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The Space of Lorentz Metrics*

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Abstract. The set of all C^2 Lorentz metrics on a non-compact four-manifold is given the Whitney fine C^2 topology. It is shown that this provides the correct framework within which to discuss the global properties of spacetime manifolds in general, and the singularity theorems in particular. The main result is a theorem showing that the Robertson-Walker big bang (global infinite density singularity in the finite past) is stable under sufficiently small, but otherwise arbitrary, finite C^2 perturbations of the metric tensor.

I. Introduction

This paper deals with the topological structure of the space of all exact solutions to the Einstein field equations on an arbitrary four-manifold. A rigorous mathematical framework emerges within which it is possible to pose and answer the following questions:

(1) Which of the well-known global properties of spacetimes are stable under sufficiently small perturbations of the metric tensor?

(2) Do the singularity theorems of Hawking, Penrose and others translate into statements concerning the topology of this space? For instance, are G-singularities *stable* in the set of "physically realistic" metrics (do they form an open set)? Are there situations in which G-singularities are *generic* (for example, when the spacetime contains a closed space section, or when it contains an object undergoing catastrophic gravitational collapse)?

(3) Do there exist situations in which it is possible to say something precise about the exact nature of the singularity (other than that an incomplete causal geodesic exists)? For example, *is the "big bang" stable*? If a Robertson-Walker model is slightly perturbed, does the new spacetime still possess an infinite density singularity over all space in the finite past?

Section II presents the necessary mathematical machinery; it reviews jet bundles, defines and to some extent motivates the Whitney mapping

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topologies, and recalls some elementary properties of these topologies. A convenient definition of a relativistic conformal structure is given. The Lorentz metrics and the conformal structures admit Whitney topologies consistent with one another (meaning that the map sending a Lorentz metric to its conformal equivalence class is continuous).

Section III contains a digression on the Einstein equations as viewed from the standpoint of Section II. It is primarily intended to give the reader some feeling for the techniques that will be used in the following sections. It is shown, in passing, that there are no isolated solutions to the field equations.

Section IV is concerned with the stability of certain global properties of spacetime manifolds. More or less obviously, the following properties are stable: (a) the existence of a space-section with diverging future normals, (b) the existence of a trapped surface, (c) the strong energy condition, and (d) a decomposition of the spacetime into $S \times R$, where $S \times \{t\}$ is spacelike for each $t \in R$, and $\{p\} \times R$ is a timelike curve for each $p \in S$. The proofs are straightforward. Less trivially, perhaps, geodesic completeness is a stable property, as is a certain useful type of incompleteness of vector fields.

In Section V, a definition is proposed for the set of all physically realistic cosmologies on a given manifold. Question 2 of the introductory paragraph is answered affirmatively in this context. Remarks are made to the effect that it is not even possible to discuss the question in any other known topology. The results of this section are definitely implicit in the singularity theorems.

Section VI deals with the third question and contains the main result of this paper. The metric tensor of an arbitrary Robertson-Walker model with a big bang singularity in the past is subjected to an arbitrary C^2 perturbation. If the perturbation is sufficiently small, the new spacetime has the property that the energy density blows up in the finite past along each flowline of the matter. The corresponding time-reversed statements concerning stability of the eventual collapse or indefinite expansion of the universe follow immediately.

II. Basic Concepts

Throughout this paper, all manifolds are C^∞, Hausdorff and paracompact; M will denote a fixed, non-compact, four-dimensional manifold without boundary. Let $S \to M$ be the vector bundle of twice-covariant symmetric tensors, and $L \to M$ the open subbundle of indefinite quadratic forms of signature -2. The set of all C^k sections of L is denoted $C^k(L)$; any element of this set is called a C^k *Lorentz metric.*

It is this set we wish to topologize. Before doing so, we note that many of the interesting geometric objects which occur in the theory of relativity are completely determined by the conformal structure of spacetime; it will clearly be to our advantage if we can topologize the set of conformal structures as well.

Delete the zero section from S, and denote the resulting bundle by $\hat{S}$; there is an obvious retraction τ of $\hat{S}$ onto a bundle Q whose fibers are diffeomorphic to the nine-sphere: simply identify, in each fiber, all elements which are positive multiples of one another (in local coordinates, τ is given by $(u^i, s_{ab}) \mapsto (u^i, s_{ab}/s)$, where $a \leqq b$ and $s = \left(\sum_{m \leqq n} (s_{mn})^2\right)^{\frac{1}{2}}$. Q is a C^∞ bundle over M, and τ is a C^∞ open surjection. The set $\tau(L)$ is an open subbundle of Q which we denote by C.

For any Lorentz metric g, $\bar{g} := \tau g$ is a section of C; if h is another Lorentz metric $\bar{h} = \bar{g}$ iff. h and g are conformally related. A C^k section of C is called a C^k *conformal structure*. For any $w \in C^k(C)$, there exists an $h \in C^k(L)$ such that $\bar{h} = w$: to see this, let μ be any positive-definite metric on M; at each $p \in M$, define $h(p)$ to be the unique element of μ-norm 1 in $\tau^{-1}(w(p))$.

Let $E \to M$ be any bundle, and let $J^k(M, E) \to M \times E$ be the bundle of k-jets of local C^k maps from M to E[1]. Recall that if f is such a map and p is in the domain of f, $j^k(f)(p)$ denotes the equivalence class of all local C^k maps g which are k-tangent to f at p. (The Taylor series for f and g agree up to and including the k^{th} derivatives in any (and all) local coordinate systems at p and $f(p)$.) By restricting attention to the subset of local C^k sections of E, we obtain a closed subbundle $J^k(E)$ called the *bundle of k-jets of sections* of E. If a C^k section f is a geometric object field with components $f^{\cdots}_{\cdots}$ relative to some local coordinates (u^a), the local coordinates of $j^k(f)(p)$ are given by

$$(u^a(p), f^{\cdots}_{\cdots}(p), f^{\cdots}_{\cdots, i_1}(p), \ldots, f^{\cdots}_{\cdots, i_1, i_2, \ldots, i_k}(p)),$$

where the comma denotes partial differentiation. Any $f \in C^r(E)$ determines a C^{r-k} section of $J^k(E)$ defined by $p \mapsto j^k(f)(p)$ and called the *k-jet extension* of f. Notice, for future reference, that $J^0(M, E) = M \times E$, and that $J^0(E) \cong E$.

Let U be open in $J^k(E)$. Define

$$N(U) = \{f \in C^k(E) : j^k(f)(M) \subseteqq U\}.$$

The W_k or Whitney fine C^k topology on $C^k(E)$ is generated by the sets $N(U)$, where U ranges over the open sets of $J^k(E)$. For $k > r \geqq 0$, the W_r

[1] For a neat, precise introduction (with proofs) to jet bundles and the Whitney topologies, the reader is referred to the first few pages of Mather [1].

topology is well-defined on $C^k(E)$ and is strictly coarser that the W_k topology. From now on, $\Gamma^k(E)$ stands for the set $C^k(E)$ in the W_k topology.

By way of motivation, suppose a collection of observers is scattered throughout M, and that each observer is directed to measure the values of some field f, together with its first k derivatives, in his neighborhood. The values reported, with judicious error bounds, determine an open $U \subseteqq J^k(E)$, and $j^k(f)(M) \subseteqq U$. Notice that if the set U has been made as small as possible, than any other field f' satisfying $j^k(f')(M) \subseteqq U$ must be regarded as a legitimate perturbation of the "real" field f. We summarize below the few facts we shall need concerning $\Gamma^k(E)$:

(2.1) For non-compact M, $\Gamma^k(E)$ is *not first-countable*. Thus we must use nets or filters to talk about convergence. In fact, convergence of a sequence $\{f_i\}$ occurs only under the very restrictive condition that there exist a compact $K \subset M$ outside of which $f_i = f_j$ for sufficiently large i, j, and inside of which $\{j^k(f_i)\}$ converges uniformly. An immediate consequence is that if E is a vector bundle, scalar multiplication is *not* continuous in $\Gamma^k(E)$; so it is not a topological vector space. It *is*, however, a topological module over the ring of real-valued C^k functions on M [2]. This is proven by Mather in [1]. Another consequence is that the well-known one (or many)-parameter families of metrics encountered in relativity theory do *not* determine continuous curves (or surfaces) in $\Gamma^k(L)$ [3].

(2.2) Let E and F be bundles over M; denote by p_1 and p_2 the projections of $J^0(F)$ to M and F respectively. Let $\tilde{\varphi}: J^k(E) \to J^0(F)$ be a continuous map inducing the identity on M (for any $s \in C^k(E)$, $p_1 \circ \tilde{\varphi} \circ j^k(s) = \mathrm{id}_M$). Then the map $\varphi: \Gamma^k(E) \to \Gamma^0(F)$ defined by setting $\varphi s(x) = p_2 \circ \tilde{\varphi}(j^k(s)(x))$ for any $s \in \Gamma^k(E)$ is continuous: if U is open in $J^0(F)$, then $\varphi^{-1}(N(U)) = N(\tilde{\varphi}^{-1}(U))$, which is open, since $\tilde{\varphi}$ is continuous. In many cases of interest, φ is a continuous partial differential operator; this will be illustrated in III for the case of the Einstein equations.

(2.3) Let E and F be bundles over M, and let $w: E \mapsto F$ be a C^r $(r \geqq k)$ map inducing the identity on M. Then the map $w: \Gamma^k(E) \mapsto \Gamma^k(F)$ defined by $s \mapsto w \circ s$ for $s \in \Gamma^k(E)$ is continuous. This is obvious.

Proposition (2.4). *Let L, C and τ be as above. Then the map $\tau: \Gamma^k(L) \mapsto \Gamma^k(C)$ sending g to $\bar{g}(=\tau g)$ is continuous. For $k=0$, τ is open.*

Proof. We have only to show openness. Let U be open in $J^0(L) = L$. Then $\tau(U)$ is open in C. Let $\bar{g}$ satisfy $\bar{g}(M) \subset \tau(U)$. We must find $g \in C^0(L)$

[2] This function space, in the W_k topology, is denoted $C^k(M)$.

[3] This is not a serious problem. In the first place, as Geroch [2] points out, the process of taking limits is not without ambiguity. Secondly, the limiting metric is often one which, when maximally extended, determines a base manifold $M'' \neq M$. Thus *the problem of limits is not well-posed in $C^k(L)$ no matter which topology one uses.* See Geroch [2] for details.

such that $g(M) \subset U$ and $\tau g = \overline{g}$. So let g' be any C^0 metric such that $g' = \overline{g}$. We must find a positive continuous function λ such that $\lambda g'(M) \subset U$. For each $p \in M$, there exists a positive number $a(p)$ such that $a(p)\, g'(p) \in U \cap L_p$, where L_p denotes the fiber of L over p. By continuity considerations, there is a neighborhood $V(p)$ such that $q \in V(p) \Rightarrow a(p)\, g'(q) \in U \cap L_q$. Thus we obtain an open cover of M, for which there exists a locally finite open refinement $\{V_i : i \in I\}$ and a subordinate partition of unity $\{f_i : i \in I\}$. For each i, choose a $V(p)$ such that $V_i \subseteqq V(p)$, and let b_i be the constant function with value $a(p)$. Put $\lambda = \sum_i f_i b_i$; then $g = \lambda g'$ is the metric we need.

We exhibit some of the open sets of $\Gamma^0(C)$:

Definition (2.5). Let $\overline{g} \in C^0(C)$, $p \in M$, $T_p =$ the tangent space at p. Define $t_p(\overline{g}) = \{Z \in T_p : g(Z, Z) > 0$ for some (and thus any) $g \in \tau^{-1}(\overline{g})\}$. Let $\overline{h}, \overline{g} \in C^0(C)$. We write $\overline{h}(p) < \overline{g}(p)$ iff. $\overline{t_p(\overline{h})} \subset t_p(\overline{g})$, and define $\overline{h} < \overline{g}$ iff. $\overline{h}(p) < \overline{g}(p)$ for all $p \in M$. If $\overline{h} < \overline{g}$, define

$$(\overline{h}, \overline{g}) = \{\overline{k} \in C^0(C) : \overline{h} < \overline{k} < \overline{g}\}\,.$$

This set is called an *interval*, and the topology on $C^0(C)$ generated by taking the intervals as a subbase, the *interval topology* (see Geroch [3]). Intuitively, $\overline{k} \in (\overline{h}, \overline{g})$ if and only if at each $p \in M$, the null cone of $\overline{k}$ lies strictly between those of $\overline{h}$ and $\overline{g}$.

Proposition (2.6). *An interval is an open set in $\Gamma^0(C)$.*

Proof. Let $\overline{h}, \overline{g}, \overline{k}$ be as above. We must find a W_0 neighborhood of $\overline{k}$ contained in $(\overline{h}, \overline{g})$. Since $J^0(C) \cong C$, it suffices to find an open set $U \subset C$ containing $\overline{k}(M)$ with the property that any $\overline{m}(p) \in U$ satisfies $\overline{h}(p) < \overline{m}(p) < \overline{g}(p)$. This is trivial, using the local product structure.

The converse, implying that the W_0 topology is identical to the interval topology, is also true, although we shall not need this fact here [4].

III. Example and Digression; the Einstein Equations

Consider the map $\widetilde{\mathrm{Ric}}: J^2(L) \mapsto J^0(S)$ sending the 2-jet of a Lorentz metric to the 0-jet of its Ricci tensor[4]. Choose a chart with coordinates (u^i). If p is in the chart and g is a local section at p, local coordinates for $j^2(g)\,(p)$ are just $(u^i(p), g_{ab}(u^i), g_{ab,c}(u^i), g_{ab,cd}(u^i))$, while those of $\widetilde{\mathrm{Ric}}(j^2(g)\,(p))$ are $(u^i(p), 2\Gamma^a_{b[a,c]} + 2\Gamma^s_{b[a}\Gamma^a_{c]s}) = (u^i(p), R_{bc}(u^i))$. Notice that the necessary derivatives have already been taken in forming the bundle $J^2(L)$, and that $\widetilde{\mathrm{Ric}}$ is a purely algebraic map which is obviously con-

[4] If g is a local section of L, and Γ^a_{bc} are the Christoffel symbols of g in some chart, then the curvature tensor of g has components $R^a{}_{bcd} = 2\Gamma^a_{b[d,c]} + 2\Gamma^s_{b[d}\Gamma^a_{c]s}$. The Ricci tensor is given by $R_{bc} = R^a{}_{bca}$, and the scalar curvature by $R = g^{bc}R_{bc}$.

tinuous. By 2.2, the map $g \mapsto \mathrm{Ric}(g)$ sending a metric to its Ricci tensor is continuous from $\Gamma^2(L)$ to $\Gamma^0(S)$. Similarly, the map $g \mapsto Rg$ is continuous. Using the fact that $\Gamma^0(S)$ is a $C^0(M)$ module, we see that the map

$$T: g \mapsto T(g) = -\{\mathrm{Ric}(g) - \tfrac{1}{2} Rg\}$$

from $\Gamma^2(L)$ to $\Gamma^0(S)$ is continuous.

The set $\mathfrak{A} = T(\Gamma^2(L))$ is *the complete set of continuous energy-momentum tensors on M for which the Einstein field equations have solutions.* Since the determination of the detailed structure of $\mathfrak{A}$ (under what conditions can we solve the field equations on some manifold M?) is one of the major problems of general relativity, it is surprising that virtually nothing is known about it, even in the case $M = R^4$. Obvious possibilities are:

(a) $\mathfrak{A}$ is open: this would be very powerful. For any exact solution g to $\mathrm{Ric}(g) - \frac{1}{2} Rg = -T_0$, exact solutions would exist for any T' sufficiently C^0 close to T_0.

(b) $\mathrm{int}(\mathfrak{A}) \neq \emptyset$: there would then exist an open set $W \subset \Gamma^0(S)$ in which $\mathfrak{A}$ is dense. For any $s \in W$, one could obtain approximate solutions to $\mathrm{Ric}(g) - \frac{1}{2} Rg = -s$ to any desired degree of accuracy.

(c) $\mathfrak{A}$ is nowhere dense: then it is generic for the field equations to be unsolvable.

Nothing is known concerning the truth or falsity of any of these. Less interesting is the fairly obvious

Proposition (3.1). *$\mathfrak{A}$ contains no isolated points.*

Proof. For any $T_0 = T(g_0)$, we must find a net $\{T_i\} \subset \mathfrak{A} \setminus \{T_0\}$ with $T_i \to T_0$. Let $\{s_i\}$ be any net of functions converging to 0 in $C^2(M)$. Putting $f_i = e^{2s_i}$, we have $T(f_i g_0) \to T_0$. We need only verify that the net $\{s_i\}$ may be chosen so that $T(f_i g_0) \neq T_0$ for all i. Define, for any C^2 function s, the functions $\Delta_1 s = g_0^{ab} s_{,a} s_{,b}$ and $\Delta_2 s = g_0^{ab} s_{;ab}$, where the semi-colon denotes covariant differentiation with respect to g_0. It is easily checked that a necessary condition for $T(e^{2s} g_0) = T(g_0)$ is that $\Delta_1 s + \Delta_2 s$ be identically zero. The map $D: s \mapsto \Delta_1 s + \Delta_2 s$ from $C^2(M)$ to $C^0(M)$ is continuous, so that $D^{-1}(0)$ is closed in $C^2(M)$. Clearly $\mathrm{int}(D^{-1}(0)) = \emptyset$, so that $C^2(M) \setminus D^{-1}(0)$ is open dense. Thus any net in $C^2(M) \setminus D^{-1}(0)$ converging to 0 will suffice.

Thus if $T_0 \in \Gamma^0(S)$ has a solution $T(g_0) = T_0$, any neighborhood of T_0 contains a $T_1 \neq T_0$ for which there is also a solution $T(g_1) = T_1$. The result is weak because it is "clear" that any neighborhood of T_0 contains many more exact solutions than those corresponding simply to metrics conformally related to g_0. However, the author is unaware of any stronger results in this direction. Many of the approximation methods of general relativity are specifically designed to deal with metrics that are in some sense close to a fixed metric g_0. Thus they are not, as such, applicable to the whole of $\mathfrak{A}$. From the point of view adopted here, these methods simply provide alternative proofs of the fact that $T(g_0)$ is not isolated. We forego the details, but the reason for this, roughly, is that certain integrability conditions must be imposed to insure the existence of the approximate solutions; and the set of "admissible functions" satisfying the integrability conditions is nowhere dense in the set of all admissible functions.

IV. Stability of Global Properties of Spacetimes

Within the present limits of astronomical observations [5, 6], there is apparently no reason to believe that the large-scale behavior of the visible universe is *not* described, to a fair degree of accuracy, by one of the following Robertson-Walker metrics

$$g(t, r, \theta, \varphi) = dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right\}.$$

It will be assumed that the elementary properties of these metrics are known. Particular $R-W$ metrics are obtained by assuming the energy-momentum tensor to be that of a perfect fluid – $T(g) = (p+\varrho)\,dt^2 - pg$ (ϱ and p are density and pressure, respectively) – and solving the field equations for $R(t)$. The $R-W$ metrics have the following properties:

(a) Each $t=$ constant surface Σ_t is a spacelike hypersurface. Each Σ_t is a Cauchy surface [3, 7] – any maximally extended timelike curve intersects Σ_t precisely once.

(b) For sufficiently small t, the future-directed orthogonal trajectories to Σ_t are diverging on Σ_t.

(c) Scaling $R(t)$ appropriately, there is an infinite curvature singularity at $t=0$ (big bang). The space is metrically inextendible (cannot be isometrically embedded as a proper subset of a larger spacetime).

(d) If $T(g)$ is assumed to satisfy $\varrho + p > 0$, $\varrho + 3p > 0$ (corresponding to the existence, at each event, of some non-zero rest mass – see Tolman [8], p. 379), then the *strong energy condition* [7, 9, 10] is satisfied:

$$g_{ab} V^a V^b \geqq 0 \Rightarrow R_{ab} V^a V^b < 0, \ (V^a \neq 0).$$

We conjecture that the "real" metric g' is close to g in some sense and ask whether or not g' also has the above properties. That is, we ask whether or not these properties are *stable* under small *perturbations* of g. Before proceeding rigorously, a few remarks about perturbations in this context:

(a) It does not make sense to perturb the metric subject to constraints; in the case of the $R-W$ metrics, we are not interested in nearby homogeneous, isotropic, perfect fluid solutions. Indeed, the whole purpose of perturbing the metric is to obtain a more realistic one with no symmetries.

(b) By the same token, it is inappropriate to consider perturbations as part of the Cauchy problem (perturbing initial data on a spacelike hypersurface and constructing a new spacetime via the time-evolution of this new 3-surface). In the first place, the Cauchy problem involves constraints; unique solutions are obtained only when the form of the four-dimensional energy-momentum tensor is specified in advance. Secondly, the solution to the Cauchy problem does not normally yield the whole spacetime (a metrically inextendible one) but only the maximal spacetime with respect to which the given initial data surface is a Cauchy surface (see Choquet-Bruhat and Geroch [11]).

(c) The point of view to be adopted here is that the manifold M is given and fixed. Starting with $g \in C^k(L)$, a perturbation of g will be given by $g' = g + h \in C^k(L)$, where h is a twice-covariant symmetric tensor appropriately close to zero.

Definition (4.1). A property $\mathscr{P}$ of Lorentz metrics is said to be *stable* in $\Gamma^k(L)$ if it holds on an open subset of $\Gamma^k(L)$. If $K \subsetneqq \Gamma^k(L)$ and $\mathscr{P}$ holds on an open dense subset of K, $\mathscr{P}$ is said to be *generic* in K. A C^k *perturbation* of $g \in C^k(L)$ is an open neighborhood of g in $\Gamma^k(L)$. Similarly for $\Gamma^k(C)$.

Proposition (4.2). *Suppose there is a diffeomorphism $\varphi: S \times R \to M$ such that for each $t \in R$, $\varphi(S, \{t\})$ is spacelike with respect to $\bar{g} \in C^0(C)$. Then this family of surfaces is also spacelike with respect to every conformal structure in some interval containing $\bar{g}$.*

Proof. Although the assertion is intuitively obvious, it will be useful to see how easily it may be proven. Choose a locally finite open cover $\{U_i : i \in I\}$ of M by sets with compact closure, and let $\{f_i : i \in I\}$ be a partition of unity subordinate to the cover. Choose any Lorentz metric $g \in \tau^{-1}(\bar{g})$, and let $(X, -X)$ be a line-element field on M timelike with respect to g. Recall that $(X, -X)$ is a vector field determined up to a factor of ± 1 at each point of M, and that such things always exist in any spacetime [12]. Using g to lower indices, we obtain a one-form field $(q, -q)$ up to sign; then $w = q \otimes q$ is a well-defined 2-covariant symmetric tensor.

Now choose $m \in \overline{U_i}$; m lies in some surface $\Sigma_t = \varphi(S, \{t\})$. There exists a real number $\lambda_m > 0$ such that (a) $g(m) \pm \lambda_m w(m)$ is Lorentzian, (b) $\overline{g(m) - \lambda_m w(m)} < \bar{g}(m) < \overline{g(m) + \lambda_m w(m)}$ [to see this, use a g-normal coordinate system at m, with time-axis tangent to $(X, -X)(m)$], and (c) both of $g(m) \pm \lambda_m w(m)$ measure Σ_t as spacelike at m. Since g and w are continuous and φ is smooth, a neighborhood $V(m)$ exists such that for any $m' \in V(m)$ the forms $g(m') \pm \lambda_m w(m')$ satisfy (a) and (b) and also measure the appropriate $\Sigma_{t'}$ containing m' as spacelike at m'. We obtain in this way an open cover of the compact set $\overline{U_i}$, and extract a finite subcover to which corresponds a finite collection of positive numbers $\{\lambda_1, \ldots, \lambda_k\}$. Put $\lambda_i = \min\{\lambda_1, \ldots, \lambda_k\}$, and define the positive continuous function $\lambda = \sum_{i \in I} \lambda_i f_i$. The interval $(\overline{g - \lambda w}, \overline{g + \lambda w})$ is the one we want.

More difficult, but in the same setting as the above, is the result of Geroch [3] that the existence of a Cauchy hypersurface is stable in $\Gamma^0(C)$. Precisely, if S is a Cauchy surface for $\bar{g}$, it is one for every $\bar{k}$ in some interval about $\bar{g}$.

Notice that if, in the above proposition, $\bar{g}$ had been C^k, g could have been chosen in $C^k(L)$. Since the partition of unity functions may be taken C^∞ (see Hicks, [13], p. 84ff.), the same construction yields two C^k Lorentz metrics $g \pm \lambda w$, and two corresponding C^k conformal structures. It follows from the definition of the Whitney topologies that the subset of $C^k(C)$ contained in $(\overline{g - \lambda w}, \overline{g + \lambda w})$ is open in $\Gamma^k(C)$; also, τ^{-1} of such a set is open in $\Gamma^k(L)$.

A metric $g \in \Gamma^0(L)$ is said to be *stably causal* (Hawking [14]) if there is a W_0 neighborhood of g no element of which admits closed timelike curves. Since the existence of such curves is a property of $\bar{g}$, and since the map τ is open, stable causality is well-defined in $\Gamma^0(C)$ and is stable there. It is also stable in $\Gamma^k(C)$ for any k. A conformal structure is said to be *acausal* if it admits closed timelike curves, *causal* if it does not. It is trivial (use the same methods as in 4.2) that the acausal conformal structures are open in $\Gamma^0(C)$. Letting A be the acausal, $\Gamma^0(C) - A$ the causal, and SC the stably causal conformal structures respectively, we observe that

(a) $SC \neq \emptyset$ on any non-compact M: on such an M, there always exists a function f with no critical points ($df(x) \neq 0$, all x). If μ is an arbitrary positive-definite metric on M and X is the (nowhere-vanishing) gradient of f with respect to μ, the Lorentz metric g defined by

$$g(Y, Z) = -\mu(Y, Z) + 2\frac{\mu(X, Y)\,\mu(X, Z)}{\mu(X, X)}$$

possesses a cosmic time function (f) and is therefore stably causal ([14]).

(b) $\Gamma^0(C) - A = \overline{SC}$: because if $\bar{h} \in \Gamma^0(C) - A$, then the open set $\{\bar{g} : \bar{g} < \bar{h}\}$ lies entirely in SC. Any interval containing $\bar{h}$ intersects this set. Among other things, this insures that there are no isolated points in $\Gamma^0(C) - A$.

Thus it is generic for a causal conformal structure to be stably causal.

Proposition (4.3). *The strong energy condition is stable in $\Gamma^2(L)$.*

Proof. We recall that the map $g \mapsto \mathrm{Ric}(g)$ is continuous. Denote by $\widehat{TM}$ the tangent bundle of M minus the zero section. If $g \in \Gamma^2(L)$ satisfies the strong energy condition, then for all $X \in \widehat{TM}$, $g(X, X) \geqq 0 \Rightarrow \mathrm{Ric}(g) \cdot (X, X) < 0$. Let $F_g \subset \widehat{TM}$ be the set of all vectors timelike or null with respect to g. For any $s \in \Gamma^0(S)$, put $N_s = \{X \in \widehat{TM} : s(X, X) < 0\}$; F_g and $N_{\mathrm{Ric}(g)}$ are closed and open respectively. By the normality of TM, there is an open set U such that $F_g \subset U \subset \bar{U} \subset N_{\mathrm{Ric}(g)}$. Moreover, since $\mathrm{Ric}(g)$ is a quadratic form, U may be chosen to satisfy $X \in U \Rightarrow aX \in U$, for any non-zero real number a. Now for each $p \in M$, there exists an open neighborhood of $\mathrm{Ric}(g)(p)$ in the fiber S_p such that if $s(p)$ lies in this neighborhood, $N_s \cap S_p \supset \bar{U} \cap S_p$. From the local triviality of S and elementary continuity considerations, it follows that there exists an open neighborhood $W(\mathrm{Ric}(g)) \subset \Gamma^0(S)$ such that s in W implies $N_s \supset \bar{U}$. Then $\mathrm{Ric}^{-1}(W)$ is an open neighborhood of g in $\Gamma^2(L)$. From considerations similar to those in 4.2, we conclude the existence of a neighborhood $V(\bar{g}) \subset \Gamma^2(C)$ such that $\bar{h} \in V \Rightarrow F_h \subset U$ for any $h \in \tau^{-1}(\bar{h})$. Then by construction, the strong energy condition holds on $\{\mathrm{Ric}^{-1}(W) \cap \tau^{-1}(V)\}$, a W_2 neighborhood of g.

Remark. The straightforward nature of the above proof should give some indication of the conceptual simplification achieved via this topological approach. One begins to see that the W_k topologies provide a mathematical framework within which many things which appear intuitively obvious become rigorously true.

It should be pointed out that 4.3 is *not* true in any of the weaker topologies frequently used on $C^2(L)$. In particular, it is not true either in the coarse C^2 topology [15, 16] or in any of the topologies of global uniform convergence [17]. If we agree that any reasonable topology on $C^2(L)$ should allow perturbations preserving the existence of non-zero rest mass, we may take this as further evidence in favor of the Whitney topologies.

(4.4) Let us denote by $\mathscr{S}\mathscr{E}$ the set of C^2 metrics on M satisfying the strong energy condition. Suppose $\{g_i : i \in I\}$ is a net in $\mathscr{S}\mathscr{E}$ converging to $g \in \Gamma^2(L)$, and let $X \in TM$ be timelike with respect to g. Since $\bar{g}_i \to \bar{g}$ in $\Gamma^0(C)$, there is an $i_0 \in I$ such that $i > i_0 \Rightarrow g_i(X, X) > 0$, and this in turn implies $\mathrm{Ric}(g_i)(X, X) < 0$. Since the map Ric is continuous, it follows that $\mathrm{Ric}(g)(X, X) \leqq 0$. Since $\mathrm{Ric}(g)$ is a continuous section, the inequality holds for X null as well. If we denote by $\mathscr{E}$ the subset of $\Gamma^2(L)$ satisfying

$$g(X, X) \geqq 0 \Rightarrow \mathrm{Ric}(g)(X, X) \leqq 0,$$

we have shown that $\overline{\mathscr{S}\mathscr{E}} \subseteqq \mathscr{E}$. The same sort of argument shows that $\mathscr{E}$ is a closed set. An element of $\mathscr{E}$ is said to satisfy the *energy condition* [7,9,10]. This condition is somewhat less stringent than the strong energy condition, and $\mathscr{E}$ contains many of the exact solutions which are not in $\mathscr{S}\mathscr{E}$, such as the vacuum and pure radiation fields.

Under the assumption that the behavior of matter and energy elsewhere is not grossly different from its observed behavior in our vicinity, it is generally agreed that any physically realistic metric should lie in $\mathscr{E}$. The converse, that any $g \in \mathscr{E}$ represents a physically realistic spacetime, is false because the source-free metrics, while extremely useful locally, cannot be expected to describe the global properties of spacetime.

What portion of E then, should we eliminate from consideration? As a first approximation, it is usual to think of the matter in the universe as being smeared out and rather smoothly distributed throughout space. Thus if an exact solution is found in this approximation, it will lie in $\mathscr{S}\mathscr{E}$. One then conceives of the actual distribution of matter and radiation as arising from such a situation via a suitable limiting process involving the gradual formation of lumps and a corresponding attenuation of the intervening matter. That is, the realistic metrics should in $\overline{\mathscr{S}\mathscr{E}}$. The existence of isotropic background radiation at 3 °K appears to support this – the "real" Ricci tensor would satisfy the strong energy condition inside matter, while in "empty space" it would satisfy $\mathrm{Ric}(X,X)<0$ for any timelike X. The following proposition shows that by neglecting metrics in $\mathscr{E}-\overline{\mathscr{S}\mathscr{E}}$, we are not eliminating too many.

Proposition (4.5). $\mathrm{int}(\mathscr{E})=\mathscr{S}\mathscr{E}$. *In particular,* $\mathscr{E}-\overline{\mathscr{S}\mathscr{E}}$ *is nowhere dense.*

Proof. Let $g \in \mathscr{E}-\mathscr{S}\mathscr{E}$. Then there exists $X \in \widehat{TM}$ such that $g(X,X) \geqq 0$ and $\mathrm{Ric}(g)(X,X)=0$. We shall exhibit a sequence of metrics converging to g each one of which violates the energy condition [implying that $g \notin \mathrm{int}(\mathscr{E})$].

Assume first that $X \in T_p$ is timelike with respect to g. Choose normal coordinates (t,x,y,z) in a neighborhood $U(p)$ with $p=(0,0,0,0)$. Choose compact 4-balls N_1 and N_2, neighborhoods of p, such that $N_1 \subset \mathrm{int}(N_2) \subset N_2 \subset U$, and let φ be a C^∞ bump function such that $0 \leqq \varphi \leqq 1$, $\varphi(N_1)=1$, $\varphi(M-\mathrm{int}(N_2))=0$. Define η in U by $\eta(t,x,y,z)=e^t$. Then at p, using the fact that $\Gamma^a_{bc}=0$, $\eta_{;11}=\eta_{;1}\eta_{;1}=1$, and $\eta_{;ab}=\eta_{;a}\eta_{;b}=0$ unless $a=b=1$, where the semi-colon denotes covariant differentiation with respect to g. The function $\varphi\eta$ is well-defined, C^∞, and equal to zero outside of N_2.

Consider the sequence $\{\sigma_n=(1/n)\varphi\eta\}$. It is immediate from 2.1 that $\sigma_n \to 0$ in $C^k(M)$ for any k, and thus that $e^{2\sigma_n} \to 1$ in $C^2_+(M)$. Putting $g_n=e^{2\sigma_n}$, we have $g_n \to g$ in $\Gamma^2(L)$. An easy calculation (see Eisenhart [18], p. 90) shows that at p, we have $\mathrm{Ric}(g_n)(X,X)=\dfrac{3}{n}g(X,X)>0$, which implies $g_n \notin \mathscr{E}$, all n.

For X null with respect to g, choose normal coordinates such that X lies in the (t,x)-plane in T_p. We may rescale X so that $X^a=(1,1,0,0)$

in these coordinates (without affecting the fact that $\mathrm{Ric}(g)(X,X)=0$). Repeat the above argument, taking $\eta(t,x,y,z)=-(x-t)^2$.

Remark. It may be that we actually *have* $\mathscr{E}=\overline{\mathscr{SE}}$, but there does not seem to be any obvious way of showing this. At any rate, we may be reasonably certain that we are not overlooking any solutions of serious cosmological significance by restricting our attention to $\overline{\mathscr{SE}}$.

(4.6) In conjunction with the singularity theorems to be discussed in the next section, we mention the following additional condition. A metric with curvature tensor R_{abcd} is said to satisfy the *generality condition* [10] if every causal (i.e. timelike or null) geodesic $\gamma(s)$ with $d\gamma/ds=k^a\dfrac{\partial}{\partial x^a}$ contains some point at which

$$k_{[a}R_{b]cd[e}k_{f]}k^c k^d \neq 0\,.$$

Denote the corresponding subset of $\Gamma^2(L)$ by $\mathscr{G}$. Hawking and Penrose have proven [10] that if $g\in\mathscr{G}\cap\mathscr{E}$ and in addition, every causal geodesic in (M,g) is complete, then each such geodesic contains a pair of points conjugate to one another. This provides one of the key tools needed to prove the singularity theorems. It is easily seen that $R_{ab}k^a k^b\neq 0$ implies $k_{[a}R_{b]cd[e}k_{f]}k^c k^d\neq 0$, so that, trivially, $\mathscr{SE}\subset\mathscr{G}$, and the generality condition is generic in $\overline{\mathscr{SE}}$.

The next proposition will be useful in showing that singularities in certain cosmologies are stable.

Proposition (4.7). *Let* $g\in\Gamma^1(L)$ *satisfy*

(a) (M,g) *is time-oriented and contains a spacelike hypersurface* Σ.

(b) $\theta>0$ *on* Σ, *where* θ *is the divergence of the (local) vector field of future-directed orthogonal geodesic trajectories to* Σ.

Then (a) *and* (b) *hold on an open neighborhood of* g *in* $\Gamma^1(L)$.

Proof. The first condition is trivial. Given a net $g_\alpha\to g$ in $\Gamma^1(L)$, the proposition follows if we can show $\theta_\alpha\to\theta$, where $\theta_\alpha,\theta\in C^0(\Sigma)$.

We let $\pi: A\to M$ be the bundle of affine connections on M, and note that the map $g\to\nabla_g$ sending g to its Riemannian connection is continuous from $\Gamma^1(L)$ to $\Gamma^0(A)$, so that $\nabla_{g_\alpha}\to\nabla_g$. Since $g_\alpha\to g$, we have $\hat{g}_\alpha\to\hat{g}$, where $\hat{g}_\alpha$ is the induced metric (first fundamental form) of the hypersurface $\Sigma\subset(M,g_\alpha)$. By (*a*), we may assume thet Σ is spacelike with respect to each g_α. Let $\underset{\alpha}{V}$ be the unit tangent vector field to the g_α hypersurface-orthogonal geodesics to Σ; for each α, $\underset{\alpha}{V}$ is defined in a neighborhood of Σ. On Σ, we clearly have $\underset{\alpha}{V}\to V$ in the W_0 topology. Introduce coordinates (x^i) in a neighborhood of a portion of Σ, so that locally Σ is given by $x^1=0$ [5]. Using the coordinates (x^a) on Σ, we have $\underset{\alpha}{\hat{g}}=\underset{\alpha}{g_{ab}}\,dx^a\,dx^b$.

[5] In this section only, the indices $a,b,c,\ldots$ go from two to four, while $i,j,k,\ldots$ go from one to four.

If $\underset{\alpha}{\Omega}$ denotes the second fundamental form of $\Sigma \subset (M, g_\alpha)$, we have, in these coordinates

$$\underset{\alpha}{\Omega}_{ab} = -\underset{\alpha}{V}_{a;b}\Big|_{\Sigma} = \underset{\alpha}{\Gamma}^{i}_{ab} V_i \tag{4.8}$$

and

$$\theta_\alpha = \underset{\alpha}{V}^{i}{}_{;i} = \underset{\alpha}{V}_{a;b}\hat{g}^{ab}_{\alpha} = -\operatorname{tr}\left(\underset{\alpha}{\Omega}\right) \tag{4.9}$$

where $\underset{\alpha}{\Gamma}^{i}_{ab}$ are Christoffel symbols for g_α, and where $\underset{\alpha}{\hat{g}}^{ab}$ are the contravariant components of $\hat{g}_\alpha$. So θ_α depends only upon the surface components of $\underset{\alpha}{V}_{i;j}$. Doing the usual things with a covering of Σ by submanifold charts, $\theta_\alpha \to \theta$ follows immediately from (4.8) and (4.9).

Definition (4.10). Let (M, g) be time-oriented and $S \subset M$ be a closed spacelike two-surface. If both families of future-directed null normals to S have strictly negative divergence on S, then S is said to be a *trapped surface* (Penrose [19]). Trapped surfaces may be expected to arise whenever an approximately spherical object (such as a star) undergoes irreversible gravitational collapse. We refer to Penrose [7] for a detailed explanation of this.

Proposition (4.11). *If $S \subset (M, g)$ is a trapped surface, there is a neighborhood of g in $\Gamma^1(L)$ for which S is trapped.*

Proof. Similar to (4.7).

Lemma (4.12). *For any C^∞ manifold M, with tangent bundle $\pi: T \to M$, the set of complete vectorfields (those whose flow exists for all time) is open in $\Gamma^1(T)$.*

Proof. See Appendix I.

Corollary. *Geodesic completeness is stable in $\Gamma^2(L)$.*

Proof. Any C^2 metric determines a C^1 vector field $\mathrm{Sp}(g)$, the spray of g, on TM. (See, for example, Lang [20], pp. 164ff.) In local coordinates, $\mathrm{Sp}(g)$ is given by $(u^i, V^j; V^j, -\Gamma^r_{st} V^s V^t)$. The spacetime (M, g) is geodesically complete iff. the vector field $\mathrm{Sp}(g)$ is complete. Since it is clear that the map from $\Gamma^2(L)$ to $\Gamma^1(T^2 M)$ sending g to $\mathrm{Sp}(g)$ is continuous, the assertion follows from the lemma.

Lemma (4.13). *Let $f: S \times (0, \infty) \mapsto M$ be a diffeomorphism, where S is some 3-manifold, and let $V = f_*(\partial/\partial t)$. Then there is a neighborhood $W(V) \subset \Gamma^0(T)$ such that for any $V' \in W$.*

(a) *each integral curve of V' is transverse to all the surfaces $f(S, \{t\})$.*

(b) *each such curve $\gamma(s)$ is incomplete in the direction of decreasing s.*

Proof. It is almost obvious that this is true. The proof may be found in Appendix II.

V. *G*-Singularities

We first define a subset $\mathscr{P} \subset \Gamma^2(L)$ containing the physically realistic cosmologies on M. Together with the most recent theorem of Hawking and Penrose [10], the machinery we have developed allows an easy proof of the fact that G-singularities are generic (a) in the spatially closed cosmologies in $\mathscr{P}$, and (b) in the subset of $\mathscr{P}$ in which local gravitational collapse occurs.

Definition (5.1) *Let* $\mathscr{Q}$ be the set of C^2 Lorentz metrics g such that $\bar{g}$ is stably causal. Define $\mathscr{P} = \overline{\mathscr{Q} \cap \mathscr{S}\mathscr{E}} \subset \Gamma^2(L)$. Then $g \in \mathscr{P}$ iff. there exists a net $g_i \to g$ such that for each i, $\bar{g}_i$ is stably causal and g_i satisfies the strong energy condition.

The assertion that $\mathscr{P}$ contains all the physically realistic cosmologies in $\Gamma^2(L)$ is clearly predicated on the unverifiable assumption that at no event in spacetime is the local behavior of matter and radiation drastically different from that observed in the vicinity of the solar system. At the moment, there is no reason to believe that this is not the case.

An element g of $\mathscr{P}$ is said to be *G-singular* [16] if (M, g) contains incomplete causal geodesics. The existence of such a singularity, coupled with metric inextendibility, is often taken as evidence that something is seriously wrong with the spacetime under consideration; for it may be interpreted as a statement that test particles or photons spontaneously appear or disappear. The proofs of most of the singularity theorems proceed by showing that certain combinations of physically plausible restrictions on the metric are incompatible with causal geodesic completeness. We state the most recent of these:

Theorem (5.2). (*Hawking and Penrose* [10])[6]. *Let* (M, g) *satisfy*

(a) $g \in \mathscr{E} \cap \mathscr{G}$ *– g satisfies the energy and generality conditions*

(b) (M, g) *has no closed timelike curves*

(c) (M, g) *contains either*

(1) *a trapped surface*

or (2) *a point p for which the convergence of all the null geodesics through p changes sign somewhere to the past of p*

or (3) *a closed spacelike hypersurface*

Then (M, g) *is G-singular.*

In view of this and 4.7–4.11, we have the following

Theorem (5.3). *Let* $\Lambda \subset \mathscr{P}$ *denote the set of metrics with trapped surfaces. Then G-singularities are generic in* Λ. *Let* $\Delta \subset \mathscr{P}$ *denote the set of metrics admitting closed spacelike hypersurfaces. Then G-singularities are generic in* Δ.

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[6] This is actually a corollary to their main theorem.

Proof. Let $g \in \Lambda$. If $g \in \operatorname{int}(\mathscr{P})$, then by the above theorem (since $g \in \mathscr{E} \cap \mathscr{G}$), g is G-singular. If $g \in \partial\mathscr{P}$, then by 4.11 there exists a net $\{g_i\} \subset \operatorname{int}(\mathscr{P} \cap \Lambda)$ with $g_i \to g$. Since each g_i is G-singular, we are finished. Similarly for the second part.

While the theorem as stated applies to a fixed M, it is invariant under extension in the following sense. If $g \in \Lambda \cup \Delta$, and there is an isometric embedding of (M, g) into $(\tilde{M}, \tilde{g})$, where $\tilde{g} \in \tilde{\mathscr{P}}$, then if g is G-singular, $\tilde{g}$ is G-singular as well. This means that the G-singularities predicted by 5.2 are *not* due to the arbitrary removal of some regular points from a non-singular spacetime. We thus have the result that *the generic spatially closed universe is G-singular and remains so under metric extension.* If we regard the existence of a trapped surface as equivalent to the existence of a gravitationally collapsing object, we have a similar statement concerning gravitational collapse. We are assuming, of course, that everything lies in the appropriate $\mathscr{P}$, but as we have indicated, this is quite reasonable.

VI. Stability of the Big Bang

The greatest drawback to Theorem 5.2 and the other singularity theorems is that they provide no insight into the *nature* of the singularity. In fact, when the theorems are stated in their full generality, there are no logical grounds for concluding that G-singularities must occur at all. The actual proofs, as we mentioned earlier, involve showing that a number of conditions, including causal completeness, are mutually inconsistent. Although it is argued that causal completeness is the condition most likely to fail, it is certainly possible for the spacetime to violate the appropriate energy or causality condition instead. (As has been observed [7], such a spacetime would clearly still be regarded as singular in some sense by most physicists.) In particular, as Hawking and Penrose point out [10], "*one cannot conclude [on the basis of the theorem(s)] that such a singularity need necessarily be of the 'infinite curvature' type*".

However, in certain specific cases of interest, the singularity is *known* to result from infinite curvature, and one can prove that the infinite curvature singularity *itself* is stable. We give one example.

Theorem (6.1). *The big bang of the Robertson-Walker models is stable in $\Gamma^2(L)$. That is, any sufficiently small C^2 perturbation of such a metric results in a spacetime with the following property: each flowline of the matter has encountered infinite energy density in the finite past.*

Proof. The energy-momentum tensor, with respect to the coordinates given in § IV, is $T(g) = (\varrho + p)\, dt^2 - pg$. It is immediate either from the field equations or from symmetry arguments that ϱ is a function of t alone. The big bang assumption is that, with an appropriate scaling of $R(t)$ (i.e., $R(0)=0$), we have $\lim_{t\to 0^+} \varrho(t) = +\infty$. Let g be such an R-W metric and M the manifold it determines. Recall that for any metric h_{ab} with non-degenerate energy-momentum tensor T_{ab}, there is, at each event x, a unique number $\lambda(x)$, and a unique future-pointing timelike unit vector $V(x)$ satisfying

$$[T_{ab}(x) - \lambda(x)\, h_{ab}(x)]\; V^a(x) = 0\,.$$

The integral curves of V are parametrized by h-proper time and are (by definition) the flowlines of the matter-energy distribution described by T_{ab}. An observer travelling on such a curve measures, at each event x, an energy density $\lambda(x)$. In the case at hand, $\lambda=\varrho$ and $V=\partial/\partial t$. Thus to prove the theorem, it is necessary to find a neighborhood $U(g)\subset \Gamma^2(L)$ such that for $g' \in U$, the associated λ and V are close to ϱ and $\partial/\partial t$ respectively. We proceed in two steps:

(1) Regard g as an element of $\Gamma^0(S)$. Fix a point $x\in M$, and let T_{ab}, g_{ab} be the components, at x, of $T(g)$ and g in any coordinate system. For small variations, the unique positive eigenvalue satisfying the equation $\det(T_{ab}-\lambda g_{ab})=0$ is (at x) a continuous function of the independent variables T_{ab} and g_{ab}. As we have seen, this pointwise behavior translates into a global statement: Choose and fix $\varepsilon>0$. Then there exist W_0 neighborhoods $A(T(g))$ and $B(g)$ in $\Gamma^0(S)$ such that for any $N\in A$, $h\in B$, $x\in M$, the unique positive λ determined by the equation $\det(N_{ab}(x)-\lambda h_{ab}(x))=0$ satisfies $|\lambda-\varrho(x)|<\varepsilon$. There is no necessary relation as yet between N and h; but $T^{-1}(A)$ is an open neighborhood of g in $\Gamma^2(L)$. Set $B'=B\cap\Gamma^2(L)$, and put $U_1(g)=B'\cap T^{-1}(A)$. Then $g'\in U_1 \Rightarrow |\lambda(x)-\varrho(x)|<\varepsilon$ for all $x\in M$, where $\lambda(x)$ is the positive eigenvalue for $T(g')(x)$ and $g'(x)$. Thus on each surface Σ_t (the $t=$ constant surfaces of our original metric) we have

$$\min\{\lambda(x): x\in\Sigma_t\} \geqq \varrho(t)-\varepsilon\,,$$

so that λ blows up on any curve forced to cross all the surfaces Σ_t.

(2) Put $V=\partial/\partial t$. By 4.13, there exists a W_0 neighborhood $W(V)\subset\Gamma^0(TM)$ such that $V'\in W$ implies that the integral curves of V' are transverse to all the surfaces Σ_t and are incomplete in the direction of decreasing t. Notice that the map β which sends g to V is well-defined and continuous on some neighborhood $D(g)\subset\Gamma^2(L)$. Put $U_2(g) = D\cap\beta^{-1}(W)$, and finally, set $U(g)=U_1\cap U_2$. *Then for any $g'\in U$, any flowline of the matter-energy distribution determined by $T(g')$ has a finite past history* (because $g'\in U_2$) *at the beginning of which the energy density was infinite* (because $g'\in U_1$).

Remarks. We have shown that the global infinite density singularity of these big band models persists under small C^2 perturbations of the metric tensor. The time-reverse version of 6.2, concerning the stability of the eventual collapse (for $k>0$) or indefinite expansion (for $k \leqq 0$) of the R-W models, is proven in an entirely similar way. It should be clear that the same techniques allow one to prove similar theorems concerning the stability of infinite curvature singularities in other classes of cosmological models as well.

With reference to the R-W models, a related question is whether or not, under sufficiently small C^2 perturbations

(a) *every* past-directed causal geodesic remains incomplete

and

(b) is *compelled* to be incomplete by the presence of infinite curvatures.

This will be treated in a more general context in a subsequent paper.

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Appendices

Preliminaries. For this section, it will be useful to have an alternative definition of the Whitney topologies on sections of TM (vector fields). Let μ be an arbitrary positive definite metric on M; μ induces a norm in the fiber of each tensor bundle over M in the standard way. For example, if $Y^{ab}_{cd}(p)$ is a tensor at p in some coordinate system, then $\|Y\|(p)=(Y^{ab}_{cd}\,Y^{ef}_{gh}\,\mu_{ae}\mu_{bf}\,\mu^{cg}\mu^{dh})^{\frac{1}{2}}$. Let ∇ denote the covariant derivative with respect to μ, let $\delta(p)$ be any positive continuous function on M, and let V be a C^r vector field on M. Define

$$W(V,\delta(p))=\{X\in C^r(TM)\colon \|X-V\|(p)<\delta(p),\ \|\nabla X-\nabla V\|(p)<\delta(p),$$
$$\cdots \|\nabla^r_s X-\nabla^r_s V\|(p)<\delta(p)\}\,,$$

where the "s" stands for symmetrization.

The reader may verify for himself (or see Munkres [15]) that $W(V,\delta(p))$ is a W_r neighborhood of V, and that the topology defined by using these sets to construct a neighborhood basis at each $V\in C^r(TM)$ is identical to the W_r topology. In particular, the topology so determined is independent of μ.

I. *The set of complete vector fields is open in* $\Gamma^1(TM)$ [7].

[7] This proof is based, in part, on a suggestion of C. Pugh. The definition of a flow and other important things may be found in Lang [20].

Proof. Write $M = \bigcup_{k=1}^{\infty} C_k$, where C_k is compact and $C_k \subset \operatorname{int}(C_{k+1})$ for each k. Let X be complete and $\psi_X : R \times M \mapsto M$ the flow of X. The set $\psi_X([-1,1] \times C_1)$ is compact, so there exists a smallest k_1 such that $\psi_X([-1,1] \times C_1) \subset \operatorname{int}(C_{k_1})$. We claim there exists $r_1 > 0$ such that if $Z \in C^1(TM)$ lies in the set $N(r_1, C_{k_1}, X) = \{Y \in C^1(TM) : \|Y(p) - X(p)\|, \|\nabla Y(p) - \nabla X(p)\| < r_1, p \in C_{k_1}\}$, then ψ_Z is defined on $[-1,1] \times C_1$, and $\psi_Z([-1,1] \times C_1) \subset \operatorname{int}(C_{k_1})$. For suppose the claim is false. Then for each n there exists $Z_n \in N(1/n, C_{k_1}, X)$ and a point $p_n \in C_1$ such that $\psi_{Z_n}(t_n, p_n) = q_n \in M - \operatorname{int}(C_{k_1})$ for some t_n in $[-1,1]$. By compactness, we may assume $p_n \to p_0 \in C_1$, $t_n \to t_0 \in [-1,1]$. Cover the curve $[-1,1] \ni t \mapsto \psi_X(t, p_0)$ [which lies in $\operatorname{int}(C_{k_1})$] by finitely many coordinate charts. Using Gronwall's inequality (Hurewicz [21], p. 6, or Lang [20], p. 67) it is easy to see that the curves $\psi_{Z_n}(t, p_n)$ converge to $\psi_X(t, p_0)$ uniformly in t in each chart and thus in their union (note that since $\nabla Z_n \to \nabla X$ uniformly on C_{k_1}, the appropriate Lipschitz constants converge as well). In particular, $\psi_{Z_n}(t_n, p_n) \to \psi_X(t_0, p_0) \in \operatorname{int}(C_{k_1})$, which is impossible. So r_1 exists, as asserted.

Set $Q_1 = C_1$, $Q_i = C_i - \operatorname{int}(C_{i-1})$, so that $M = \bigcup_{i=1}^{\infty} Q_i$. Then $C_{k_1} =$ a finite union of the Q_i's. As above, there exists, for each i, a smallest subcollection of $\{Q_j\}$ such that $\psi_X([-1,1] \times Q_i) \subset \operatorname{int}(Q_{i_1} \cup \cdots \cup Q_{i_m})$; and there exists $r_i > 0$ such that $Z \in N(r_i, Q_{i_1} \cup \cdots \cup Q_{i_m}, X)$ implies that $\psi_Z([-1,1] \times Q_i)$ is defined and contained in $\operatorname{int}(Q_{i_1} \cup \cdots \cup Q_{i_m})$. Notice that since $[-1,1] \times Q_i$ is compact, each Q_j can appear only finitely many times in the above enumeration (as a Q_{i_n} for some i). Put $s_j = \min\{r_i : \psi_X([-1,1] \times Q_i) \cap Q_j \neq \emptyset\}$. Finally, choose any strictly positive continuous function $\delta(p)$ such that $\delta(p) < s_j$ on Q_j, and put $W = W(X, \delta(p))$. If $Z \in W$, then Z is complete, since by construction, $\psi_Z(\pm 1, p)$ is defined for all $p \in M$.

II. *Let $f : S \times (0, \infty) \mapsto M$ be a diffeomorphism, where S is some 3-manifold, and let $V = f_*(\partial/\partial t)$. Then there is a neighborhood $W(V) \subset \Gamma^0(TM)$ such that for any $V' \in W$*

(a) *each integral curve of V' is transverse to all the surfaces $f(S, \{t\}) = \Sigma_t$, and*

(b) *each such curve $c(s)$ is incomplete in the direction of decreasing s.*

Proof. Introduce an auxilliary positive definite metric μ as follows. Let g be any positive 3-metric on one of the hypersurfaces Σ_t, and form the 4-metric $dt^2 + g$ on Σ_t. Lie propagate this throughout M via the action of V to obtain μ. For any $x \in S$, the curve $t \mapsto f(x, t)$ is a geodesic with respect to μ; it is parametrized by arc length and orthogonal to all the hypersurfaces Σ_t. For any $Y \in T_p M$, set $\|Y\|(p) = (\mu_{ab} Y^a Y^b)^{\frac{1}{2}}$. Let $\delta(p)$ be a positive continuous function on M, and consider the W_0

neighborhood

$$W(V, \delta(p)) = \{Y \in C^0(TM) : \|Y - V\|(p) < \delta(p)\}.$$

Clearly, if $\delta(p)$ is small enough at each p, then any $Y \in W$ will be transverse to all the surfaces Σ_t. Moreover, if we require that for all p, $\delta(p) < a < 1$ for some fixed a, we have $1 - a < \|Y\|(p) < 1 + a$. Then if $c(s)$ is any integral curve of Y which is complete to the past [i.e. $c(s)$ exists for any $s \in (-\infty, 0)$], we have

$$\int_{s=-\infty}^{s=0} \|c'(s)\| \, ds > \int_{s=-\infty}^{s=0} (1-a) \, ds = +\infty$$

and the μ-arc length is infinite. Similarly, $c(s)$ is past-incomplete iff. its μ-arc length to the past is finite. We will therefore be finished if we can choose $\delta(p)$ such that the latter case holds for each such curve.

Let Y be any vector field transverse to all the hypersurfaces Σ_t, and let Y' be the vector field obtained from Y by reparametrizing the integral curves with t. We have $Y' = hY$ for some positive continuous function h. If $c(s)$ satisfies $c(0) = p \in \Sigma_{t_0}$, $c'(0) = Y(p)$, then $Y'(p) = c'(s(t_0))$ $= \dfrac{ds}{dt}(t_0)\, Y(p) = h(p)\, Y(p).\Big)$

Sublemma. *With notation as above, there exists a W_0 neighborhood $W(V, \delta(p))$ such that $Y \in W$ implies $1 - \varepsilon < h < 1 + \varepsilon$, for fixed $\varepsilon \in (0, 1)$.*

Proof. Fix $p \in \Sigma_{t_0}$. There exists a number $b > 0$ such that if $Z \in T_p M$ satisfies $\|Z - V(p)\| < b$, then any curve $c(s)$ tangent to Z may be reparametrized by t near p, and $\dfrac{ds}{dt}(t_0) \in (1 - \varepsilon, 1 + \varepsilon)$. This is trivial. By the continuity of $\|\cdot\|$ and V, there is a neighborhood $U(p)$ such that if $q \in U \cap \Sigma_{t_1}$ and $\|Z - V\|(q) < b$, then $\dfrac{ds}{dt}(t_1) \in (1 - \varepsilon, 1 + \varepsilon)$ for any curve tangent to $Z(q)$ and reparametrized by t. This is sufficient local information; we now use a partition of unity to construct the required function $\delta(p)$.

We may, in addition, demand that $\delta(p)$ be always less than some fixed number a. Now let $Y \in W(V, \delta(p))$, and let $c(s)$ be any integral curve of Y with $c(0) \in \Sigma_{t_0}$. Suppose $c(s)$ is complete to the past; then its μ-arc length is infinite, as we saw above, and

$$\infty = \int_{s=-\infty}^{s=0} \|Y(c(s))\| \, ds = \int_{t=0}^{t=t_0} \|Y'(c(s(t)))\| \, dt < \int_{t=0}^{t=t_0} (1+a)(1+\varepsilon) \, dt$$

which is impossible.

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