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Elastic Perturbation Theory in General Relativity and a Variation Principle for a Rotating Solid Star

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Abstract. Perturbation analysis is applied to the theory of a General Relativistic perfectly elastic medium as developed by Carter and Quintana (1972). Formulae are derived for the Eulerian variations of the principal fields (density, pressure tensor, etc.) on which the description of such a medium is based, where the perturbations are induced both by infinitesimal displacements of the medium and by infinitesimal variations of the metric tensor. These formulae will be essential for problems such as the study of torsional vibration modes in a neutron star.

As examples of their application, the variation formulae are used in the derivation firstly of a simple (dynamic) action principle for a perfectly elastic medium (this principle being a generalisation of the one given by Taub (1954) for a perfect fluid) and secondly in the derivation of a rather more sophisticated mass variation principle for a stationary rotating solid star (this principle being a generalisation of the one given by Hartle and Sharp (1967) for a perfect fluid star).

1. Introduction

In the general theory of perfectly elastic solids (as compared with the special subcase of a perfectly elastic fluid) the *linearised* theory of small deformations plays a disproportionately important role, since in most solid materials the behaviour will be on the point of ceasing to be perfectly elastic (due to fracture or hysteresis effects) when the deformations are sufficiently large for deviations from linearity to be important.

The primary purpose of the present article is to show how to calculate the linearised perturbations in the fundamental tensor fields used to describe a perfectly elastic medium in the General Relativistic theory recently developed by Carter and Quintana (1972) [1]. In Newtonian elasticity theory it is usual to consider all elastic perturbations as being due to displacements with respect to the (flat) background space. In general relativity theory it is necessary also to take into account the effect of geometric changes due to absolute variations of the space-time metric tensor. Indeed in a fully covariant theory it is possible in principle to consider *all* variations as being due to changes in the geometry of space-time, since continuous displacements can always be transformed away by co-ordinate transformations. However it is not always convenient to make such transformations since in particular applications there will often be reasons (due to the presence of symmetries for example) for preferring to use some special reference system which is not transported with the displacements of the medium. Therefore in the following discussion we shall consider deformations due both to space-time metric variations *and* to relative displacements.

It is to be remarked that it was due essentially to its inability to deal with *absolute* strain variations (as is done here) as compared with *time-rates* of strain variations that the earlier approach to General Relativistic elasticity theory of Bennoun (1965) was not fully successful.

In Section 2 of this paper we shall consider the general properties of perturbations in an elastic medium. This will enable us to give a list of Eulerian variation formulae for the principal fields, on which the description of such a medium is based, and in particular for the energy momentum tensor, in Section 3. Such formulae will be essential in the treatment of problems such as those arising in the theory of vibrations in the crust of a neutron star, where effects both of rigidity and of General Relativistic deviations from Newtonian gravitational theory are important simultaneously.

As a simple example of the we shall apply the perturbation formulae to *stationary* variations of a rotating star. After a preparatory discussion of integral variations in Section 4, we shall go on to apply the formulae of Section 3 in the derivation of two variational principles which are stated in Section 5. The first of these is a straightforward generalisation of the simple action principle given originally by Taub (1954) [3] for a perfect fluid. The second is a generalisation of the mass variation principle for a stationary axisymmetric solid star which was given originally for a rigidly rotating perfect fluid by Hartle and Sharp [4] (1967). (An alternative generalisation of the Hartle-Sharp principle has been made by Bardeen (1970) [5] to cover the case of a differential rotation which can occur in a perfect fluid, although not of course in a solid.) The proof of this variation principle is given in Sections 6 and 7. As in the variation principles of Hartle and Sharp and Bardeen, it turns out to be necessary to invoke the causality requirement that the speed of sound in the solid (cf. Carter, 1972) [6] should not be greater than the speed of light.

In the variational principles given by Taub and by Hartle and Sharp it was necessary to impose the condition of baryon conservation explicitly. The variational principles which will be given here are formally simpler in that baryon conservation is not even mentioned, since it holds automatically in the more elaborate formalism which is necessary to describe a solid as compared with a fluid. Indeed for a solid star there is no straightforward way to discuss the variation of baryon number even if one wished to do so. For rigidly rotating perfect fluid stars with a well defined (single parameter) equation of state the equilibrium states corresponding to a given angular momentum form a family which depends only on one parameter which may be taken to be total baryon number. However when the material of the star is solid there will be an infinitely richer range of possibilities characterized by differing surface topographies and internal stress structures. Whereas in the case of the fluid star it makes sense to ask how the total mass-energy and other quantities vary when the total baryon number is altered, on the other hand such a question can have no well defined answer in so far as a solid star is concerned since the outcome would depend, for example on whether the additional baryons were attached in the form of a mountain at the north or south pole or in the form of an equatorial ridge.

What one can ask however, is how the equilibrium mass-energy etc. of a star with a *given* solid structure (and by implication a given baryon number) will vary when the angular momentum J is changed. For the mass-energy M itself, the variation principle which will be given here leads rigorously to the simple general formula $dM = \Omega\, dJ$ where Ω is the angular velocity, a result which would have been expected from general physical considerations, by the arguments given by Zeldovich and Thorne (cf. the discussion given by Hartle (1970) [7]; see also Zeldovich and Novikov (1971) [8]). This is discussed in the final section.

The notation and terminology used here will be exactly in accordance with those of Carter and Quintana (1972) [1]. The sign conventions in the definition of the Ricci tensor etc. are those of Landau and Lifshitz (1962) [9]. Units are such that the speed of light c and Newtons constant G are equal to unity.

2. Basic Principles and Lagrangian Variations

We shall start by recapitulating the fundamental principles on which the elasticity theory of Carter and Quintana (1972) [1] is based. We consider a 4-dimensional space-time manifold $\mathcal{M}$, with a pseudo-Riemannian metric tensor g_{ab}, and with a projection operator $\mathcal{P}: \mathcal{M} \to \mathcal{X}$ of $\mathcal{M}$ onto a 3-dimensional manifold $\mathcal{X}$ whose points represent idealised particles of the medium. The inverse image $\mathcal{P}^{-1}(X) \subset \mathcal{M}$ of a point $X \in \mathcal{X}$ is interpreted as the world-line of the particle represented by X. We denote the tangent vector field of the world lines by u^a the magnitude of this vector being fixed by the normalisation condition

$$u^a u_a = -1 . \tag{2.1}$$

The projection $\mathscr{P}$ determines a canonical one-one mapping $\mathscr{P}^{-1}$ between the set of *material tensors* (i.e. tensors in $\mathscr{X}$) at any point $x \in \mathscr{X}$ and the corresponding set of *orthogonal space-time tensors* (i.e. tensors in $\mathscr{M}$ all of whose contractions with u^a vanish) at any point $x \in \mathscr{P}^{-1}(X)$. This mapping enables us to define various orthogonal tensor fields on $\mathscr{M}$ (notably the density scalar ϱ, the pressure tensor p^{ab}, and the elasticity tensor E^{abcd}) as *functions of state* (or more precisely of the *strain-state*) in the sense that their images in $\mathscr{X}$ under $\mathscr{P}$ are well defined functions of the image in $\mathscr{X}$ under $\mathscr{P}$ of the projection tensor

$$\gamma_{ab} = g_{ab} + u_a u_b \tag{2.2}$$

(which of course is automatically orthogonal).

We wish here to consider the (linearised) variations of such functions of state due to the effect both of displacements of the world lines in $\mathscr{M}$ due to position-coordinate displacements of the form $x^a \to x^a + \varDelta x^a$ and of alterations $g_{ab} \to g_{ab} + \delta g_{ab}$ of the metric tensor at fixed points in $\mathscr{M}$. Any such variation can be considered from two points of view: we can either consider the *Lagrangian variation* (i.e. the variation of the field in terms of a co-ordinate system which is itself dragged along by the displacement $\varDelta x^a$) and which we shall denote by the symbol $\varDelta$, or we can consider the *Eulerian variation* (i.e. the variation of the field at a fixed point in $\mathscr{M}$) which we shall denote by δ. For any field quantity whatsoever the difference between these two kinds of variation is (by definition) the Lie derivative of the field with respect to the displacement, i.e. we have

$$\varDelta - \delta = \underset{\xi}{\mathscr{L}} \tag{2.3}$$

where $\underset{\xi}{\mathscr{L}}$ is the operation of Lie differentiation with respect to the vector ξ^a defined by

$$\varDelta x^a = \xi^a . \tag{2.4}$$

In particular, if we denote the variation of g_{ab} at a fixed point in $\mathscr{M}$ by

$$\delta g_{ab} = h_{ab} \tag{2.5}$$

then using the standard formula

$$\underset{\xi}{\mathscr{L}}\, g_{ab} = 2\xi_{(a;b)} \tag{2.6}$$

we obtain the Lagrangian variation of g_{ab} in the form

$$\varDelta g_{ab} = h_{ab} + 2\xi_{(a;b)} . \tag{2.7}$$

The partial derivative $\partial(T_a{}^b{}_{\cdots}^{\cdots})/\partial\gamma_{cd}$ of an orthogonal tensor function of strain $T_a{}^b{}_{\cdots}^{\cdots}$ in $\mathscr{M}$ is itself an orthogonal tensor function of strain in $\mathscr{M}$ whose strict definition is given in terms of tensors on $\mathscr{X}$ by

$$\mathscr{P}\left(\frac{\partial(T_a{}^b{}_{\cdots}^{\cdots})}{\partial\gamma_{cd}}\right)=\frac{\partial\mathscr{P}(T_a{}^b{}_{\cdots}^{\cdots})}{\partial\mathscr{P}(\gamma_{cd})} \tag{2.8}$$

and hence the (Eulerian) variation in $\mathscr{X}$ of the projection of any orthogonal tensor function of strain is given by

$$\delta\mathscr{P}(T_a{}^b{}_{\cdots}^{\cdots})=\mathscr{P}\left(\frac{\partial(T_a{}^b{}_{\cdots}^{\cdots})}{\partial\gamma_{cd}}\right)\delta\mathscr{P}(\gamma_{cd})\,. \tag{2.9}$$

The Lagrangian Variation

In calculating the variation in $\mathscr{M}$ of the image under $\mathscr{P}^{-1}$ of a material tensor in $\mathscr{X}$ whose variation is known it is necessary to exercise some care, since the mapping $\mathscr{P}^{-1}$ itself varies *not only* due to the displacement directly (an effect which must be taken into account in the analogous Newtonian theory) but also due to the fact (which has no Newtonian analogue) that in so far as general tensors are concerned the mapping $\mathscr{P}^{-1}$ is affected by the change in the metric. However in the special case of *covariant* tensors, it can be seen that the orthogonality condition and the projection mapping are defined independently of the metric tensor, and hence for such a tensor the simple relation

$$\delta\mathscr{P}(T_{ab\ldots})=\mathscr{P}(\Delta T_{ab\ldots}) \tag{2.10}$$

will hold. In particular we shall have

$$\delta\mathscr{P}(\gamma_{cd})=\mathscr{P}(\Delta\gamma_{cd}) \tag{2.11}$$

and hence by substituting in (2.9) and taking the inverse image under $\mathscr{P}$ we obtain, for the Lagrangian variation of a *covariant* orthogonal tensor function of strain, the simple relation

$$\Delta(T_{ab\ldots})=\frac{\partial(T_{ab\ldots})}{\partial\gamma_{cd}}\,\Delta\gamma_{cd}\,. \tag{2.12}$$

In order to obtain a convenient formula for the Lagrangian variation of a general orthogonal tensor, we proceed as follows. Using the identity

$$\frac{\partial\gamma_{cd}}{\partial\gamma_{ef}}=\gamma_c^{(e}\gamma_a^{f)} \tag{2.13}$$

we obtain

$$\begin{aligned}\frac{\partial(T_{ab\ldots})}{\partial\gamma_{ef}} &= \frac{\partial}{\partial\gamma_{ef}}(\gamma_{bd}\ldots T_a{}^{d\cdots}_{\cdots})\\ &= \gamma_{bd\ldots}\frac{\partial(T_a{}^{d\cdots}_{\cdots})}{\partial\gamma_{ef}} + \Sigma\gamma_b^{(e}T_a{}^{f)\cdots}_{\cdots}\end{aligned}\tag{2.14}$$

where the summation includes one term for each of the contravariant indices of $T_a{}^{b\cdots}_{\cdots}$. Using the identity

$$\Delta g^{bc} = -g^{be}g^{cf}\Delta g_{ef}\tag{2.15}$$

we obtain

$$\begin{aligned}\Delta(T_a{}^{b\cdots}_{\cdots}) &= \Delta(g^{bc}\ldots T_{ac\ldots})\\ &= g^{bc}\ldots\Delta(T_{ac\ldots}) - \Sigma g^{be}T_a{}^{f\cdots}_{\cdots}\Delta g_{ef}\end{aligned}\tag{2.16}$$

where again the summation includes one term for each of the contravariant indices. Now from the definition of the projection tensor we have

$$\Delta\gamma_{ef} = \Delta g_{ef} + u_e\Delta u_f + u_f\Delta u_e\tag{2.17}$$

and hence, using the orthogonality property of the partial derivative tensor in (2.12), and substituting into (2.16) we obtain

$$\Delta(T_a{}^{b\cdots}_{\cdots}) = \left(g^{bc}\ldots\frac{\partial(T_{ac\ldots})}{\partial\gamma_{ef}} - \Sigma g^{be}T_a{}^{f\cdots}_{\cdots}\right)\Delta g_{ef}\,.\tag{2.18}$$

Finally, substituting from (2.14) and again making use of the orthogonality property of the partial derivative function, we obtain the desired formula for the Lagrangian variation of a general orthogonal tensor function of strain in the form

$$\boxed{\Delta(T_a{}^{b\cdots}_{\cdots}) = \left(\frac{\partial(T_a{}^{b\cdots}_{\cdots})}{\partial\gamma_{ef}} + u^e\,\Sigma u^b T_a{}^{f\cdots}_{\cdots}\right)\Delta g_{ef}}\tag{2.19}$$

where again the summation includes one term for each of the contravariant indices of $T_a{}^{b\cdots}_{\cdots}$. We note that independently of the value of the partial derivative function, the Lagrangian variation has the orthogonality properties

$$u^a\,\Delta(T_a{}^{b\cdots}_{\cdots}) = 0\,,\tag{2.20}$$

$$u_b\,\Delta(T_a{}^{b\cdots}_{\cdots}) = -u^e T_a{}^{f\cdots}_{\cdots}\Delta g_{ef}\,.\tag{2.21}$$

Using the formula (2.13) we obtain as a trivial example of the application of (2.19) the expression

$$\Delta\gamma_{ab} = \gamma_a^c\gamma_b^d\Delta g_{cd}\,.\tag{2.22}$$

Similarly, using

$$\frac{\partial \gamma^{ab}}{\partial \gamma_{cd}} = -\gamma^{a(c}\gamma^{d)b} \tag{2.23}$$

we obtain

$$\Delta \gamma^{ab} = (-\gamma^{ac}\gamma^{db} + 2u^c u^{(a}\gamma^{b)d})\Delta g_{cd}\,. \tag{2.24}$$

Alternatively, this last expression could have been obtained directly from the definition of γ^{ab}, using the formula

$$\Delta u^a = \tfrac{1}{2} u^a u^c u^d \Delta g_{cd} \tag{2.25}$$

[obtained from the normalization condition (2.1), using the fact that the Lagrangian variation does not change the direction of u^a] together with (2.15).

3. Eulerian Variation Formulae

Once the Lagrangian variation formula has been derived we can obtain the corresponding Eulerian (fixed point) variation immediately by application of (2.3). Thus using (2.1) in conjunction with (2.19) we obtain the general formula for the Eulerian variation of a general orthogonal tensor function of strain $T_a{}^{b\cdots}_{\cdots}$ due to an Eulerian change $\delta g_{ab} = h_{ab}$ of the metric and a displacement $\Delta x^a = \xi^a$ of the flow lines in the form

$$\boxed{\begin{aligned}\delta(T_a{}^{b\cdots}_{\cdots}) = {} & \left(\frac{\partial (T_a{}^{b\cdots}_{\cdots})}{\partial \gamma_{ef}} + u^e \Sigma u^b T_a{}^{f\cdots}_{\cdots}\right)(h_{ef} + 2\xi_{(e;f)}) \\ & - \underset{\xi}{\mathcal{L}}[T_a{}^{b\cdots}_{\cdots}]\end{aligned}} \tag{3.1}$$

where the summation includes one term for each contravariant index of $T_a{}^{b\cdots}_{\cdots}$. The corresponding orthogonality conditions can be expressed by

$$u^a\,\delta(T_a{}^{b\cdots}_{\cdots}) = (T_a{}^{b\cdots}_{\cdots})\,[\xi, u]^a \tag{3.2}$$

and

$$u_b\,\delta(T_a{}^{b\cdots}_{\cdots}) = (T_a{}^{f\cdots}_{\cdots})\,([\xi, u]_f - u^e h_{ef}) \tag{3.3}$$

where we have used the notation

$$[\xi, u]^a = \underset{\xi}{\mathcal{L}}[u^a] = -\underset{u}{\mathcal{L}}[\xi^a] \tag{3.4}$$

for the commutator of u^a and ξ^a.

The formulae corresponding to (2.22), (2.24) and (2.25) are

$$\delta\gamma_{ab} = \gamma_a^c \gamma_b^d h_{cd} - 2u_{(a}\gamma_{b)c}[\xi, u]^c, \tag{3.5}$$

$$\delta\gamma^{ab} = (-\gamma^{ac}\gamma^{db} + 2u^c u^{(a}\gamma^{b)d}) h_{cd} - 2u^{(a}\gamma^{b)c}[\xi, u]_c, \tag{3.6}$$

$$\delta u^a = \tfrac{1}{2} u^a u^b u^c h_{bc} - \gamma^{ac}[\xi, u]_c. \tag{3.7}$$

We remark that the general equation of motion of an orthogonal space-time tensor function of strain given by Carter and Quintana (1972) can be expressed in the form

$$\begin{aligned} 0 = &\left(\frac{\partial(T_a{}^{b\cdots}_{\cdots})}{\partial\gamma_{cd}} + u^c \Sigma u^b T_a{}^{d\cdots}_{\cdots}\right)(2u_{(c;d)}) \\ &- \mathop{\mathscr{L}}_u [T_a{}^{b\cdots}_{\cdots}]. \end{aligned} \tag{3.8}$$

By comparing this with the right hand side of the variation formula (3.1), it may be checked that for a variation with $h_{ab} = 0$ and $\xi^a = \sigma u^a$ (for some scalar function σ), i.e. for a variation due purely to a *displacement along the world lines*, the Eulerian variation of an orthogonal space time tensor function of strain must be zero, as could have been seen directly from first principles.

We shall conclude this section by giving examples of the application of the preceding formulae to the Eulerian variations of some of the most important tensor fields in general relativistic elasticity theory.

The partial derivatives of the baryon number density n, the mass density ϱ, the pressure tensor p^{ab}, the Lagrangian strain tensor e_{ab} and the constant volume shear tensor s_{ab} with respect to γ_{cd}, are given (cf. Carter and Quintana, 1972) [1] by

$$\frac{\partial n}{\partial\gamma_{cd}} = -\tfrac{1}{2} n\gamma^{cd}, \tag{3.9}$$

$$\frac{\partial\varrho}{\partial\gamma_{cd}} = -\tfrac{1}{2}(p^{cd} + \varrho\gamma^{cd}), \tag{3.10}$$

$$\frac{\partial p^{ab}}{\partial\gamma_{cd}} = -\tfrac{1}{2}(E^{abcd} + p^{ab}\gamma^{cd}), \tag{3.11}$$

$$\frac{\partial e_{ab}}{\partial\gamma_{cd}} = \tfrac{1}{2}\gamma_a^{(c}\gamma_b^{d)}, \tag{3.12}$$

$$\frac{\partial s_{ab}}{\partial\gamma_{cd}} = \tfrac{1}{2}(\gamma_a^{(c}\gamma_b^{d)} - (\tfrac{1}{3}\eta_{ab} - \tfrac{1}{2}\tau_{ab})\gamma^{cd}) \tag{3.13}$$

respectively. Hence by application of (3.1) we obtain for the corresponding Eulerian variations, the formulae

$$\delta n = -\tfrac{1}{2} n \gamma^{cd} h_{cd} - (n\xi^c)_{;c} - n u^c u^d \xi_{c;d}, \tag{3.14}$$

$$\delta \varrho = -\tfrac{1}{2}(p^{cd} + \varrho\gamma^{cd}) h_{cd} - (\varrho\xi^c)_{;c} - T^{cd}\xi_{c;d}, \tag{3.15}$$

$$\begin{aligned}\delta p^{ab} = {} & (2p^{c(a}u^{b)}u^d - \tfrac{1}{2} p^{ab}\gamma^{cd} - \tfrac{1}{2} E^{abcd})(h_{cd} + 2\xi_{(c;d)}) \\ & - \underset{\xi}{\mathscr{L}}[p^{ab}],\end{aligned} \tag{3.16}$$

$$\delta e_{ab} = \tfrac{1}{2}\gamma_a^c \gamma_b^d (h_{cd} + 2\xi_{(c;d)}) - \underset{\xi}{\mathscr{L}}[e_{ab}], \tag{3.17}$$

$$\begin{aligned}\delta s_{ab} = {} & \tfrac{1}{2}\{\gamma_a^c\gamma_b^d - (\tfrac{1}{3}\eta_{ab} - \tfrac{1}{2}\tau_{ab})\gamma^{cd}\}(h_{cd} + 2\xi_{(c;d)}) \\ & - \underset{\xi}{\mathscr{L}}[s_{ab}],\end{aligned} \tag{3.18}$$

where

$$T^{ab} = \varrho u^a u^b + p^{ab} \tag{3.19}$$

is the energy momentum tensor.

It is not possible to give the full variation of the elasticity tensor E^{abcd} without going to third order in the partial derivatives of the energy density. However it may sometimes occur that the orthogonality property of the variation of the elasticity tensor is useful even in problems where only first and second derivatives are involved. From (3.3) we see that this orthogonality property can be expressed in the form

$$u_d\, \delta E^{abcd} = E^{abcf}([\xi, u]_f - u^e h_{ef}). \tag{3.20}$$

In many practical applications the variation of the energy momentum tensor T^{ab} itself will be of primary importance. Since the energy momentum tensor is not orthogonal its variation cannot be calculated directly from (3.1). However it can easily be obtained by using (3.15) and (3.16) in conjunction with (3.7). Thus we find

$$\begin{aligned}\delta T^{ab} = {} & \{-\tfrac{1}{2}\varrho u^a u^b u^c u^d + 2T^{c(a}u^{b)}u^d - \tfrac{1}{2} u^a u^b T^{cd} \\ & - \tfrac{1}{2} T^{ab}\gamma^{cd} - \tfrac{1}{2} E^{abcd}\}(h_{ab} + 2\xi_{(c;d)}) \\ & - \underset{\xi}{\mathscr{L}}[T^{ab}].\end{aligned} \tag{3.21}$$

In particular, for the study of perturbations of Einstein's equations, this last formula will be used in conjunction with the corresponding standard Eulerian variation formula

$$\delta R_{ab} = h_{(a\ ;b);c}^{\ \ c} - \tfrac{1}{2}(h^c{}_{c;a;b} + h_{ab;c}{}^{;c}) \tag{3.22}$$

for the Ricci tensor. In the applications which follow it will only be necessary to consider the variation of the Ricci scalar which may be obtained from the full Ricci tensor variation (3.22) by contraction in the form

$$\delta R = -2h_c{}^{[c;b]}{}_{;b} - R^{cb} h_{cb}\,. \tag{3.23}$$

We shall also have occasion to use the formula

$$\delta(\sqrt{-g}) = \tfrac{1}{2}\sqrt{-g}\, h_c^{\,c} \tag{3.24}$$

for the Eulerian variation of the volume density factor $\sqrt{-g}$ where g is the determinant of the covariant metric tensor g_{ab}.

4. Lagrangian and Eulerian Variations of Integrals

In the following sections we shall consider variational integrals of two forms, namely *action integrals* of the form

$$S = \int_\tau L\, d\tau \tag{4.1}$$

taken over a volume τ where L is a scalar Lagrangian function and $d\tau$ is the metric 4-volume measure, and *flux integrals* of the form

$$I = \int_\Sigma F^a\, d\Sigma_a \tag{4.2}$$

taken over a hypersurface Σ, where F^a is a flux vector and $d\Sigma_a = n_a\, d\Sigma$ where $d\Sigma$ is the metric 3-volume measure on Σ and n_a is a unit normal to Σ. We shall have frequent occasion to use Green's theorem and Stokes theorem in the forms

$$\int_\tau V^a{}_{;a}\, d\tau = \oint_{\partial\tau} V^d\, d\Sigma_a \tag{4.3}$$

for any vector V^a, where $\partial\tau$ is the 3-surface bounding τ, and

$$\int_\Sigma F^{ab}{}_{;b}\, d\Sigma_a = \oint_{\partial\Sigma} F^{ab}\, dS_{ab} \tag{4.4}$$

for any *antisymmetric* tensor F^{ab}, with $dS_{ab} = n^{(1)}{}_{[a} n^{(2)}{}_{b]}\, dS$ where dS is the metric 2-surface measure on the boundary of Σ and where $n^{(1)}{}_a$ and $n^{(2)}{}_a$ are unit vectors orthogonal to $\partial\Sigma$ and to each other.

Since the metric 4-volume element can be expressed by

$$d\tau = \sqrt{-g}\, d^{(4)}x \tag{4.5}$$

in terms of the co-ordinate 4-volume element $d^{(4)}x$ where g is the determinant of the metric tensor, we can write the comoving and fixed point variations of the scalar action in the form

$$\Delta S = \int_\tau \Delta(L\sqrt{-g})\frac{d\tau}{\sqrt{-g}} \tag{4.6}$$

$$\delta S = \int_\tau \delta(L\sqrt{-g})\frac{d\tau}{\sqrt{-g}} \tag{4.7}$$

Hence by (2.3)

$$\Delta S - \delta S = \int_\tau \mathop{\mathscr{L}}_{\xi}[L\sqrt{-g}]\frac{d\tau}{\sqrt{-g}}. \tag{4.8}$$

Using the standard formulae

$$\mathop{\mathscr{L}}_{\xi}[L] = L_{,a}\xi^a \tag{4.9}$$

$$\mathop{\mathscr{L}}_{\xi}[\sqrt{-g}] = \sqrt{-g}\,\xi^a{}_{;a} \tag{4.10}$$

we obtain

$$\Delta S - \delta S = \int_\tau (L\xi^a)_{;a}\,d\tau \tag{4.11}$$

and hence, using the Green's theorem (2.3),

$$\Delta S - \delta S = \oint_{\partial\tau} L\xi^a\,d\Sigma_a. \tag{4.12}$$

Thus we see that it is unnecessary to make any distinction between the Lagrangian and Eulerian variations of a scalar action integral S, provided *either* the action L *or* the displacement ξ vanishes on the boundary of the volume τ in which variations take place.

For a flux integral of the form I the distinction between Lagrangian and Eulerian variations must be taken more seriously even when F^a or ξ^a vanish on the boundary of Σ. Let us introduce a co-ordinate system of the form t, x^1, x^2, x^3 in such a way that Σ is determined by the condition $t=0$ and let $d^{(3)}x$ denote the coordinate volume element on Σ. Then we can express the metric element $d\Sigma_a$ on Σ in the form

$$d\Sigma_a = \sqrt{-g}\,t_{,a}\,d^{(3)}x. \tag{4.13}$$

Thus we can write

$$\Delta I = \int_\Sigma \Delta(F^a\sqrt{-g})\frac{d\Sigma_a}{\sqrt{-g}}, \tag{4.14}$$

$$\delta I = \int_\Sigma \delta(F^a\sqrt{-g})\frac{d\Sigma_a}{\sqrt{-g}} \tag{4.15}$$

from which we obtain

$$\Delta I - \delta I = \int_{\Sigma} \underset{\xi}{\mathcal{L}}[F^a \sqrt{-g}] \frac{d\Sigma_a}{\sqrt{-g}} . \tag{4.16}$$

Using the standard formula,

$$\underset{\xi}{\mathcal{L}}[F^a] = 2 F^{[a}{}_{;b} \xi^{b]} \tag{4.17}$$

together with (4.10) we can re-write this as

$$\Delta I - \delta I = \int_{\Sigma} \{(2F^{[a} \xi^{b]})_{;b} + \xi^a F^b{}_{;b}\} \, d\Sigma_a . \tag{4.18}$$

Hence, using Stokes theorem (4.4), we obtain

$$\Delta I - \delta I = \int_{\Sigma} F^b{}_{;b} \xi^a \, d\Sigma_a + 2 \oint_{\partial \Sigma} F^a \xi^b \, dS_{ab} . \tag{4.19}$$

Thus the Lagrangian and Eulerian variations of I will be equivalent, subject to the condition that ξ^a or F^a vanishes on the boundary of Σ, *only* when the divergence of F^a is zero, i.e., when

$$F^a{}_{;a} = 0 . \tag{4.20}$$

This is, of course, the same condition that is necessary and sufficient for; the unvaried action integral I to be independent of the choice of Σ.

The application which follows will be entirely based on the consideration of *Eulerian* variations. As far as the action principle described at the beginning of the next section is concerned one could just as well use Lagrangian variations, but for the variation principle the logical distinction is significant. It will be shown that the vanishing of the Eulerian variation of the mass integral defined in Section 5 is both necessary and sufficient for the appropriate field equations to hold. As is usually the case in such variation principles, the relevant flux vectors are chosen so that when these field equations hold (but not in general otherwise) they satisfy divergence conditions of the form (4.20) so that the integrals will be independent of Σ. This implies that the vanishing of the Lagrangian variation is *also necessary* for the field equations to hold, but it does *not* imply that it is sufficient.

5. Statement of the Variation Principle

Before describing the *mass variation principle* for a stationary star, which will be the main topic of this section, we shall first present a simpler general purpose *action variation principle* which is a straightforward generalisation of the variation principal for a perfect fluid given by Taub (1954) [3] (a somewhat different version was given by Taub (1969) [10]). We consider an action integral of the form

$$S = \int_{\tau} \left(\varrho - \frac{1}{16\pi} R \right) d\tau \tag{5.1}$$

this integral being formally identical to that of Taub (1954) [3], the only difference being that here the density ϱ is to be regarded as a general function of strain whereas in Taub's case it was regarded as a function of baryon number density only. Using (4.7) and the Greens theorem (4.3), together with the explicit variational expressions (3.15), (3.23), (3.24) we find that the Eulerian variation of this integral can be expressed in the form

$$\begin{aligned} \delta S = & \frac{1}{16\pi} \int_{\tau} \{R^{cb} - \tfrac{1}{2} R g^{cb} - 8\pi T^{cb}\} h_{cb} \, d\tau \\ & + \int_{\tau} T^{cb}{}_{;b} \xi_c \, d\tau \\ & + \oint_{\partial\tau} \left\{ \frac{1}{8\pi} h_c{}^{[c;b]} - (p^{cb} + \varrho \gamma^{cb}) \xi_c \right\} d\Sigma_b . \end{aligned} \tag{5.2}$$

From this equation we can immediately deduce the following action principle: *if the Einstein equations*

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab} \tag{5.3}$$

and hence also the conservation equations

$$T^{cb}{}_{;b} = 0 \tag{5.4}$$

are satisfied, then it follows that for any displacement ξ^a and metric perturbation h_{ab} which vanish on the boundary, $\partial\tau$, the consequent Eulerian variation δS (and hence also, by the results of the previous section, the Lagrangian variation ΔS) *will be zero;* conversely *if δS* (or equivalently ΔS) *vanishes for any displacement ξ^a which is zero on $\partial\tau$ then the conservation Eq.* (5.4) *must be satisfied,* and *if δS* (or equivalently ΔS) *also vanishes for any metric perturbation h_{ab} which vanishes on $\partial\tau$ then the Einstein Eqs.* (5.3) *must be satisfied.*

The main purpose of this section is to describe a related but more sophisticated variation principle (whose proof will take up the two subsequent sections) which is a generalisation of the one given by Hartle and Sharp (1957) [4]. This principle applies in the special case, to which we shall henceforth restrict our attention, of a spacetime which is stationary, axisymmetric, topologically Euclidean, and assymptotically flat, in the sense of Papapetrou (1949) [11]. We shall denote the Killing vector generator of the stationary action by k^a, this vector being specified uniquely by the normalisation condition that $k^a k_a \to 1$ in the assymptotic limit at spacial infinity. We shall denote the Killing vector generator of the axisymmetric action by m^a, this vector, whose trajectories are circles, being uniquely specified by the normalisation condition $\oint d\varphi = 2\pi$ where φ is any scalar defined (modulo 2π) by $\varphi_{,a} m^a = 1$, and where the integral is taken around any one of the circular trajectories. By their definition as Killing vectors, k^a and m^a satisfy

$$\left.\begin{aligned} k_{a;b} &= k_{[a;b]} \\ m_{a;b} &= m_{[a;b]} \,. \end{aligned}\right\} \tag{5.5}$$

Moreover there is no loss of generality (cf. Carter, 1970) [12] in supposing that they satisfy the commutation conditions

$$[k, m]^a = 0 \tag{5.6}$$

(using the bracket notation defined by (3.4)). We suppose that the system consists of an isolated star with assymptotically defined (Lenz-Thirring) mass M_∞ and angular momentum J_∞. The Papapetrou assymptotic flatness conditions consist of the requirement that in a standard assymptotically Cartesian co-ordinate system, x^0, x^1, x^2, x^3 with $k^a = \delta^a_0$ the metric tensor components g_{ab} should be well behaved functions of $1/r$, where

$$r^2 = \delta_{ij} x^i x^j \,, \tag{5.7}$$

such that

$$\left.\begin{aligned} g_{00} &= -1 + \frac{2M_\infty}{r} + O\left(\frac{1}{r^2}\right) \\ g_{0i} &= \frac{2J_\infty}{r^3} m_i + O\left(\frac{1}{r^3}\right) \\ g_{ij} &= \left(1 + \frac{2M_\infty}{r}\right)\delta_{ij} + O\left(\frac{1}{r^2}\right) \end{aligned}\right\} \tag{5.8}$$

and where the indices i, j run from 1 to 3; it is evident that in terms of the same co-ordinate system, the metric perturbation h_{ab} must satisfy

$$h_{ab} = 2\frac{\delta M_\infty}{r}\delta_{ab} + O\left(\frac{1}{r^2}\right). \tag{5.9}$$

The definitions of the assymptotic mass and angular momentum may be cast into co-ordinate independent form as integrals over a space-like 2-sphere S surrounding the star in the limit as S goes to an assymptotically large distance. Thus it follows directly from the assymptotic boundary conditions that we shall have

$$4\pi M_\infty = -\mathop{\mathscr{L}t}_{\infty} \oint_S k^{a;b}\, dS_{ab}, \tag{5.10}$$

$$8\pi J_\infty = \mathop{\mathscr{L}t}_{\infty} \oint_S m^{a;b}\, dS_{ab}. \tag{5.11}$$

For future reference, we note that in consequent of the boundary conditions satisfied by h_{ab} we shall also have the co-ordinate independent identity

$$2\pi\,\delta M_\infty = \mathop{\mathscr{L}t}_{\infty} \oint_S k^a h_c{}^{[c;b]}\, dS_{ab}. \tag{5.12}$$

Taking advantage of the Killing antisymmetry conditions (5.5) we can use the Stokes theorem (4.4) to cast the definitions (5.10), (5.11) into the alternative forms

$$M_\infty = \frac{1}{4\pi} \int_\Sigma k^{b;a}{}_{;b}\, d\Sigma_a \tag{5.13}$$

$$J_\infty = -\frac{1}{8\pi} \int_\Sigma m^{b;a}{}_{;b}\, d\Sigma_a \tag{5.14}$$

which were first given by Komar (1959) [13] where Σ is any well behaved space-like hypersurface extending to infinity. Using the identitites

$$\left.\begin{aligned} k^{b;a}{}_{;b} &= R^a{}_b\, k^b \\ m^{b;a}{}_{;b} &= R^a{}_b\, m^b \end{aligned}\right\} \tag{5.15}$$

(which follow from the Killing Eqs. (5.5)) together with the integral identity

$$\int_\Sigma R m^a\, d\Sigma_a = 0 \tag{5.16}$$

which follows from the fact that the flux Rm^a is invariant under the axisymmetry action and from the fact that the integral curves of m^a are circles, which must therefore cross Σ an equal number of times in the positive and negative senses) we can at once deduce that *when the Einstein field Eqs.* (5.3) *are satisfied* (but not in general otherwise) we shall have

$$M_\infty = M, \tag{5.17}$$

$$J_\infty = J \tag{5.18}$$

where the *variational mass* M and the *variational angular momentum* J are defined by

$$M = \int_{\Sigma} \left\{ T^a{}_b k^b + \frac{1}{16\pi} (R k^a - 2k^{a;b}{}_{;b}) \right\} d\Sigma_a , \tag{5.19}$$

$$J = - \int_{\Sigma} T^a{}_b m^b \, d\Sigma_a . \tag{5.20}$$

These expressions for the variational mass and angular momentum are the obvious generalisations of the corresponding expressions given by Hartle and Sharp (1967) [4] and Bardeen (1970) [5] in the perfect fluid case. The expression (5.19) for M differs at first sight from the form given by these authors in that they used a term of the form $k^a W^b{}_{;b}$ in place of the Komar term $-1/8\pi \, k^{a;b}{}_{;b}$. In fact however this difference is illusory since both terms are functions of the metric and its derivatives only, chosen to give the same contribution $\frac{1}{2} M_\infty$ when integrated over Σ. The vector W^a can be specially contrived so that the terms containing second derivatives of the metric tensor cancel out of the integrand, but unfortunately it cannot conveniently be given an explicit co-ordinate independent definition. For the purposes of the present discussion the more straightforwardly defined (but ultimately equivalent) Komar-type term is perfectly adequate.

From this point onwards we shall suppose that the flow is non-convective, in the sense that

$$u^{[a} k^b m^{c]} = 0 \tag{5.21}$$

which means that there exist scalar functions U, Ω defined within the star such that

$$u^a = U(k^a + \Omega m^a) \tag{5.22}$$

where Ω is the *angular velocity*. We shall also suppose that the flow is *rigid* in the sense that Ω is constant throughout the star, i.e.

$$\Omega_{,a} = 0 . \tag{5.23}$$

(The significance of these assumptions will be discussed in the final section.)

We are now in a position to state the following mass variation principle: *the Einstein equations* (5.3) *[and hence also the conservation equations* (5.4)*] will be satisfied in a stationary axisymmetric assymptotically flat system subject to* (5.22) *and* (5.23) *if and only if the Eulerian variation of* M (as defined by 5.19) *is zero for any displacement* ξ^a *and any metric perturbation* h_{ab} *which preserve the group invariance under the action of* k^a *and* m^a, *as well as the assymptotic flatness conditions and the* *conditions* (5.22) *and* (5.23), *subject to the restriction that the Eulerian variation of J [as defined by* (5.20)*] should be zero.*

We shall conclude this section by specifying explicit restrictions which it will be convenient to impose on h_{ab} and ξ^a during the following proof, which will be sufficient to ensure that the group invariance under k^a and m^a is preserved. As far as the metric g_{ab} is concerned, it is clearly both necessary and sufficient that

$$\underset{k}{\mathscr{L}}[h_{ab}] = 0, \tag{5.24}$$

$$\underset{m}{\mathscr{L}}[h_{ab}] = 0. \tag{5.25}$$

To preserve the group invariance of a *Lagrangian* variation of an orthogonal tensor function of strain it is clear from (2.7) and (2.19) that it is sufficient to have

$$\underset{k}{\mathscr{L}}[\xi_{(a;b)}] = 0, \tag{5.26}$$

$$\underset{m}{\mathscr{L}}[\xi_{(a;b)}] = 0 \tag{5.27}$$

also. In fact this is also sufficient for the Eulerian variation to preserve the symmetry as can be seen from the following considerations. By (3.1) the Eulerian variation will preserve the stationary axisymmetry group if and only if in addition to (5.24) and (5.25), the Lie derivative with respect to ξ^a of the function of strain under consideration is itself invariant under the actions generated by k^a and m^a. Now using the standard operator commutator identity

$$\underset{k}{\mathscr{L}}\,\underset{\xi}{\mathscr{L}} = \underset{\xi}{\mathscr{L}}\,\underset{k}{\mathscr{L}} + \underset{[k,\xi]}{\mathscr{L}} \tag{5.28}$$

(see e.g. Yano (1955) [14]) and noting that the first operator on the right hand side gives zero when acting on the (unperturbed) functions of strain under consideration (in consequence of the original symmetry generated by k^a) we see that the Lie derivatives with respect to ξ^a of general orthogonal functions of strain will be invariant under the action generated by k^a *provided* $[k, \xi]^a$ is itself a killing vector, i.e. provided

$$\underset{k}{\mathscr{L}}[\xi^a] = c_{11}k^a + c_{12}m^a \tag{5.29}$$

where c_{11} and c_{12} are scalar *constants*. Similarly we require

$$\underset{m}{\mathscr{L}}[\xi^a] = c_{21}k^a + c_{22}m^a \tag{5.30}$$

where c_{21} and c_{22} are two more scalar constants. However since $\xi_{(a;b)}$ is itself the Lie derivative with respect to ξ^a of the (unperturbed) metric tensor it is clear (by the same argument) that the conditions (5.26) and (5.27) are completely equivalent to (5.29) and (5.30).

The restriction that the variation of angular momentum be zero can be expressed in terms of a Lagrange multiplier, Λ say. In what follows we shall show rigourously that the condition that

$$\delta(M - \Lambda J) = 0 \tag{5.31}$$

for all metric variations preserving the asymptotic flatness conditions and satisfying (5.24) and (5.25) and for all displacements satisfying (5.26) and (5.27) [or equivalently (5.29) and (5.30)] is necessary and sufficient to ensure that the field Eqs. (5.3) must hold.

6. The Variation of Angular Velocity

The conditions (5.24)–(5.27) on the variations were required simply in order to ensure that the variations preserve the stationary-axisymmetry conditions. However they also automatically preserve both the no-convection condition (5.22) *and* the rigididity condition (5.23), although the angular momentum Ω itself can have a constant variation $\delta\Omega$. In so far as a variation due to a metric change h_{ab} *without* displacement is concerned, this is obvious, and in this case Ω itself is invariant, i.e. $\delta\Omega$ is zero. In order to see the effect of a displacement ξ^a we use the fact that the general solution of Eqs. (5.29) and (5.30) can be expressed in the form $\xi^a = \eta^a + \alpha m^a + \beta k^a$ where η^a satisfies

$$[k, \eta]^a = 0, \tag{6.1}$$

$$[m, \eta]^a = 0 \tag{6.2}$$

and where α and β are scalars whose Lie derivatives with respect to m^a and k^a are scalar *constants*; and where moreover (in order that the solution be *globally* well defined) the constant values of the Lie derivatives with respect to m^a must actually be zero since the trajectories of m^a are circles. We have remarked in Section 3 that a displacement parallel to the flow vector u^a gives zero Eulerian displacement for orthogonal tensor functions of strain (and also, a fortiori, for the metric g_{ab} and the flow u^a itself) so that it is therefore possible to arrange without effective loss of generality (by the addition of such a displacement) that β is zero. Hence the solution to the Eqs. (5.29), (5.30) can be taken to have the simple form

$$\xi^a = \eta^a + \alpha m^a \tag{6.3}$$

where

$$\underset{k}{\mathcal{L}}[\alpha] = c_{12} \tag{6.4}$$

$$\underset{m}{\mathcal{L}}[\alpha] = 0 \tag{6.5}$$

(the constants c_{11}, c_{21}, c_{22} being all zero). Since these conditions are unaltered if α is changed by the addition of a scalar whose Lie derivative with respect to k^a is zero, we can choose η^a so as to arrange without loss of generality that α is zero on Σ. Now it is clear that the displacement η^a leaves the value of Ω unchanged, since by (6.1), (6.2), and (5.23) it commutes with the flow vector u^a i.e.

$$[\eta, u]^a = 0\,. \tag{6.6}$$

Therefore the only possible source of a variation in Ω is the displacement αm^a and it can easily be checked that a displacement of this form will indeed produce a variation $\delta\Omega$ which will be given by

$$\delta\Omega = \alpha_{,a} k^a\,. \tag{6.7}$$

On comparison with (5.4) we see that in the present case this variation must be constant, i.e. we have

$$\delta\Omega = c_{12} \tag{6.8}$$

which confirms that a displacement ξ^a satisfying (5.26) and (5.27) will preserve the rigidity condition (5.23).

[If we wished to derive a Bardeen-type variational principle for a star with differential rotation, it would be necessary to consider displacements of the form (6.3) subject to (6.4) are (6.5) but with *non-constant* $\delta\Omega$. This would violate Eqs. (5.26) and hence would not preserve the stationarity condition for *general* functions of strain. However it *is* compatible with stationarity in the special case of a perfect fluid for which ϱ and p^{ab} are isotropic functions of baryon number density only.]

In the remainder of this section we shall consider the effect on the variational integral $M - \Lambda J$ of a pure angular velocity changing displacement of the form

$$\xi^a = \alpha m^a \tag{6.9}$$

(i.e. we shall temporarily take η^a and h_{ab} to be zero) where α is defined in terms of the (constant) variation $\delta\Omega$ by the Eqs. (6.5) and (6.7) and by the condition that it vanish on the hypersurface Σ over which the integrals are to be taken, i.e.

$$\alpha(\Sigma) = 0\,. \tag{6.10}$$

Now since m^a is a Killing vector, (6.9) implies

$$\xi_{(a;b)} = \alpha_{,(a} m_{b)} \tag{6.11}$$

and since the Lie derivative of the energy momentum tensor T^{ab} with respect to m^a is zero we shall also have

$$\underset{\xi}{\mathcal{L}}[T^{ab}] = -2T^{c(a} m^{b)} \alpha_{,c}\,. \tag{6.12}$$

Therefore using Eq. (3.21) we find that the Eulerian variation in T^{ab} due to the displacement (6.9) is given by

$$\begin{aligned}\delta T^{ab} = \{ & -\tfrac{1}{2}\varrho u^a u^b u^c u^d + 2T^{c(a}u^{b)}u^d - \tfrac{1}{2}T^{cd}u^a u^b \\ & -\tfrac{1}{2}T^{ab}\gamma^{cd} - \tfrac{1}{2}E^{abcd}\}\, 2\alpha_{,(c}m_{d)} \\ & + 2T^{c(a}m^{b)}\alpha_{,c}\,.\end{aligned} \tag{6.13}$$

Now since we are keeping h_{ab} zero for the time being, (so that raising and lowering of tensor suffices commutes with Eulerian variation) the corresponding variation in $M - \Lambda J$ will be given simply by

$$\delta(M - \Lambda J) = +\int_{\Sigma} (k_b + \Lambda m_b)\,\delta(T^{ab})\,d\Sigma_a\,. \tag{6.14}$$

Now in consequence of (6.10) we can replace $d\Sigma_a$ by $\sigma\alpha_{,a}d\Sigma$ where $d\Sigma$ is the scalar metric measure in Σ and σ is a positive scalar (whose explicit value is given by $\sigma^2 = -\alpha_{,a}\alpha^{;a}$). Hence using (5.23) we can recast (6.14) in the form

$$\delta(M - \Lambda J) = \int_{\Sigma} \{U^{-1}u_a + (\Lambda - \Omega)m_a\}\,\alpha_{,c}\,\delta(T^{ac})\sigma\,d\Sigma\,. \tag{6.15}$$

Now, on substituting from (6.13) we obtain

$$\alpha_{,c}\,\delta T^{ac} = \{u^b u^d(\varrho\gamma^{ac} + p^{ac}) - A^{abcd}\}\,\alpha_{,b}\alpha_{,d}m_c \tag{6.16}$$

where A^{abcd} is the relativistic Hadarmard elasticity tensor, which was introduced by Carter (1972) [6] in a discussion of sound wave propagation, and which is defined by

$$A^{abcd} = E^{abcd} - \gamma^{ac}p^{bd}\,. \tag{6.17}$$

Using the orthogonality properties of the various functions of strain involved we immediately deduce that

$$\alpha_c\,\delta(T^{ac})u_c = 0\,. \tag{6.18}$$

This is a valuable result which will be useful in more general contexts: in effect we have shown that the *variation of angular velocity gives no contribution to the flux of* $u_c\delta T^{ac}$ *across* Σ. In consequence (6.15) reduces to

$$\delta(M - \Lambda J) = \int_{\Sigma} (\Omega - \Lambda)\alpha_{,c}\,\delta(T^{ac})m_a\sigma\,d\Sigma\,. \tag{6.19}$$

By expressing $\alpha_{,c}$ in the form

$$\alpha_{,c} = \lambda u_c + \kappa v_c \tag{6.20}$$

where v_c satisfies the orthonormality conditions

$$v_c u^c = 0 \tag{6.21}$$

$$v^c v_c = 1 \tag{6.22}$$

and where the scalars λ and κ must be such that

$$\lambda^2 > \kappa^2 \tag{6.23}$$

since Σ is spacelike, we obtain

$$\alpha_{,c}\,\delta(T^{ac})\,m_a = \{\lambda^2(\varrho\gamma^{ac} + p^{ac}) - \kappa^2 A^{abcd} v_b v_d\}\, m_c m_a\,. \tag{6.24}$$

Now it was shown by Carter (1972) [6] that the squared speeds v^2 of sound propagation in a direction v^a orthogonal to the flow u^a are the eigenvalues of the eigenvector equation

$$\{v^2(\varrho\gamma^{ac} + p^{ac}) - A^{abcd} v_b v_d\}\, \iota_b = 0 \tag{6.25}$$

where the eigenvectors ι_b are the possible directions of polarization. Since local causality requires that the eigenvalues v^2 be *not* greater than unity, it follows that $\{v^2(\varrho\gamma^{ac} + p^{ac}) - A^{abcd} v_b v_d\}\, m_a m_c$ must be *strictly positive* for any spacelike vector m_a whenever v^2 *is* greater than unity. Thus by (6.23) and (6.24) local causality also implies

$$\alpha_{,c}\,\delta(T^{ac})\,m_a > 0 \tag{6.26}$$

everywhere.

Hence, by (6.19), we arrive at the important conclusion that the *variation in* $M - \Lambda J$ *induced by the variation* $\delta\Omega$ *in the angular velocity due to the displacement* (6.9) *can be zero only if*

$$\Omega = \Lambda\,. \tag{6.27}$$

This result could have been expected from general thermodynamic considerations since the angular velocity and angular momentum are dynamically conjugate. However the author would be interested to know of a general physical reason why it should apparently be necessary (as it was also in the more specialised derivation given by Hartle and Sharp for a perfect fluid) to invoke causality in order to establish it rigorously.

7. The Angular Velocity Conserving Variation

In this section we consider the variations in the integral $M - \Lambda J$ due to a metric variation h_{ab} satisfying (5.24) and (5.25) and a displacement ξ^a of the form

$$\xi^a = \eta^a \tag{7.1}$$

where η^a satisfies (6.1) and (6.2). As has already been remarked variations of this form will leave the angular velocity Ω invariant. We shall suppose that the variations take place with respect to an unperturbed flow which has been restricted to satisfy the necessary condition $\Omega = \Lambda$. Thus in consequence of (7.1) the perturbed flow will continue to satisfy this condition, and hence the variational integral will reduce to the form

$$M - \Lambda J = -\int_\Sigma \left\{ \varrho(k^a + \Omega m^a) - \frac{1}{16\pi}(Rk^a - 2k^{a;b}{}_{;b}) \right\} d\Sigma_a \tag{7.2}$$

both before *and* after variation, so that as far as the present section is concerned we can treat this expression as a definition of the variational integral. We can use the identity

$$\int_\Sigma \varrho m^a \, d\Sigma_a = 0 \tag{7.3}$$

(which holds for the same reason as (5.16)) to replace (7.2) by the even simpler expression

$$M - \Lambda J = -\int_\Sigma \left\{ \left(\varrho - \frac{1}{16\pi} R \right) k^a + \frac{1}{8\pi} k^{a;b}{}_{;b} \right\} d\Sigma_a . \tag{7.4}$$

This form being closely analogous to the Taub-type action integral given by Eq. (5.1).

Now by substituting the expressions (3.23) and (3.24) from section 3 into the general formula (4.15) we obtain

$$\begin{aligned} \delta \int_\Sigma Rk^a \, d\Sigma_a = & -\int_\Sigma (R^{bc} - \tfrac{1}{2} Rg^{bc}) \, h_{bc} k^a \, d\Sigma_a \\ & - 2 \int_\Sigma h_c{}^{[c;b]}{}_{;b} k^a \, d\Sigma_a . \end{aligned} \tag{7.5}$$

Since in the present case we are requiring that h_{ab} satisfy (5.24) we also have the identity

$$h_c{}^{[c;b]}{}_{;b} k^a = -\{ k^a h_c{}^{[b;c]} - k^b h_c{}^{[a;c]} \}_{;b} \tag{7.6}$$

so that Stokes Theorem (4.4) can be applied in conjunction with (5.11) to give

$$\int_\Sigma h_c{}^{[c;b]}{}_{;b} k^a \, d\Sigma = 4\pi \, \delta M_\infty . \tag{7.7}$$

Also it follows directly from (5.12) that

$$\delta \int_\Sigma k^{a;b}{}_{;b} \, d\Sigma^a = -4\pi \, \delta M_\infty . \tag{7.8}$$

Thus we obtain the important identity

$$\delta \int_\Sigma (Rk^a - 2k^{a;b}{}_{;b}) \, d\Sigma_a = -\int_\Sigma (R^{bc} - \tfrac{1}{2} Rg^{bc}) h_{bc} k^a \, d\Sigma_a . \tag{7.9}$$

Substituting the expressions (3.15), (3.24) from Section 3 into the general expression (4.15), we obtain

$$\begin{aligned}\delta \int \varrho k^a \, d\Sigma_a = & - \int (\tfrac{1}{2} T^{cb} h_{cb} - T^{cb}{}_{;b} \eta_c) k^a \, d\Sigma_a \\ & - \int \{(p^{cb} + \varrho \gamma^{cb}) \eta_c\}_{;b} k^a \, d\Sigma_a \, .\end{aligned} \tag{7.10}$$

Now since the Lie derivatives with respect to k^a of p^{cb}, ϱ, γ^{cb} and η_c are all zero, we have the identity

$$\{(p^{cb} + \varrho \gamma^{cb}) \eta_c\}_{;b} k^a = 2\{(p^{c[b} + \varrho \gamma^{c[b}) \xi_c k^{a]}\}_{;b} \tag{7.11}$$

and hence Stokes Theorem (4.4) gives

$$\begin{aligned}\int_\Sigma \{(p^{cb} + \varrho \gamma^{cb}) \eta_c\}_{;b} k^a \, d\Sigma_a & = \mathop{\mathscr{L}t}_{\infty} \oint_S (p^{cb} + \varrho \gamma^{cb}) \eta_c k^a \, dS_{ab} \\ & = 0\end{aligned} \tag{7.12}$$

since the star is bounded.

Thus we finally obtain

$$\begin{aligned}\delta(M - \Lambda J) = & - \frac{1}{16\pi} \int_\Sigma (R^{bc} - \tfrac{1}{2} R g^{bc} - 8\pi T^{bc}) h_{bc} k^a \, d\Sigma_a \\ & - \int_\Sigma T^{cb}{}_{;b} \, \eta_c k^a \, d\Sigma_a \, .\end{aligned} \tag{7.13}$$

Since R^{bc}, g^{bc}, T^{bc} satisfy the same restrictions of invariance under Lie transport by the fields k^a and m^a as have been imposed on η^c and h_{bc} it follows that the condition that $\delta(M - \Lambda J)$ be zero for a general displacement η^a subject to (6.1) and (6.2) is *sufficient*, as well as obviously being necessary for the conservation Eqs. (5.4) to hold, and similarly the condition that $\delta(M - \Lambda J)$ be zero for a general metric variation satisfying the assymptotic flatness conditions and subject to (5.24) and (5.25) is *sufficient* as well as necessary for the full Einstein field Eqs. (5.3) to hold.

This completes our demonstration of the variation principle as it is stated at the end of Section 5.

8. Discussion

The conclusion of the preceeding sections may be summed up briefly as follows: if all possible metric variations h_{ab} subject to (5.24) and (5.25), and satisfying the assymptotic flatness conditions as well as all possible displacement fields ξ^a subject to (5.26) and (5.27) are taken into account then we have

$$\delta M - \Lambda \delta J = 0 \Rightarrow \begin{cases} \Lambda = \Omega \\ R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab} \end{cases} \tag{8.1}$$

and conversely

$$R_{ab} - \tfrac{1}{2} R g_{ab} = 8\pi T_{ab} \Rightarrow \begin{cases} \delta M - \Omega \delta J = 0 \\ \Delta M - \Omega \Delta J = 0 \,. \end{cases} \tag{8.2}$$

(The last conclusion, expressed in terms of the Lagrangian variation, follows from the fact that when the Einstein field equations are satisfied the integrands in M and J automatically satisfy divergence conditions of the form (4.20) so that Eulerian and Lagrangian variations coincide.) It is to be remarked that in this mass variation principle (unlike the action principle described at the beginning of Section 5) it is essential to take displacements (not merely metric perturbations) into account in order to introduce the variation of angular velocity on which (8.1) depends.

It is intuitively evident (although the author does not have a rigorous proof) that the assumption (5.1) that there is no convection does not involve any loss of generality so long as one is considering a *simply* connected star (or a simply connected part of a star such as a neutron star crust) which is *strictly* solid in the sense of having non zero rigidity. (In a multiply connected body e.g. an annulus, it is possible to conceive of convective motions consistent with stationary axisymmetry even in a solid.) This contrasts with the situation in the discussion of the perfect fluid case by Bardeen (1972) [5], where the assumption of purely circular motion clearly does imply a significant loss of generality. The further restriction to the case of *rigid* flow, which was made in the earlier discussion of the perfect fluid case by Hartle and Sharp (1967) [4], also involves no loss of generality in the case of simply connected medium which is strictly solid.

Under the perfect fluid conditions which apply in the variational principles of Hartle and Sharp (1967) [4] and Bardeen (1970) [5], the assumption of the circular flow condition (5.21) automatically implies that the space-time must be invariant under a discrete isometry mapping which simultaneously inverts the direction of the stationary and axisymmetry killing vector fields. This follows from a generalisation (Kundt and Trumper, 1966) [15], Carter (1969) [16] of the theorem of Papapetrou (1966) [17] which applied originally only to the pure vacuum exterior of the star. However although Papapetrou's theorem can also be generalised to the case when an electromagnetic field is present (Carter, 1969) [16] it certainly cannot be extended to apply to a solid, not even a perfect solid in the sense defined by Carter and Quintana (1972) [1]. One might of course impose the existence of the simultaneous time and rotation angle inversion isometry as a simplifying assumption in a study of stationary axisymmetric solid bodies, but it is to be emphasized that none of the results of the present work depend on such a condition.

An additional discrete symmetry condition which can be expected to hold automatically (although the author does not know of a rigorous proof) in a stationary axisymmetric star composed of a non-convective perfect fluid, provided it satisfies the same one-parameter equation of state throughout, but which need not hold more generally, is invariance under reflection in an equatorial hypersurface. Such a condition has not been assumed here, and therefore the variation principle which has just been derived applies not only to cigar shaped and doughnut shaped stars, but also, for example, to pear shaped stars.

We conclude by pointing out (as has previously been remarked by Hartle (1970) in a discussion of the special case of a rigidly rotating perfect fluid star) that the conclusion of (8.2) applies, in particular, for variations through neighbouring equilibrium states. It therefore follows rigourously that if a perfectly elastic star undergoes (adiabatic) variations through a one-parameter family of stationary axisymmetric equilibrium states, characterised by varying angular velocity, then the differential relation

$$\frac{dM}{dJ} = \Omega \tag{8.3}$$

will be satisfied. [This result could have been predicted by the heuristic physical argument of Thorne and Zeldovich (see Hartle (1970), Zeldovich and Novikov (1971)) which can be applied to any material body which undergoes thermodynamically reversible variations between stationary axisymmetric equilibrium states.]

For many purposes it will be useful to expand the mass which must clearly be an even function of Ω, as a power series in the form

$$M = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2} E_n \Omega^{2n} \tag{8.4}$$

where n runs over integer values, and where the coefficients E_n are constants. This leads, by (8.3), to the expression

$$J = \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2} E_{n+1} \Omega^{2n+1} \tag{8.5}$$

in terms of the *same* coefficients E_n. The zeroth coefficient E_0, is the energy of the star in the zero angular velocity state. The first coefficient E_1 is the moment of inertia in the zero angular velocity state. The higher coefficients E_2, E_3 etc. characterise the way in which the star undergoes elastic deformations under the influence of centrifugal force as the angular velocity is increased.

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