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On the Uniqueness of the Equilibrium State for Antiferromagnetic Ising Spin System in the Phase Transition Region

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Abstract. We show that at low temperature an Ising spin system with antiferromagnetic interaction in a small enough external magnetic field has only one translationally invariant state.

Introduction

We consider an Ising antiferromagnet with nearest neighbour interaction in a finite box Λ on a two-dimensional lattice $\mathbb{Z}^2$ i.e. at each point x_i of the lattice there is a spin $\sigma_{x_i} = \pm 1$. The conditional probability of a spin-configuration $\{\underline{\sigma}\}$ in Λ for a given boundary configuration $\underline{\tau}$ is proportional $e^{-\beta H_\Lambda(\underline{\sigma})}$ where

$$H_\Lambda(\underline{\sigma}) = J \sum_{\substack{\langle i,j\rangle \\ i \neq j}} (\sigma_i \sigma_j + 1) - h \sum_i \sigma_i + J \sum_{\substack{\langle i,j\rangle \\ i \in \Lambda \\ j \notin \Lambda}} (\sigma_i \tau_j + 1) \tag{0.1}$$

τ_j belongs to the first external layer, J is pair interaction, h is an external magnetic field, β is the reciprocal temperature. A boundary condition for the system in the box Λ is specified by giving a probability distribution $P_\Lambda(\underline{\tau})$ for the boundary configuration $\underline{\tau}$.

An (equilibrium) state of the infinite system is defined to be a family of correlation functions $\langle \sigma_s \rangle_{h,\beta}$, for the finite subset S of $\mathbb{Z}^2$, obtained as suitable thermodynamical limit of

$$\langle \sigma_s \rangle_{\Lambda,h,\beta,P_\Lambda} \equiv \Big\langle \prod_{x_i \in S} \sigma_{x_i} \Big\rangle_{\Lambda,h,\beta,P_\Lambda} \quad \text{with} \quad S \subseteqq \Lambda \quad \text{and}$$

$$\langle \sigma_S \rangle_{\Lambda,h,\beta,P_\Lambda} = \sum_{\underline{\tau}} P_\Lambda(\underline{\tau}) \, \langle \sigma_S \rangle_{\Lambda,\beta,h,\underline{\tau}}$$

i.e. of spin-correlation functions for a sequence of finite boxes with some boundary condition P_Λ.

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The study of the limiting properties of $\langle\sigma_S\rangle_{\Lambda,h,\beta,p_\Lambda}$ for ferromagnetic interaction has been done so far using Griffith's inequalities as an essential tool. In the antiferromagnetic case such inequalities (or their generalizations [1]) do not seem useful to us.

Our method will be rather different and will consist in expressing the spin correlation functions in terms of 'outer contours correlation functions'.

In Sec. I we will use only 'closed' boundary conditions: $P_\Lambda(\underline{\tau}) \equiv \Sigma_1$ defined by putting $\tau = +1(-1)$ for all spin in even (odd) sites; Σ_2 the reversed condition.

For any given spin configuration $\{\underline{\sigma}\}$ in Λ we draw all the unit segments which separate nearest neighbours with equal spin; we find then a set of compatible[1] self-avoiding lines, i.e. closed contours[2] Among the contours associated to $\{\underline{\sigma}\}$ we call 'outer' those which can be connected to the boundary of Λ by a broken line without crossing other contours.

Then we define equations for outer contours correlation functions in an external magnetic field, for antiferromagnetic interaction. The equations are similar to the ones used by Minlos and Sinai [3] in the ferromagnetic case with non-zero field, but they are different in some essential aspect, in such a way that an argument of Dobruschin [6] for the existence of the antiferromagnetic phases can be used to estimate the magnitude of the kernel of the equations.

In Sec. 2 we prove the uniqueness of the translation invariant state for β large enough and h fixed and small.

The proof follows the outline of [2] but in a different context.

Section 1: Outer Contours Correlation Functions Equations for the Ising Antiferromagnet in an External Field

Let Λ be a rectangular box in a two dimensional lattice $\mathbb{Z}^2$ with closed condition Σ_1 or Σ_2. By (0.1) the energy of the configuration $\{\underline{\sigma}\}$ to which is associated a set of closed contours $\{\gamma\}$ is

$$H_\Lambda(\underline{\sigma}) = 2J \sum_\gamma |\gamma| - h(N^+ - N^-) \tag{1.1}$$

where

$$|\gamma| = \text{length of } \gamma$$

$$N^\pm = \text{number of spins } \pm \text{ in } \{\underline{\sigma}\} .$$

[1] Two different contours are compatible if they have in common at most corners.
[2] For a much more extended discussion of this point see [5].

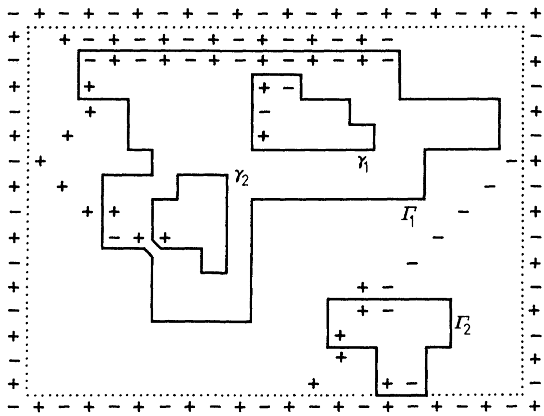

Fig. 1

Moreover, if $\{\underline{\sigma}\}$ in Λ has $X=\{\Gamma_1, \Gamma_2 \dots \Gamma_n\}$ as a set of outer contours, (1.1) can be written:

$$H_\Lambda(\underline{\sigma}) = 2J \sum_{\Gamma\in X} |\Gamma| + 2J \sum_{\Gamma\in X}\sum_{\gamma\subset\Gamma} |\gamma| - h(N^+ - N^-) \tag{1.2}$$

where $\gamma\subset\Gamma$ means that the enclosed regions $\theta(\gamma), \theta(\Gamma)$ are such that $\theta(\gamma)\subset\theta(\Gamma)$ (see Figure)

Using the equivalence relation between "lattice gas" and spin system (cfr. [8]) we are able to find the geometrical identity:

$$(N^+ - N^-) = \sum_{\Gamma\in X} (|\Gamma|^+ - \tfrac{1}{2}|\Gamma|) + \sum_{\Gamma\in X}\sum_{\gamma\subset\Gamma} (|\gamma|^+ - \tfrac{1}{2}|\gamma|) - \tfrac{1}{2}(|\Gamma_\Lambda|^+ - \tfrac{1}{2}|\Gamma_\Lambda|) \tag{1.3}$$

where

$|\Gamma_\Lambda|$ = length of the boundary Γ_Λ

$|\Gamma_\Lambda|^+$ number of spins $\tau=+1$ along Γ_Λ

$|\cdot|^+$ = length of the subset of segments of the contour $(\cdot)$ which separate pairs of positive nearest neighbours.

The last term in (1.3) is a constant which depends only on the region Λ. So inserting (1.3) in (1,2) we find:

$$H_\Lambda(\underline{\sigma}) = \sum_{\Gamma\in X}\Big[|\Gamma|\,(2J - h\eta(\Gamma)) + \sum_{\gamma\subset\Gamma} |\gamma|\,(2J - h\eta(\gamma))\Big] + \text{const.}(\Lambda) \tag{1.4}$$

where

$$\eta(\gamma) = \frac{|\gamma|^+}{|\gamma|} - \frac{1}{2} \quad \text{and} \quad -\frac{1}{2} \leqq \eta(\gamma) \leqq \frac{1}{2}\,.$$

Notice that $\eta(\gamma)$ depends also on Γ and on the number of contours which contain γ. We will always take into account this dependence only implicitly, in order not to burden the notation.

We shall now write down explicit expressions for outer contour correlation functions and then, derive the equations for such functions by a slight (but, for our purposes, crucial) modification of the method of Minlos and Sinai.

Let $\mathscr{P}_\Lambda(X)$ be the probability for a spin configuration in Λ for which the outer contours are $\Gamma_1, \Gamma_2 \ldots \Gamma_n$.

We can write:

$$\mathscr{P}_\Lambda(X) = \frac{\exp\left(-\beta \sum_{\Gamma \in X} (2J - h\eta(\Gamma))\right) \zeta(\theta(X))}{Z(\Lambda)} \equiv \frac{\tilde{\zeta}(\theta(X))}{Z(\Lambda)} \tag{1.5}$$

where $Z(\Lambda) = \sum_X \exp\left(-\beta \sum_{\Gamma \in X} |\Gamma| (2J - h\eta(\Gamma))\right) \zeta(\theta(X))$ is the partition function for Λ and

$$\zeta(\theta(X)) = \prod_{i=1}^{n} \sum_{\substack{\{\gamma\} \\ :\theta(\gamma) \subset \theta(\Gamma_i) \\ \gamma \cap \Gamma_i = \phi}} \exp\left[-\beta \sum_{\gamma \in \{\gamma\}} |\gamma| (2J - h\eta(\gamma))\right] \tag{1.6}$$

The correlation function for the set X is precisely by definition:

$$\varrho_\Lambda(X) = \sum_Y \frac{\tilde{\zeta}(\theta(X \cup Y))}{Z(\Lambda)} = \sum_Y \frac{\tilde{\zeta}(\theta(X)) \cdot \tilde{\zeta}(\theta(Y))}{Z(\Lambda)} \quad \text{(using 1, 4)} \tag{1.7}$$

where the sum is over the set Y of outer contours such that $X \cup Y$ is a collection of outer compatible contours.

Definition. Let $\mathscr{B}_\Lambda(X)$ be the collection of spin configurations in Λ such that $X = \{\Gamma_1, \Gamma_2 \ldots \Gamma_n\}$ is a subset of the set of contours.

The set $\{\Gamma_2 \ldots \Gamma_n\}$ will be denoted by $X^{(1)}$ and $\mathscr{B}_\Lambda^{\Gamma_1}(X^{(1)})$ will be the subset of elements of $\mathscr{B}_\Lambda(X)$ such that $X^{(1)}$ is a subset of outer and that no outer contour "intersects" or embraces[3] the curve Γ_1.

We say that Γ is "intersecting the curve Γ_1" when Γ crosses Γ_1 or, when $\Gamma \cap \Gamma_1 \neq \phi$ and $\theta(\Gamma) \cap \theta(\Gamma_1) = \phi$.

Notice that our set $\mathscr{B}_\Lambda^{\Gamma_1}(X^{(1)})$ differs from the corresponding one in Ref. [3] which does not include the configurations containing contours which intersect the curve Γ_1 and lie inside $\theta(\Gamma_1)$.

This point will make our result different from that of Minlos and Sinai. Let now

$$\varrho_\Lambda^{\Gamma_1}(X^{(1)}) \equiv \mathscr{P}_\Lambda(\mathscr{B}_\Lambda^{\Gamma_1}(X^{(1)})) = \sum_Y \frac{\zeta^0(\theta(\Gamma_1)) \cdot \tilde{\zeta}(\theta(X^{(1)} \cup Y))}{Z(\Lambda)} \tag{1.8}$$

[3] Among the embracing contours there is Γ_1 itself.

where Y is just the same as above and $\zeta^0(\theta(\Gamma_1))$ is given by the contribution of all the spin-configurations in $\theta(\Gamma_1)$ for which no outer contour coincide with the curve Γ_1.

In addition since $X=\Gamma_1\cup X^{(1)}$, (1.7) can be written:

$$\varrho_\Lambda(X)=\sum_Y \frac{\tilde{\zeta}(\theta(\Gamma_1))\cdot\tilde{\zeta}(\theta(X^{(1)}\cup Y))}{Z(\Lambda)}. \tag{1.9}$$

Comparing (1.8) and (1.9) we obtain:

$$\varrho_\Lambda(X)=\nu(\Gamma_1)\,\varrho_\Lambda^{\Gamma_1}(X^{(1)}) \tag{1.10}$$

where

$$\nu(\Gamma_1)=\frac{\tilde{\zeta}(\theta(\Gamma_1))}{\zeta^0(\theta(\Gamma_1))}. \tag{1.11}$$

Now we shall write down (following Minlos and Sinai [3]) the equations for the outer contours correlation functions.

Let $X''=$ set of outer contours 'intersecting' the curve Γ_1 and such that $\theta(X'')\subset\Lambda$

$X'=$ set of outer contours embracing the curve Γ_1 and such that $\theta(X')\subset\Lambda$.

From the previous definition it follows:

$$\mathscr{B}_\Lambda^{\Gamma_1}(X^{(1)})=\mathscr{B}_\Lambda(X^{(1)})-\bigcup_{x''\in X''}\mathscr{B}_\Lambda(X^{(1)}\cup x'')-\bigcup_{x'\in X'}\mathscr{B}_\Lambda(X^{(1)}\cup x') \tag{1.12}$$

Then, since the elements of X'' are not mutually incompatible, for the related probabilities we have:

$$\varrho_\Lambda^{\Gamma_1}(X^{(1)})=\varrho_\Lambda(X^{(1)})+\sum_{k=1}\sum_{\substack{T:N(T)=k\\ T\subset X''}}(-1)^k\varrho_\Lambda(X^{(1)}\cup T)-\sum_{x'\in X'}\varrho_\Lambda(X^{(1)}\cup x')$$

This relation together with (1.10) gives

$$\varrho_\Lambda(X)=\nu(\Gamma_1)\Big[\varrho_\Lambda(X^{(1)})+\sum_{k=1}\sum_{\substack{T:N(T)=k\\ T\subset X''}}(-1)^k\varrho_\Lambda(X^{(1)}\cup T)-\sum_{x'\in X'}\varrho_\Lambda(X^{(1)}\cup x')\Big] \tag{1.14}$$

We will write for an infinite box:

$$\varrho(x)=\nu(\Gamma_1)\Big[\varrho(X^{(1)})+\sum_{k=1}\sum_{\substack{T:N(T)=k\\ T\subset X''}}(-1)^k\varrho(X^{(1)}\cup T)-\sum_{x'\in X'}\varrho(X^{(1)}\cup x')\Big] \tag{1.15}$$

where now X, X', X'' are in an infinite box.

Relation (1.15) can be written as an integral equation

$$\varrho = \alpha + A\varrho \tag{1.16}$$

in a Banach space $\mathfrak{M}$ whose elements are infinite sequences:

$$f = \{f_1(\Gamma_1), f_2(\Gamma_1\Gamma_2) \ldots f_k(\Gamma_1, \Gamma_2 \ldots \Gamma_k) \ldots\}$$

with the following norm:

$$\|f\| = \sup_X \frac{|f(X)|}{\zeta^{|X|}(e^{-\beta t})^{|X|}}$$

where ζ denote a constant greater then 1 and $t = 2J - \frac{h}{2}$.

In (1.16)

$$\begin{aligned} \alpha(X) &= 0 \qquad \text{if} \quad N(X) > 1 \\ &= \nu(\Gamma_1) \quad \text{if} \quad N(X) = 1 \quad \text{and} \quad X = \{\Gamma_1\} \end{aligned}$$

and A acts in the following manner on the generic element f

$$\begin{aligned} (Af)(X) = \nu(\Gamma_1) \Bigg[f(X^{(1)}) + \sum_{k=1} \sum_{\substack{T: N(T)=k \\ T \subset X''}} (-1)^k f(X^{(1)} \cup T) \\ - \sum_{x' \in X'} f(X^{(1)} \cup x') \Bigg]^4 \end{aligned} \tag{1.17}$$

Observe that in the kernel of (1.16) the term $\nu(\Gamma_1)$ is slightly different from the corresponding one in the equations of Minlos and Sinai [3].

Moreover in their case ($J < 0$ and $h \neq 0$) it is rather hard to construct an upper bound for the kernel.

In Appendix I we will show that in our case ($J > 0$ and $h \neq 0$) for our modified kernel the use of the $\mathfrak{T}$ transformation (introduced by Dobruschin [6]) gives us quite easily a good estimate, which then ensures, in a standard way, the right convergence properties for the solutions of (1.16).

Section 2: Uniqueness of the Translationally Invariant State

Here we prove our main result.

First let us give a basic result due to Lanford and Ruelle [7].

Proposition I. *If* $\langle \sigma_{x_1} \ldots \sigma_{x_s} \rangle_{\beta,h} \equiv \langle \sigma_S \rangle_{\beta,h}$ *is a translation invariant state, then one can find a suitable sequence of* $P_\Lambda(\underline{\tau})$ *such that*

$$\langle \sigma_S \rangle_{\beta,h} = \lim_{\Lambda \to \infty} \overline{\langle \sigma_S \rangle}_{P_\Lambda, \beta, h} \tag{2.1}$$

[4] If $X = \{\Gamma_1\}$ the term $X' = \phi$ and $T = \phi$ is missing in the sum because it is included in α.

where

$$\overline{\langle\sigma_S\rangle}_{P_\Lambda,\beta,h} = \sum_{\underline{\tau}} P_\Lambda(\underline{\tau})\,\overline{\langle\sigma_S\rangle}_{\underline{\tau},\beta,h}$$

and

$$\overline{\langle\sigma_S\rangle}_{\underline{\tau},\beta,h} = \frac{1}{|\Lambda|} \sum_{\substack{a\in\mathbb{Z}^2\\ :S+a\subset\Lambda}} \langle\sigma_{x_1+a}\ldots\sigma_{x_s+a}\rangle_{\underline{\tau},\beta,h} \tag{2.2}$$

We are now ready to formulate the theorem:

Theorem: *Let $\langle\sigma_S\rangle^*_{\beta,h}$ be an arbitrary translationally invariant state on $\mathbb{Z}^2$ and $P^*_\Lambda(\underline{\tau})$ a sequence of boundary conditions such that*

$$\langle\sigma_S\rangle^*_{\beta,h} = \lim_{\Lambda\to\infty} \overline{\langle\sigma_S\rangle}_{P^*_\Lambda,\beta,h}$$

then if h is fixed and β is large enough, the following relation holds:

$$\begin{aligned}\overline{\langle\sigma_S\rangle}_{P^*_\Lambda,\beta,h} &= \alpha(P^*_\Lambda,\beta,h)\,\langle\sigma_S\rangle_{\Sigma_1,\beta,h} + (1-\alpha(P^*_\Lambda,\beta,h))\,\langle\sigma_S\rangle_{\Sigma_2,\beta,h}\\ &\quad + \mu(\Lambda,\beta,h,P^*_\Lambda)\end{aligned}$$

where $\mu(\Lambda,\beta,h,P^*_\Lambda)$ decreases to zero as Λ goes to infinity and $\lim_{\Lambda\to\infty}\alpha(P^*_\Lambda,\beta,h)=\frac{1}{2}$

Proof. Observe first that for any fixed $\underline{\tau}$, each spin-configuration $\{\underline{\sigma}\}$ in Λ will be associated to a set of closed contours $\{\gamma\}$ and a set of lines (open contours): $(\lambda_1,\lambda_2,\ldots,\lambda_k)$ which begin and end on the boundary, dividing the box Λ into disjoint regions $\theta_1\ldots\theta_p$; $(p\leq k+1)$ each with closed boundary condition Σ_1 or Σ_2.

The energy for $\{\underline{\sigma}\}$ will now be

$$H_\Lambda(\underline{\sigma}) = \sum_\gamma |\gamma|\,(2J-h\eta(\gamma)) + \sum_\lambda |\lambda|\,(2J-h\eta(\lambda)) + \text{const.} \tag{2.3}$$

Notice that $\eta(\lambda)$ depends only on λ and on the boundary condition $\underline{\tau}$ whereas $\eta(\gamma)$ depends also on the boundary condition $\Sigma_i (i=1,2)$ relative to θ_g $(g\in(1\ldots p))$ to which γ belongs, and on the number of countours which contain γ.

As in Sec. 1 we will always take into account this dependence only implicitly.

Moreover a simple algebraic calculation based on the defining relation

$$\langle\sigma_S\rangle_{\underline{\tau},\beta,h} = \sum_{\lambda_1\ldots\lambda_k}\sum_{\gamma_1\ldots\gamma_n} \cdot\frac{(\sigma_S)\exp\left[-\beta\sum_i|\lambda_i|\,(2J-h\eta(\lambda_i)) - \beta\sum_j|\gamma_j|\,(2J-h\eta(\gamma_j))\right]}{Z_{\underline{\tau}}(\Lambda)} \tag{2.4}$$

(where $Z_{\underline{\tau}}(\Lambda) = \sum_{\lambda_1\ldots\lambda_k}\sum_{\gamma_1\ldots\gamma_n}$

$$\exp\left[-\beta\sum_i|\lambda_i|\,(2J-h\eta(\lambda_i)) - \beta\sum_j|\gamma_j|\,(2J-h\eta(\gamma_j))\right] \tag{2.5}$$

is the partition function for the box Λ with boundary condition $\underline{\tau}$) and relations (2.1) and (2.2), provides the following expression:

$$\overline{\langle\sigma_S\rangle}_{P^*_\Lambda,\beta,h} = \sum_{\underline{\tau}} P^*_\Lambda(\underline{\tau}) \frac{1}{|\Lambda|} \sum_{\substack{a\in\mathbb{Z}^2\\ :S+a\subset\Lambda}} \sum_{\lambda_1\ldots\lambda_k} P_\Lambda(\lambda_1 \ldots \lambda_k) \prod_{i=1}^{p} \langle\sigma_{S\cap\theta_i+a}\rangle_{\theta_i,\Sigma_i,\beta,h} \tag{2.6}$$

where $\bigcup_{i=1}^{p} (S\cap\theta_i) = S$; $\Sigma_i = \Sigma_1$ or Σ_2 $|\Lambda| =$ volume of Λ and

$$P_\Lambda(\lambda_1 \ldots \lambda_k) = \sum_{\gamma_1\ldots\gamma_n} \frac{\exp\left[-\beta\sum_i |\lambda_i|\,(2J-h\eta(\lambda_i)) - \beta\sum_j |\gamma_j|\,(2J-h\eta(\gamma_j))\right]}{Z_{\underline{\tau}}(\Lambda)} \tag{2.7}$$

Our proof needs some preliminary statements:

Lemma 1. *Given a boundary condition $\underline{\tau}$ on the box Λ with volume $|\Lambda| = L^2$ the probability of having a set of open contours $\lambda_1\lambda_2 \ldots \lambda_k$ such that $\sum_i |\lambda_i| \geqq \omega L$ with β and h such that $\beta(2J-\frac{h}{2}) \geqq \ln 7$ and $\omega > 128 + 27h\beta$ is less than $\varepsilon(L)$ where $\varepsilon(L)$ decreases to zero as L goes to infinity.*

i.e.

$$P_\Lambda(\underline{\tau}, \omega L) \equiv \sum_{\substack{\lambda_1\ldots\lambda_k\\ :\Sigma|\lambda_i|\geqq\omega L}} P_\Lambda(\lambda_1 \ldots \lambda_k) \leqq \varepsilon(L)\,.$$

Lemma 2. *For an arbitrary simply connected region D with closed boundary condition Σ_1 or Σ_2, and for β large enough the following relation holds*

$$|\langle\sigma_S\rangle_{D,\Sigma_1,\beta,h} - \langle\sigma_S\rangle_{\Sigma_1,\beta,h}| \leqq f(S, \sqrt{|D|}, \beta, h)$$

where $f(S, \sqrt{|D|}, \beta, h)$ is translationally invariant and decreases to zero as $\sqrt{|D|}$ goes to infinity.

Proofs of Lemma 1 and Lemma 2 are given in Appendix II.

Using Lemma 1 the relation (2.6) can be written

$$\overline{\langle\sigma_S\rangle}_{P^*_\Lambda,\beta,h} = \sum_{\underline{\tau}} P^*_\Lambda(\underline{\tau}) \frac{1}{|\Lambda|} \sum_{\substack{a\in\mathbb{Z}^2\\ :S+a\subset\Lambda}} \sum_{\substack{\lambda_1\ldots\lambda_k\\ :\Sigma|\lambda_i|<\omega L}} P_\Lambda(\lambda_1 \ldots \lambda_k) \cdot \prod_{i=1}^{p} \langle\sigma_{S\cap\theta_i+a}\rangle_{\theta_i,\Sigma_i,\beta,h} + c_1\varepsilon(L) \tag{2.8}$$

where c_1 is a constant.

Let $A_{\lambda_1\ldots\lambda_k}$ be the set of points at distance not exceeding $\frac{1}{2}L^{1/3}$ from the set of contours $\lambda_1 \ldots \lambda_k$.

The fraction of points in $A_{\lambda_1\ldots\lambda_k}$ is not larger than $\dfrac{\omega L^{4/3}}{L^2}$ if $\Sigma|\lambda_i| < \omega L$.

So the fraction of translations which bring some point of the set S in $A_{\lambda_1 \dots \lambda_k}$ is less or equal to $\frac{\omega L^{4/3}}{L^2}$.

Furthermore if L is big enough, the remaining translations (i.e. such that $(S+a) \cap A_{\lambda_1 \dots \lambda_k} = \phi$) will bring the entire set S in the interior of one of the regions $\theta_i (i = 1 \dots p)$.

Let $\mathbb{Z}_i^2$ be the set of translations which take S into the region θ_i, we shall have therefore:

$$\frac{1}{|\Lambda|} \sum_{\substack{a \in \mathbb{Z}^2 \\ :S+a \subset \Lambda \\ S+a \cap A_{\lambda_1 \dots \lambda_k} = \phi}} = \frac{1}{|\Lambda|} \sum_{i=1}^{p} \sum_{\substack{a \in \mathbb{Z}^2 \\ :S+a \subset \theta_i}}$$

Observe now that for L big enough we can write

$$\sum_{a \in \mathbb{Z}_i^2} \langle \sigma_{S+a} \rangle_{\theta_i, \Sigma_i, \beta, h} = \sum_{a \in (\mathbb{Z}_i^2)_{\text{even}}} (\langle \sigma_{S+a} \rangle_{\theta_i, \Sigma_i, \beta, h} + \langle \sigma_{S+a+\underline{e}} \rangle_{\theta_i, \Sigma_i, \beta, h}) \quad (2.9)$$

where $(Z_i^2)_{\text{even}}$ is the subset of even translations in $\mathbb{Z}_i^2$ and $\underline{e}$ is a unit translation.

Relation (2.8) can now be written

$$\overline{\langle \sigma_S \rangle}_{P_\Lambda^*, \beta, h} \leqq \frac{1}{|\Lambda|} \sum_{\underline{\tau}} P_\Lambda^*(\underline{\tau}) \sum_{\substack{\lambda_1 \dots \lambda_k \\ :\Sigma |\lambda_i| < \omega L}} P_\Lambda(\lambda_1 \dots \lambda_k) \sum_{i=1}^{p}$$
$$\cdot \sum_{a \in (\mathbb{Z}_i^2)_{\text{even}}} (\langle \sigma_{S+a} \rangle_{\theta_i, \Sigma_i, \beta, h} + \langle \sigma_{S+a+\underline{e}} \rangle_{\theta_i, \Sigma_i, \beta, h}) + c_1 \varepsilon(L) + \frac{\omega L^{4/3}}{L^2} \quad (2.10)$$

Also by Lemma 2 and the relation (AII, 19) of Appendix II we have

$$\langle \sigma_{S+a} \rangle_{\theta_i \Sigma_i \beta, h} + \langle \sigma_{S+a+\underline{e}} \rangle_{\theta_i, \Sigma_i, \beta, h} \leqq \langle \sigma_{S+a} \rangle_{\Sigma_1, \beta, h} + \langle \sigma_{S+a} \rangle_{\Sigma_2, \beta, h} + 2f(S+a, \beta, h, \theta_i)$$

Finally using the translation invariance of $\langle \sigma_S \rangle_{\Sigma_1, \beta, h} + \langle \sigma_S \rangle_{\Sigma_2, \beta, h}$ and of $f(s, \beta, h, \theta)$ together with

$$\lim_{L \to \infty} \varepsilon(L) = \lim_{\sqrt{\theta} \to \infty} f(S, \beta, h, \theta) = 0$$

$$\lim_{L \to \infty} \frac{1}{|\Lambda|} \sum_{i=1}^{p} \sum_{a \in (\mathbb{Z}_i^2)_{\text{even}}} = \frac{1}{2}$$

and $\sum_{\underline{\tau}} P_\Lambda^*(\underline{\tau}) = 1$ for every finite box Λ, we have:

$$\langle \sigma_S \rangle_{\beta, h}^* = \lim_{\Lambda \to \infty} \overline{\langle \sigma_S \rangle}_{P_\Lambda^*, \beta, h} = \tfrac{1}{2} \langle \sigma_S \rangle_{\Sigma_1, \beta, h} + \tfrac{1}{2} \langle \sigma_S \rangle_{\Sigma_2, \beta, h} \cdot \qquad \text{Q.E.D.}$$

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Appendix I

We give now an upper bound for

$$\nu(\Gamma_1) = \frac{\tilde{\zeta}(\theta(\Gamma_1))}{\zeta^0(\theta(\Gamma_1))} \tag{AI.1}$$

and for the kernel of (1.16). We then show how these imply in a standard way the right convergence properties for the solutions of the equations for outer contours correlation functions.

Let Λ_1 be the union of $\theta(\Gamma_1)$ and its first external layer. Let $\mathscr{K}_{\Lambda_1}$, be defined as the set of spin-configurations in Λ_1 such that Γ_1 is an outer contour.

If $k \in \mathscr{K}_{\Lambda_1}$, using (1.4), we can write:

$$H_{\Lambda_1}(k) = |\Gamma_1|\,(2J - h\eta(\Gamma_1)) + \sum_{\gamma \in k} |\gamma|\,(2J - h\eta(\gamma)) \tag{AI.2}$$

Let also $\bar{k} = \mathfrak{T}_{\Lambda_1} k$ i.e. $\{\bar{\sigma}\} = \mathfrak{T}_{\Lambda_1}\{\sigma\}$ *where the Dobruschin transformation* $\mathfrak{T}_{\Lambda_1}$ [6] is defined by

a) $\bar{\sigma}_i = \sigma_i$ if $i \notin \theta(\Gamma_1)$

b) $\bar{\sigma}_i = \sigma_{ji}$ if $i \in \theta(\Gamma_1)$ and $j_i \in \theta(\Gamma_1)$ where j_i (AI.3)

is the neighbour of i from below

c) $\bar{\sigma}_i = -\sigma_{j_i}$ if $i \in \theta(\Gamma_1)$ and $j_i \notin \theta(\Gamma_1)$.

Since $\mathfrak{T}_{\Lambda_1}$ deletes the contour Γ_1 and rises upward by one step the inner contours we have:

$$H_{\Lambda_1}(k) = |\Gamma_1|\,(2J - h\eta(\Gamma_1)) + H_{\Lambda_1}(\mathfrak{T}_{\Lambda_1} k) \tag{AI.4}$$

Remark now that $\mathfrak{T}_{\Lambda_1}$ is a one-to-one mapping between $\mathscr{K}_{\Lambda_1}$ and a subset of the configurations contributing to $\zeta^0(\theta(\Gamma_1))$. Replacing in (AI.1) $\zeta^0(\theta(\Gamma_1))$ by a smaller number we have for $\nu(\Gamma_1)$ the following upper bound:

$$\nu(\Gamma_1) \leqq \frac{\sum_{k \in \mathscr{K}_{\Lambda_1}} \exp(-\beta H_{\Lambda_1}(k))}{\sum_{k \in \mathscr{K}_{\Lambda_1}} \exp(-\beta H_{\Lambda_1}(\mathfrak{T}_{\Lambda_1} k))} \tag{AI.5}$$

and since $\mathfrak{T}_{\Lambda_1}$ is one-to-one

$$\nu(\Gamma_1) \leqq \exp(-\beta|\Gamma_1|\,(2J - h\eta(\Gamma_1)) \leqq \exp(-\beta t\,|\Gamma_1|) \tag{AI.6}$$

where

$$t = 2J - \frac{h}{2}.$$

Inequality (AI.6) gives us a good estimate for the norm of A in (1.16).

In fact let us equip $\mathfrak{M}$ with the following norm

$$\|f\| = \sup_X \frac{|f(X)|}{\zeta^{|X|}(e^{-\beta t})^{|X|}} \tag{AI.7}$$

where ζ denotes a constant greater than 1.

Then by standard methods (cfr. [4]) one obtains that $\|A\| < 1$ for

$$\zeta \geqq \tfrac{3}{2} \cdot 2^{1/4} \quad \text{and} \quad 3e^{-\beta t} \leqq \tfrac{1}{3} \cdot 2^{1/4}\,. \tag{AI.8}$$

These conditions assure that the series which gives the iterated solutions of (1.16) i.e.

$$\varrho(X) = \left(\sum_{k=0}^{\infty} A^k \alpha\right)(X) \tag{AI.9}$$

is convergent and satisfies the inequality

$$|\varrho(X)| \leqq \text{const.} \cdot (\zeta e^{-\beta t})^{|X|} \tag{AI.10}$$

(which is obvious for finite volume correlation functions). Similar properties hold for the kernel of the finite volume equation:

$$\varrho_\Lambda = \chi_\Lambda \alpha + \chi_\Lambda A \varrho_\Lambda \tag{AI.11}$$

where χ_Λ is such that

$$\begin{aligned} (\chi_\Lambda f)(X) &= 0 \qquad \text{for} \quad X \not\subset \Lambda \\ &= f(X) \quad \text{for} \quad X \subset \Lambda\,. \end{aligned}$$

Finally, we now give a sketch of how inequality (AI.6) can be used to prove the following result (cfr. [4]).

Theorem: *Let Λ be an arbitrary region on the lattice $\mathbb{Z}^2$, then, for an arbitrary collection of outer contours X enclosed in Λ, the hypotheses* (AI.8) *assure that*

$$|\delta(X)| \equiv |\chi_\Lambda \varrho(X) - \varrho_\Lambda(X)| \leqq \text{const.}\,(\zeta e^{-\beta t})^{|X|} \cdot (3\zeta e^{-\beta t})^{\tau(X,\Gamma_\Lambda)} \tag{AI.12}$$

where $\tau(X, \Gamma_\Lambda) = \min_{\Gamma_i \in X}.\, d(\Gamma_i, \Gamma_\Lambda)$ and $d(\Gamma_i, \Gamma_\Lambda)$ = distance from the boundary Γ_Λ of the outer contour Γ_i.

In fact acting with χ_Λ on (1.16) and inserting the terms $\pm \chi_\Lambda A \chi_\Lambda \varrho$ one has:

$$\chi_\Lambda \varrho = \chi_\Lambda \alpha + \chi_\Lambda A \chi_\Lambda \varrho + \chi_\Lambda A(\varrho - \chi_\Lambda \varrho) \tag{AI.13}$$

then subtracting relation (AI.11) from the last one, we have:

$$\begin{gathered} \delta = \xi + \chi_\Lambda A \delta \\ \text{where } \delta = \chi_\Lambda \varrho - \varrho_\Lambda \quad \text{and} \quad \xi = \chi_\Lambda A(\varrho - \chi_\Lambda \varrho)\,. \end{gathered} \tag{AI.14}$$

It is a matter of standard manipulation to show (cfr. [4]) how inequality (AI.8) implies that

$$|\xi(X)| \leqq c(\zeta e^{-\beta t})^{|X|} (3\zeta e^{-\beta t})^{\tau(X, \Gamma_\Lambda)} \tag{AI.15}$$

where c is a constant which is the same for all components of ξ. The final results follows if one shows that the series $\delta = \sum_k A^k \xi$ converges and that the vector δ satisfies the relation (AI.12). But this is an immediate consequence of the following Lemma (cfr. [4]).

Lemma. *Let Λ be some set of lattice points and let the vector $\xi \in \mathfrak{M}$ satisfy the condition* (AI.15) *for each set of outer contours X.*

Then the vector $\xi' = A\xi$ for ζ and β which satisfy conditions (AI.8), *is such that*

$$|\xi'(X)| \equiv |(A\xi)(X)| \leqq Bc(\zeta e^{-\beta t})^{|X|} (3\zeta e^{-\beta t})^{\tau(X, \Gamma_\Lambda)}$$

for any set X, where $B = const. < 1$.

This concludes the sketch.

Appendix II

Here we prove Lemma 1 and Lemma 2.

Let us begin with the proof of Lemma 1.

Observe first that since the open contours $\lambda_1, \lambda_2 \dots \lambda_k$ divide the box Λ into disjoint regions $\theta_1 \dots \theta_p$ $(p \leqq k+1)$ with closed boundary conditions, we can write

$$P_\Lambda(\underline{\tau}, \omega L) = \sum_{\substack{\lambda_1 \dots \lambda_k \\ : \sum |\lambda_i| \geqq \omega L}} \sum_{\{\Gamma\}_1 \dots \{\Gamma\}_p} \frac{\exp\left[-\beta \sum_i |\lambda_i| (2J - h\eta(\lambda_i))\right] \cdot \prod_{q=1}^{p} \tilde{\zeta}(\theta(\Gamma_{1q})) \dots \tilde{\zeta}(\theta(\Gamma_{nq}))}{Z_{\underline{\tau}}(\Lambda)}$$

$$\equiv \sum_{\substack{\lambda_1 \dots \lambda_k \\ : \sum |\lambda_i| \geqq \omega L}} \sum_{\{\Gamma\}_1 \dots \{\Gamma\}_p} \frac{Z(\lambda_1 \dots \lambda_k \{\Gamma\}_1 \dots \{\Gamma\}_p)}{Z_{\underline{\tau}}(\Lambda)} \tag{AII.1}$$

$$= \sum_{\substack{\lambda_1 \dots \lambda_k \\ : \sum |\lambda_i| \geqq \omega L}} \frac{Z(\lambda_1 \dots \lambda_k)}{Z_{\underline{\tau}}(\Lambda)}$$

where $\{\Gamma\}_q$ is a set of outer contours in $\theta_q (q=1 \dots p)$ associated with a spin-configuration $\{\underline{\sigma}\}$ in Λ. $\tilde{\zeta}(\theta(\Gamma_{iq}))$ is defined as in the (1.5) and (1.6) for the region $\theta(\Gamma_{iq})$.

This relation defines $Z(\lambda_1 \dots \lambda_k, \{\Gamma\}_1 \dots \{\Gamma\}_p)$ and $Z(\lambda_1 \dots \lambda_k)$. $Z_{\underline{\tau}}(\Lambda)$ (previously defined in (2.5)), is the partition function for the region Λ with boundary condition $\underline{\tau}$.

A clear physical picture of what is behind the following formal proof can be obtained by observing that our goal is to give an upper bound to $\dfrac{Z(\lambda_1 \dots \lambda_k)}{Z_{\underline{\tau}}(\Lambda)}$ for every fixed set $\lambda_1 \dots \lambda_k$ such that $\Sigma|\lambda_i| \geqq \omega L$. One way to do so is by taking as denominator of $\dfrac{Z(\lambda_1 \dots \lambda_k, \{\Gamma\}_1 \dots \{\Gamma\}_p)}{Z_{\underline{\tau}}(\Lambda)}$ in place of $Z_{\underline{\tau}}(\Lambda)$ the term $Z(\bar{\lambda}_1, \bar{\lambda}_2 \dots \bar{\lambda}_k, \{\Gamma\}_1 \dots \{\Gamma\}_p)$ to which contributes only one set of open contours $\bar{\lambda}_1, \bar{\lambda}_2 \dots \bar{\lambda}_k$, lying along the boundary, and of closed ones $\{\Gamma\}_1 \dots \{\Gamma\}_p$ equal to those present at the numerator.

Obviously $Z(\bar{\lambda}_1 \dots \bar{\lambda}_k, \{\Gamma\}_1 \dots \{\Gamma\}_p) \leqq Z_{\tau}(\Lambda)$.

If we apply Dobruschin's transformation (AI.3) to $\{\sigma\}_{\theta_g \Sigma_1}$ (i.e. restriction to θ_g of the spin-configuration contributing to $Z(\lambda_1 \dots \lambda_k, \{\Gamma\}_1 \dots \{\Gamma\}_p)$ for each region $\theta_g (1 \leqq g \leqq p)$ which has boundary condition Σ_1, we bring the open contours $\lambda_1, \lambda_2 \dots \lambda_k$ to $\bar{\lambda}_1, \bar{\lambda}_2 \dots \bar{\lambda}_k$ and shift by one step in a suitable lattice direction (for ex. upward) the closed ones $\{\Gamma\}_g$ (and $\{\gamma\}_g$ inside $\{\Gamma\}_g$).

This procedure will be meaningful provided the shifted positions of $\{\Gamma\}_g$ are compatible with $\bar{\lambda}_1, \bar{\lambda}_2 \dots \bar{\lambda}_k$.

But this is not always the case, so we need a preliminary step. We must take away from $\{\underline{\sigma}\}$ contributing to $Z(\lambda_1 \dots \lambda_k, \Gamma_1 \dots \Gamma_s \Gamma'_{s+1} \dots \Gamma'_n)$ the closed contours $\{\Gamma'\}$ that touch the inner layer surrounding the boundary of Λ and then give an upper bound $f(\Gamma'_{s+1} \dots \Gamma'_n)$ for their contribution.

After that we shall write:

$$Z(\lambda_1 \dots \lambda_k, \Gamma_1 \dots \Gamma_s \Gamma'_{s+1} \dots \Gamma'_n) \leqq f(\Gamma'_{s+1} \dots \Gamma'_n) \cdot \bar{\bar{Z}}(\lambda_1 \dots \lambda_k, \{\Gamma\}) \quad \text{(AII.2)}$$

where now $\bar{\bar{Z}}$ depends on the closed outer contours $\{\Gamma\} = (\Gamma_1 \dots \Gamma_s$ and other new ones not reaching the inner layer).

Then

$$\begin{aligned} \frac{Z(\lambda_1 \dots \lambda_k)}{Z_{\underline{\tau}}(\Lambda)} &\leqq \sum_{(\Gamma_1 \dots \Gamma_s \Gamma'_{s+1} \dots \Gamma'_n)} \frac{f(\Gamma'_{s+1} \dots \Gamma'_n)\, \bar{\bar{Z}}(\lambda_1 \dots \lambda_k, \{\Gamma\})}{\bar{\bar{Z}}(\bar{\lambda}_1 \dots \bar{\lambda}_k, \{\Gamma\})} \\ &\leqq \sum_{(\Gamma'_{s+1} \dots \Gamma'_n)} \frac{f(\Gamma'_{s+1} \dots \Gamma'_n) \exp\left[-\beta \sum_i |\lambda_i| (2J - h\eta(\lambda_i))\right]}{\exp\left[-\beta \sum_i |\bar{\lambda}_i| (2J - h\eta(\bar{\lambda}_i))\right]} \end{aligned} \quad \text{(AII.3)}$$

putting $\eta(\bar{\lambda}_i) = -\frac{1}{2}$ and $\Sigma|\bar{\lambda}_i| = \varepsilon 4L$ with $0 < \varepsilon < 1$ we have

$$\leqq \sum_{(\Gamma'_{s+1} \dots \Gamma'_n)} f(\Gamma'_{s+1} \dots \Gamma'_n) \frac{\exp\left(-\beta t \sum_i |\lambda_i|\right)}{\exp[-\beta(t+h)\, 4L]} \tag{AII.4}$$

Let us now perform the preliminary step.

Let Λ_0 be the region Λ subtracted from the inner layer along the boundary Γ_Λ and L_0 the length of Γ_{Λ_0}. Obviously $L_0 < 4L$.

We call $\{\Gamma'\}$ the set of closed outer contour intersecting Γ_{Λ_0} [5]. From the defining relation (AII.1) we can write the $\{\Gamma\}$ dependence of

$$Z(\lambda_1 \dots \lambda_k, \Gamma_1 \dots \Gamma_s \Gamma'_{s+1} \dots \Gamma'_n) \quad \text{as} \quad \tilde{\zeta}(\theta(\Gamma_1)) \dots \tilde{\zeta}(\theta(\Gamma_s))\, \tilde{\zeta}(\theta(\Gamma'_{s+1})) \dots \tilde{\zeta}(\theta(\Gamma'_n)).$$

Let us also define on the spin-configurations $\mathscr{K}_{\theta(\Gamma')}$ contributing to $\tilde{\zeta}(\theta(\Gamma'))$ a suitable Dobruschin transformation $\mathfrak{T}'$ (for ex. to the right) which deletes contour Γ' and shifts by one step (to the right) any contour inside Γ'.

Then using the relations (AI.4) we obtain

$$\tilde{\zeta}(\theta(\Gamma')) \leqq \exp[-\beta|\Gamma'|\,(2J - h\eta(\Gamma'))] \sum_{k \in \mathscr{K}_{\theta(\Gamma')}} \exp[-\beta H_{\theta(\Gamma')}(\mathfrak{T}'_k)] \tag{AII.6}$$

We must remark that it may happen that some new outer contour Γ'' contributing to $\sum_{k \in \mathscr{K}_{\theta(\Gamma')}} \exp[-\beta H_{\theta(\Gamma')}(\mathfrak{T} k)]$ *can intersect* Γ_{Λ_0}. *In such* a case we can apply again a suitable $\mathfrak{T}''$ to the configurations $\{\mathfrak{T}'_k\}$.

It is clear that in this manner we take away from the spin-configurations contributing to $\tilde{\zeta}(\theta(\Gamma'))$ all the unwanted contours. So we must replace inequality (AII.6) with the much more general

$$\begin{aligned}\tilde{\zeta}(\theta(\Gamma')) \leqq \exp[-\beta|\Gamma'|\,(2J - h\eta(\Gamma')) - \beta|\Gamma''|\,(2J - h\eta(\Gamma''))]& \\ \sum_{k \in \mathscr{K}_{\theta(\Gamma')}} \exp[-\beta H_{\theta(\Gamma')}(\mathfrak{T}''\mathfrak{T}' k)]&\end{aligned} \tag{AII.7}$$

Such procedure will be repeated for all $\tilde{\zeta}(\theta(\Gamma'))$ each time with the proper Dobruschin transformations.

So we have

$$f(\Gamma'_{s+1} \dots \Gamma'_n) \leqq \exp[-\beta t \Sigma|\Gamma'| - \beta t \Sigma|\Gamma''|] \tag{AII.8}$$

where Γ'' are contours which in the primitive configuration where inside Γ'.

This is the end of the preliminary step.

Inserting result (AII.8) into the relation (AII.4) we-have:

$$\frac{Z(\lambda_1 \dots \lambda_k)}{Z_{\mathfrak{T}}(\Lambda)} \leqq \frac{\exp\left(-\beta t \sum_i |\lambda_i|\right)}{\exp(-\beta 4L(t+h))} \sum_{(\Gamma'_s+1 \dots \Gamma'_n, \{\Gamma''\})} \exp(-\beta t \Sigma|\Gamma'| - \beta t \Sigma|\Gamma''|) \tag{AII.9}$$

[5] Intersection, since Γ_{Λ_0} does not have inner vertex, cannot take place in corner points.

Observe that we have

$$\sum_{(\Gamma'_{s+1}\ \Gamma'_n,\{\Gamma''\})} e^{-\beta t\Sigma|\Gamma'|-\beta t\Sigma|\Gamma''|} \leqq \sum_{r=0}^{L_0} \binom{L_0}{r} \sum_{\gamma'\ni 0} e^{-\beta t|\gamma'|\,r} \tag{AII.10}$$

which together with the fact that the number of contours of length s passing through a fixed point does not exceed 3^S and that $\beta t > \ln 3$ gives:

$$\text{right side of (AII.10)} \leqq \sum_{r=0}^{L_0} \binom{L_0}{r} \left(\sum_{s=1}^{\infty} (3e^{-\beta t})^s \right)^r \leqq \left(\frac{1}{1-3e^{-\beta t}} \right)^{L_0}$$
$$\leqq \left(\frac{1}{1-3e^{-\beta t}} \right)^{4L}$$

$$\text{and for } \beta t \geqq \ln 4 \text{ gives: right side of (AII.10)} \leqq e^{4\beta t L}. \tag{AII.11}$$

Inserting (AII.11) and (AII.9) in (AII.1) we have

$$P_\Lambda(\underline{\tau}, \omega L) \leqq \exp[4\beta t L + 4L\beta(t+h)] \sum_{\substack{\lambda_1 \dots \lambda_k \\ :\Sigma|\lambda_i| \geqq \omega L}} \exp\left(-\beta t \sum_i |\lambda_i|\right) \tag{AII.12}$$

Remember that the number of ways of choosing k end points among the $2k$ which are possible for $\lambda_1, \lambda_2 \dots \lambda_k$ is $\binom{2k}{k} \leqq 2^{2k}$. So it follows:

$$\sum_{\substack{\lambda_1 \dots \lambda_k \\ :\Sigma|\lambda_i| \geqq \omega L}} e^{-\beta t \Sigma|\lambda_i|} \leqq 2^{2k} \sum_{l_1 \dots l_k} (3e^{-\beta t})^{l_1+l_2+\dots+l_k} \tag{AII.13}$$

Put $l_1 + l_2 + \dots + l_k = s$ then

$$\text{right side of (AII.13)} \leqq 2^{2k} \sum_{s=\omega L}^{\infty} (3e^{-\beta t})^s \binom{s-1}{k-1} \leqq 2^{2k} \sum_{s=\omega L}^{\infty} (3e^{-\beta t})^s\, 2^s$$

Finally if $\beta t \geqq \ln 7$ and $\omega \geqq 128 + 27\beta h$ since $2k < 4L$ we will find:

$$p_\Lambda(\underline{\tau}, \omega L) \leqq c e^{-\psi L} \equiv \varepsilon(L) \tag{AII.14}$$

where $\psi > 0$ for every $\underline{\tau}$ and c is a constant. Now we prove Lemma 2.

First remember the defining relation:

$$\langle \sigma_S \rangle_{D,\Sigma_1,\beta,h} = \sum_{\gamma_1 \dots \gamma_n} \frac{(\sigma_S) \exp\left[-\beta \sum_i |\gamma_i|\,(2J - h\eta(\gamma_i))\right]}{Z_{\Sigma_1}(\Lambda)} \tag{AII.15}$$

Now using, as in Sec. I, capital and small letters Γ, γ for outer and inner contours, by simple calculations one finds:[6]

$$\langle \sigma_S \rangle_{D,\Sigma_1,\beta,h} = \sum_{R \in \mathscr{P}(S)} \sum_{p=1}^{p_{\max}} \sum_{\substack{X:N(X)=p \\ :\theta(X)\cap S = S\backslash R}} \varrho_{R\,D\Sigma_1}(X) \prod_{\Gamma_i \in X} \langle \sigma_{S_i} \rangle_{\theta(\Gamma_i),\Sigma_i,h,\beta} (\sigma_R)_{\Sigma_1} \tag{AII.16}$$

[6] We omit the β, h dependence in outer contour correlation functions in order not to burden the notation.

where $X=\{\Gamma_1 \dots \Gamma_n\}$ with $\theta(\Gamma_i)\subset D$ $S_i=S\cap\theta(\Gamma_i)$; $R=S\setminus\bigcup_i S_i$ $\mathscr{P}(S)=$ parts of S; $p_{\max}\leqq N(S\setminus R)$; $(\sigma_R)_{\Sigma_1}=\prod_{x_i\in R}\sigma_{x_i}$ where σ_{x_i} is the spin value in x_i when it is outside every contour and Σ_1 is the boundary condition, and

$$\underset{D}{\varrho_R}(X)=\varrho_D(X)+\sum_{k=1}^{k_{\max}}\sum_{\substack{T:N(T)=k\\ T\subset X'}}(-1)^k\varrho_D(X\cup T) \tag{AII.17}$$

where $k_{\max}\leqq N(R)$ and $X'=\{\Gamma:\theta(\Gamma)\cap R\neq\phi, \theta(\Gamma)\subset D\}$. Observe that in (AII.17) as $R=S$ then $X=\{\phi\}$ and $\varrho_D(\phi)=1$ so (AII.17) becomes:

$$\underset{D}{\varrho_S}(X)=1+\sum_{k=1}^{k_{\max}}\sum_{\substack{T:N(T)=k\\ T\subset X'}}(-1)^k\varrho_D(\phi\cup T).$$

In similar way for an infinite box we can write:

$$\langle\sigma_S\rangle_{\Sigma_1,\beta,h}=\sum_{R\in\mathscr{P}(S)}\sum_{p=1}^{p_{\max}}\sum_{\substack{X:N(X)=p\\ \theta(X)\cap S=S\setminus R}}\underset{\Sigma_1}{\varrho_R}(X)\prod_{\Gamma_i\in X}\langle\sigma_{S_i}\rangle_{\theta(\Gamma_i)\Sigma_i\beta h}(\sigma_R)_{\Sigma_1} \tag{AII.18}$$

where now the sets X, T are not restricted to any finite region.

From now on we will call $\bar{X}$, $\bar{T}$ the sets of contours extending outside D.

It is easy to recognize by inspection of (AII.18) that the following relation holds:

$$\langle\sigma_{S+\underline{e}}\rangle_{\Sigma_1,\beta,h}=\langle\sigma_S\rangle_{\Sigma_2,\beta,h} \tag{AII.19}$$

where $\underline{e}$ is the unit translation.

Assume now that the right side of (AII.16) and of (AII.18) are absolutely convergent series[7], then we can write:

$$\begin{aligned}|\langle\sigma_S\rangle_{D,\Sigma_1\beta h}-\langle\sigma_S\rangle_{\Sigma_1\beta h}|\leqq&\sum_{R\in\mathscr{P}(S)}\sum_{p=1}^{p_{\max}}\sum_{\substack{X:N(X)=p\\ \theta(X)\cap S=S\setminus R}}\\ &\cdot|\underset{D\Sigma_1}{\varrho_R}(X)\prod_{\Gamma_i\in X}\langle\sigma_{S_i}\rangle_{\theta(\Gamma_i),\Sigma_i,\beta,h}(\sigma_R)_{\Sigma_1}\\ &-\underset{\Sigma_1}{\varrho_R}(X)\prod_{\Gamma_i\in X}\langle\sigma_{S_i}\rangle_{\theta(\Gamma_i),\Sigma_i,\beta,h}(\sigma_R)_{\Sigma_1}|\\ &+\sum_{R\in\mathscr{P}(S)}\sum_{p=1}\sum_{\substack{\bar{X}:N(\bar{X})=p\\ \theta(\bar{X})\cap S=S\setminus R}}|\underset{\Sigma_1}{\varrho_R}(\bar{X})\prod_{\Gamma_i\in\bar{X}}\langle\sigma_{S_i}\rangle_{\theta(\Gamma_i)\Sigma_i\beta h}(\sigma_R)_{\Sigma_1}|\end{aligned} \tag{AII.20}$$

Since the factor $\prod_{\Gamma_i\in X}\langle\sigma_{S_i}\rangle_{\theta(\Gamma_i)\Sigma_i\beta h}(\sigma_R)_{\Sigma_1}$ is equal in (AII.16) and (AII.18) for the finite region D, it can be factorized in (AII.20), in any case its

[7] The following proof is an indirect test for this assumption.

upper bound is 1, so we can omit it from (AII.20). Splitting the second term in the right-side of (AII.20) for contours X, T and $\bar{X}, \bar{T}$ by (AII.17) one has:

$$|\langle\sigma_S\rangle_{D\Sigma_1\beta h} - \langle\sigma_S\rangle_{\Sigma_1\beta h}| \leqq$$

$$\leqq \sum_{R\in\mathscr{P}(S)} \sum_{p=1}^{p_{\max}} \sum_{\substack{X:N(X)=p \\ \theta(X)\cap S=S\setminus R}} \sum_{k=0}^{k_{\max}} \sum_{\substack{T:N(T)=k \\ T\subset X'}} |\varrho_{D\Sigma_1}(X\cup T) - \varrho_{\Sigma_1}(X\cup T)|$$

$$+ \sum_{R\in\mathscr{P}(S)} \sum_{p=1}^{p_{\max}} \sum_{\substack{X:N(X)=p \\ \theta(X)\cap S=S\setminus R}} \sum_{k=0}^{k_{\max}} \sum_{\substack{\bar{T}:N(\bar{T})=k \\ \bar{T}\subset \bar{X}'}} |\varrho_{\Sigma_1}(X\cup \bar{T})| \tag{AII.21}$$

$$+ \sum_{R\in\mathscr{P}(S)} \sum_{p=1}^{p_{\max}} \sum_{\substack{X:N(X)=p \\ \theta(X)\cap S=S\setminus R}} \sum_{k=0}^{k_{\max}} \sum_{\substack{T:N(T)=k \\ T\subset X'\cup X'}} |\varrho_{\Sigma_1}(\bar{X}\cup T)|$$

Observe that from (AI.18) and related hypotheses

$$|\varrho_{\Sigma_1}(X\cup T) - \varrho_{D\Sigma_1}(X\cup T)| \leqq \mathrm{const}\,(\zeta e^{-\beta t})^{|X|+|T|} \cdot (3\zeta e^{-\beta t})^{\tau(X\cup T, \Gamma_D)}$$

where $\tau(X\cup T, \Gamma_D) = \min_{\Gamma_i\in X\cup T} \tau(\Gamma_i, \Gamma_D)$ and also by (AI.10) that:

$$|\varrho_{\Sigma_1}(X\cup T)| \leqq \mathrm{const.}\,(\zeta e^{-\beta t})^{|X|+|T|}$$

Notice finally that the following inequality is true:

$$\sum_{R\in\mathscr{P}(S)} \sum_{p=1}^{p_{\max}} \sum_{\substack{X:N(X)=p \\ \theta(X)\cap S=S\setminus R}} \sum_{k=0}^{k_{\max}} \sum_{\substack{T:N(T)=k \\ T\subset X'}} \leqq 2^{N(S)} \sum_{k=1}^{N(S)} \sum_{Y:N(Y)=k} \tag{AII.22}$$

where $Y = \{\Gamma : \theta(\Gamma)\cap S \neq \phi\}$. We are now ready to obtain the following bound to (AII.21)

$$\begin{aligned} \text{(AII.21)} &\leqq 2^{N(S)} \sum_{k=1}^{N(S)} \sum_{\substack{Y:N(Y)=k \\ Y\subset D}} \mathrm{const.}\,(\zeta e^{-\beta t})^{|X|}\,(3\zeta e^{-\beta t})^{\tau(Y,\Gamma_D)} \\ &\quad + 2^{N(S)+1} \sum_{k=1}^{N(S)} \sum_{\substack{Y:N(Y)=k \\ Y\not\subset D}} \mathrm{const.}\,(\zeta e^{-\beta t})^{|Y|} \end{aligned} \tag{AII.23}$$

In order to perform an explicit calculation of the right-hand side of (AII.29) let D be so big that the square box $\Lambda = l^2$ containing S and centered at the center of mass of S, can lie in D at a distance l from Γ_D.

If this is the case, l increases to infinity together with $\sqrt{|D|}$. Some more remarks:

I) $$\forall x_j \in S \sum_{\Gamma:\theta(\Gamma)\ni x_j} (\zeta e^{-\beta t})^{|\Gamma|} \leqq \sum_q \sum_{r\geqq 2q} (3\zeta e^{-\beta t})^r \tag{AII.24}$$

where q is the distance, on a straight line starting at x_j, of an arbitrary point Q from x_j itself.

II) if Γ_j crosses Γ_D then in (AII.24) $q \geqq l$

III) if Γ_j is at distance less or equal to $\frac{l}{2}$ from Γ_D then $q \geqq \frac{l}{2}$.

Now we split up the first sum in the right-hand side of (AII.23) in two terms: one relative to Γ s.t. $\tau(\Gamma, \Gamma_D) \geqq \frac{l}{2}$ and the other to Γ s.t. $\tau(\Gamma, \Gamma_D) < \frac{l}{2}$ (in the latter one $\tau(\Gamma, \Gamma_D) \simeq 0$). Moreover since the number of ways of choosing k points among $N(S)$ is $\binom{N(S)}{k}$ we have:

$$\begin{aligned}
(\mathrm{AII.23}) \leqq\, & 2^{N(S)} \sum_{k=1}^{N(S)} \binom{N(S)}{k} \Big(\sum_{q=0} \sum_{r \geqq 2q} (3\zeta e^{-\beta t})^r \Big)^k \\
& + 2^{N(S)} \sum_{k=1}^{N(S)} \binom{N(S)}{k} k \Big(\sum_{q=0} \sum_{r \geqq 2q} (3\zeta e^{-\beta t})^r \Big)^{k-1} \\
& \qquad \cdot \sum_{q \geqq \frac{l}{2}} \sum_{r \geqq 2q} (3\zeta e^{-\beta t})^r (3\zeta e^{-\beta t})^{\tau=0} \\
& + 2^{N(S)+1} \sum_{k=1}^{N(S)} k \binom{N(S)}{k} \Big(\sum_{q=0} \sum_{r \geqq 2q} (3\zeta e^{-\beta t})^r \Big)^{k-1} \\
& \qquad \cdot \sum_{q \geqq l} \sum_{r \geqq 2q} (3\zeta e^{-\beta t})^r .
\end{aligned} \tag{AII.25}$$

Now using again hypothesis (AI.8) we obtain:

$$\begin{aligned}
|\langle \sigma_S \rangle_{D, \Sigma_1, \beta, h} - \langle \sigma_S \rangle_{\Sigma_1 \beta h}| \leqq\, & c_1 2^{N(S)} (3\zeta e^{-\beta t})^{l/2} + c_1 2^{N(S)} N(S) (3\zeta e^{-\beta t})^l \\
& + 2^{N(S)+1} N(S)\, c_1 (3\zeta e^{-\beta t})^{2l}
\end{aligned} \tag{AII.26}$$

where

$$c_1 = \left(1 + \frac{1}{(1 - 3\zeta e^{-\beta t})(1 - 9\zeta^2 e^{-2\beta t})} \right)^{N(S)} .$$

The right-hand side of (AII.26) decreases to zero as $\sqrt{|D|}$ goes to infinity.

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