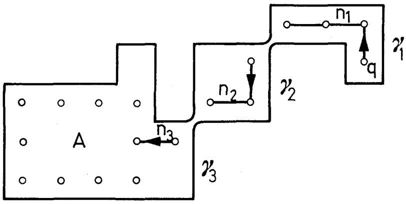

Fig. 3. A path from q to A inside γ

functions $\varphi_1, \cdots, \varphi_n$ with disjoint supports then the joint limit distribution will be gaussian with independent components with variances $\frac{\partial^2 F(\beta, h)}{\partial h^2} \int \varphi_i^2(x)\, dx$. This follows directly by applying the theorem to the variables $t_1 X_1^\varepsilon + \cdots + t_n X_n^\varepsilon$ for arbitrary $t_1, \cdots, t_n$. Hence the limit distribution is the same as if the spins were independent with mean

$$\langle \sigma_0 \rangle_h = \frac{\partial F(\beta, h)}{\partial h} \quad \text{and variance} \quad \frac{\partial^2 F(\beta, h)}{\partial h^2},$$

and the limit theorem makes precise when the description of the system as a union of macroscopically infinitesimal independent subsystems often used in fluctuation theory is valid.

Appendix A. Some Estimates of the Lengths of Certain Contours

Here we prove the following estimate used in Section 1.

Lemma A1. *Let A be a rectangular region and q a point on the lattice at distance d from A. Then for any contour γ surrounding A and q we have*

$$|\gamma| \geqq |\partial A| + 2d\,. \tag{A. 1}$$

Proof. Draw a path on the lattice from q to A going inside γ except possibly near points where γ touches itself as indicated in Fig. 3.

Let k be the number of segments in the path, let each segment be as short as possible consisting of say n_i points $i = 1, \cdots, k$ and let γ_i $i = 1, \cdots, k$ be the parts of γ surrounding the segments of the path. We then have $\sum_1^k (n_i - 1) + 2(k-1) \geqq d$, since d is the shortest distance from q to A. For each segment except the last we have the relation $|\gamma_i| \geqq 2(n_i + 1)$. This can be seen as follows: To each point p of the segment except the last associate two units of γ_i by drawing a line through p perpendicular to the next step of the segment and then assigning to p those two units of

γ_i that are first cut by the line, one in each direction. In this process no pair of units are assigned to two different points, since if that happened for e.g. p and p' with p coming before p' in the segment they would both lie on the same line with no unit of γ_i separating them, and the segment could be shortened by going directly from p to p' along the line instead. Similarly, to the last point of the segment we can associate the four units of γ_i cut out by the two lines that can be drawn through it without conflict with the earlier assignments. Hence $|\gamma_i| \geqq 2(n_i - 1) + 4 = 2(n_i + 1)$. For the last segment we have similarly $|\gamma_k| \geqq |\partial A| + 2(n_k - 1)$. This can be seen if we perform the assignments as above for all points of the segment except the last, and then assign to each point p in ∂A that unit of γ_k first cut out by the halfline going out of A through p and its neighbour in A. Again, in the assignment of units to the points of ∂A no conflicts arise, since if one occurred with e.g. p on the segment it could be shortened by following the line through p to A instead.

Summing up we see that

$$|\gamma| \geqq |\partial A| + \sum_{1}^{k} 2(n_i - 1) - 4 \geqq |\partial A| + 2d$$

and the lemma is proved.

Also if γ surrounds A and a straight segment (b, q) of length l going away from A, then since $d(q, A) = l + d(b, A)$ we have by the lemma $|\gamma| \geqq |\partial A| + 2d(q, A) > |\partial A| + 2l$ as claimed in the proof of Lemma 4.

Appendix B. Estimates of the Generalized Ursell Functions

Here we prove the estimate of the $U_n(A_1, \cdots, A_n)$ used in (20):

Lemma B1. *Suppose that we have a bound*

$$|U_2(A_1, A_2)| \leqq C \cdot e^{-\varkappa d(A_1, A_2)}$$

for all A_1, A_2, then for any partitioning of $\bigcup_1^n A_i$ into e.g.

$$A' = \bigcup_1^{n'} A_i \quad \textit{and} \quad A'' = \bigcup_{n'+1}^{n} A_i$$

we have

$$|U_n(A_1, \cdots, A_n)| \leqq C_n e^{-\varkappa d(A', A'')} \tag{B.1}$$

for all $A_1, \cdots, A_n$ and some constant C_n. Moreover if the diameters of the A_i are all $\leqq a$ then d, the maximum of $d(A', A'')$, is bounded by

$$d \geqq \frac{d\left(\bigcup_1^n A_i\right)}{n} - a \tag{B.2}$$

so that (20) *is true with* $C_n(a) = C_n e^{\varkappa a}$. *(d(A) denotes the diameter of the set A.)*

Proof. By induction on n we first prove that for any partitioning described above $U_n(A_1, \cdots, A_n)$ can be written as a finite sum of terms of the form $\pm P \cdot U_2(B', B'')$, where P is a product of $\langle \sigma_A \rangle$: s and $B' \subseteqq A'$, $B'' \subseteqq A''$. For $n=2$ this is trivially true.

Suppose it is true for all $m<n$ and that $A' \cup (A'' \cup A_n)$ is the given partitioning, where $A' \cup A''$ is a partitioning of $\bigcup_1^{n-1} A_i$.

From (19) $U_n(A_1, \cdots, A_n) = \dfrac{\partial U_{n-1}(A_1, \cdots, A_{n-1})}{\partial J_{A_n}}$, and by the induction hypothesis U_{n-1} is a sum of terms as described above. From such a term similar terms are generated when P is differentiated. Finally when $U_2(B', B'')$ is differentiated we get the following terms [using (18)]:

$$\begin{aligned} \pm P[\langle \sigma_{B'} \sigma_{B''} \sigma_{A_n} \rangle - \langle \sigma_{B'} \sigma_{B''} \rangle \langle \sigma_{A_n} \rangle - \langle \sigma_{B'} \rangle (\langle \sigma_{B''} \sigma_{A_n} \rangle - \langle \sigma_{B''} \rangle \langle \sigma_{A_n} \rangle) \quad \text{(B.3)} \\ - \langle \sigma_{B''} \rangle (\langle \sigma_{B'} \sigma_{A_n} \rangle - \langle \sigma_{B'} \rangle \langle \sigma_{A_n} \rangle)] = \pm P[\langle \sigma_{B'} \sigma_{B'' \Delta A_n} \rangle - \langle \sigma_{B'} \rangle \langle \sigma_{B'' \Delta A_n} \rangle] \\ \mp P \langle \sigma_{A_n} \rangle [\langle \sigma_{B'} \sigma_{B''} \rangle - \langle \sigma_{B'} \rangle \langle \sigma_{B''} \rangle] \mp P \langle \sigma_{B''} \rangle [\langle \sigma_{B'} \sigma_{A_n} \rangle - \langle \sigma_{B'} \rangle \langle \sigma_{A_n} \rangle] \end{aligned}$$

which are all of the desired form for the partitioning of $\bigcup_1^n A_i$. Hence (B. 1) follows from the bound of U_2 and the fact that $|P| \leqq 1$.

To prove (B. 2) let p and q be any two points in $\bigcup_1^n A_i$ with $p \in A_1$, $q \in A_n$ e.g. Take $A' = A_1$ and $A'' = \bigcup_2^n A_i$. Then some member of A' has distance at most d from A'. If it is A_n the distance $d(p, q)$ is at most $2a + d$. If not, move that member to A' instead. Any point in A' has then at most the distance $2a + d$ from p. Repeat the argument until after at most $(n-1)$ steps A_n is reached. We can then conclude that $d(p, q) \leqq n(d+a)$, and (B. 2) follows.

Appendix C. A Continuity Theorem for Generating Functions of Probability Distributions

Lemma C1. *Let* $\{G_n\}$ *be a sequence of probability distributions on* R^d *for which the generating functions* $g_n(t) = \int e^{t \cdot x} G_n(dx)$ *all are defined in some convex set* $\Omega \subset R^d$ *which contains* 0 *and some open ball of* R^d. *If* $\lim_{n \to \infty} g_n(t) = g(t)$ *in* $\operatorname{int} \Omega \cup \{0\} \equiv \Omega'$ *and* $g(t)$ *is continuous at the origin then* G_n *converge weakly to a distribution* G, *and* $g(t) = \int e^{t \cdot x} G(dx)$ *in* Ω'.

(I.e. $\lim_{n\to\infty} \int \varphi(x)\, G_n(dx) = \int \varphi(x)\, G(dx)$ *for any bounded continuous function, which we denote by* $G_n \Rightarrow G$.*)*

Proof. Consider the case $d=1$ first, and let Ω' be an interval $[0,\omega)$ $\omega>0$ e.g. Consider a point $t\in\Omega'$, $t\neq 0$, and choose $\varepsilon>0$ so that $t+\varepsilon\in\Omega'$ too. Then the tails of the integrals $\int e^{tx}\, G_n(dx)$ can be bounded as follows:

$$e^{\varepsilon A} \int_A^\infty e^{tx}\, G_n(dx) \leqq \int e^{(t+\varepsilon)x}\, G_n(dx) = g_n(t+\varepsilon) \tag{C. 1}$$

so that

$$\int_A^\infty e^{tx}\, G_n(dx) \leqq \text{const}\, e^{-\varepsilon A} \tag{C. 2}$$

uniformly in n. Similarly

$$\int_{-\infty}^{-A} e^{tx}\, G_n(dx) \leqq e^{-tA} \tag{C. 3}$$

so the tails $\int_{|x|\geqq A} e^{tx}\, G_n(dx)$ can be made arbitrarily small uniformly in n by choosing A large. By Helly's theorem there exists a subsequence, $\{n'\}$, and a distribution G (possibly having total mass <1) so that

$$\int_A^B \varphi(x)\, G_{n'}(dx) \to \int_A^B \varphi(x)\, G(dx)$$

for all continuous $\varphi(x)$ if A and B are points of continuity of the distribution function $G(x)$. From this and the uniform bound (C. 2), (C. 3) of the tail integrals follows that

$$\int e^{tx}\, G(dx) \quad \text{is finite and that} \quad \int e^{tx}\, G_{n'}(dx) \to \int e^{tx}\, G(dx)\,,$$

so that

$$g(t) = \int e^{tx}\, G(dx) \quad \text{for all} \quad t\in(0,\omega)\,.$$

Letting t go to zero we see by the monotone convergence theorem and the continuity of g that $g(0)=1=\int G(dx)$, so G is not defective which implies that $G_n \Rightarrow G$ [1]. If $\tilde{G}$ is the limit distribution of another subsequence we see that

$$\int e^{tx}\, G(dx) = \int e^{tx}\, \tilde{G}(dx) = g(t) \quad \text{for} \quad t\in[0,\omega)\,.$$

But both generating functions are analytic in the strip $\operatorname{Re} t\in[0,\omega)$ and are continuous when $\operatorname{Re} t\to 0$, so we see that they coincide when t is imaginary too, which implies that $G=\tilde{G}$ by the uniqueness theorem for Fourier-Stieltjes transforms. Hence $G \Rightarrow G$ and

$$g(t) = \int e^{tx}\, G(dx) \quad \text{for} \quad t\in\Omega'\,.$$

In particular the sequence $\{G_n\}$ is tight, i.e. for any $\varepsilon>0$ there is a compact interval K_ε such that $\int_{K_\varepsilon} G_n(dx) \geqq 1-\varepsilon$ uniformly in n [1].

Consider now the case $d>1$. Since Ω contains an open ball of full dimension we can find $t_1, \cdots, t_d \in \Omega'$ such that K_A, the intersection of the strips $\{x; |t_i \cdot x| \leqq A\}$ is compact for all $A \geqq 0$. Applying the theorem for $d=1$ to the random variables $t_i \cdot x$ $i=1, \cdots, d$ we see that they are all tight, i.e. for any $\varepsilon > 0$ we can choose A_ε such that

$$G_n(|t_i \cdot x| > A_\varepsilon) \leqq \frac{\varepsilon}{d} \quad \text{for} \quad i=1, \cdots, d$$

and all n. Hence $G_n(K_{A_\varepsilon}) \geqq 1-\varepsilon$ uniformly in n, and the family $\{G_n\}$ is tight also. Hence it is relatively compact i.e. there is a subsequence $\{n'\}$ such that $G_{n'} \Rightarrow G$ [1]. As in the proof for $d=1$ we see that for any $t \in \Omega' \setminus \{0\}$ the tail integrals $\int_{|t \cdot x| \geqq A} e^{t \cdot x} G_n(dx)$ can be made arbitrarily small uniformly in n if A is made large, so that $\int e^{t \cdot x} G_{n'}(dx) \to \int e^{t \cdot x} G(dx)$ and $g(t) = \int e^{t \cdot x} G(dx)$ for $t \in \Omega'$. The equality of any two limit distributions G and $\tilde{G}$ then follows as before, and we conclude that $G_n \Rightarrow G$ with $g(t) = \int e^{t \cdot x} G(dx)$ for $t \in \Omega'$. If as in Section 3 $g(t) = \exp\left(\frac{t \cdot Qt}{2}\right)$ with Q non negative definite then because this is an entire function we can conclude that $g(t) = \int e^{t \cdot x} G(dx)$ when t is imaginary too, and G is the gaussian distribution determined by Q.

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