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# Thermodynamics of Particle Systems in the Presence of External Macroscopic Fields

## II. Quantal Case

C. Marchioro*

Istituto Matematico, Università di Roma, Rome, Italy

E. Presutti

Istituto di Fisica, Università dell'Aquila, L'Aquila, Italy

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**Abstract.** In a previous paper the statistics of a system of identical particles moving in an external field depending on a scale factor has been studied in the classical framework. In particular the case in which the scale factor increases to infinity (macroscopic limit) has been considered.

In the present paper the quantum extension is discussed.

## 1. Introduction

In some recent papers the thermodynamic behaviour of particle systems in the presence of external macroscopic fields has been discussed in the framework of rigorous statistical mechanics [1–4]. (In the sequel Ref. [4] will be denoted as I.) In this paper we want to obtain the quantum extension of the previous results.

The approach is similar to the classical one. As well known an external field containing a particle gas in thermodynamic equilibrium is considered macroscopic if it is possible to divide the whole space in subregions small enough for the potential to be approximately constant in them, but large enough to consider in each region statistically independent systems.

We simulate a similar situation considering a system of identical interacting particles in an external field depending on a scale factor. The macroscopic limit is achieved letting the scale factor go to infinity. We study the grand partition function and as a result we again find a link between the so obtained pressure and the usual one (barometric formula).

Let us now discuss the features typical of the quantum case. As in I, we want to divide the whole space in subregions and to express the total

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pressure as a sum of pressure relative to these different subregions. It is however well known that the quantum pressure depends, for finite volume at least, on the walls of the container[1]; this reflects on the different boundary conditions we must impose on the wave functions: for instance, infinite repulsive walls impose vanishing boundary conditions, while elastic walls produce normal boundary conditions. Choosing the vanishing ones we increase, by the uncertainty principle, the energy eigenvalues, so that the total pressure decreases. On the other hand if we impose normal boundary conditions, a partition of the space produces a decrease of the energy spectrum (Minimax principle) and so the total pressure increases [5].

We use the former choice to obtain a lower bound for the "macroscopic" pressure and the latter to find an upper bound. Finally the barometric formula is derived from the equality of the lower and the upper bound. This holds if the pressure does not depend on the boundary conditions. This independence has been proved in some case (as will be seen in Section 3); nevertheless, though quite acceptable from a physical point of view, it has not yet been obtained in the most general case.

As in I we consider particles interacting via superstable potentials. This allows to exclude too high local densities [6].

In Section 2 we state the general framework in which we work. In Section 3 we find a lower and an upper bound to the pressure and finally the barometric formula. In Section 4 we discuss an application to the model of a one dimensional gas of identical particles interacting pairwise via a repulsive inversely quadratic potential.

## 2. General Framework

We consider a system of quantal identical point particles, fermions or bosons, interacting via a potential $\Phi$ and moving in the whole space $\mathbb{R}^\nu$ under the action of an external potential V. For the sake of simplicity we treat only pair interactions and we suppose that the potentials $\Phi$ and V are not infinite apart from zero-measure regions (see below).

**Definition 1.** Let $\Lambda \subset \mathbb{R}^\nu$ be Lebesgue measurable[2]. For every $n \in \mathbb{Z}^+$, $x_1, \ldots, x_n \in \Lambda$ let $X = (x_1, \ldots, x_n)$ denote the generic configuration in $\Lambda$.

[1] The problem evidently arises also in classical mechanics, but it is generally assumed that the range of the external potential which confines the particles in a box is vanishing and with this assumption the surface effects are uniquely determined.

[2] In the sequel we will only consider Lebesgue measurable regions in $\mathbb{R}^\nu$ so that we will omit to mention it explicitely. Further the surfaces of these regions will be sufficiently regular so that what we say in this section applies [7].

We define the Hilbert space $\mathfrak{H}(\Lambda)$ to be the linear space of square integrable functions of X, properly symmetrized, equipped with the scalar product defined by

$$(\psi, \varphi) = \int_{\Lambda} dX\, \overline{\psi(X)}\, \varphi(X) = \overline{\psi(\emptyset)}\, \varphi(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\Lambda} dx_1 \dots dx_n \cdot \overline{\psi(x_1, \dots, x_n)}\, \varphi(x_1, \dots, x_n). \tag{2.1}$$

We then define $\mathfrak{H}^n(\Lambda)$ to be the subspace of $\mathfrak{H}(\Lambda)$ of the functions with a support on the configurations X such that $Card\,X = n$. We will write in the sequel $\mathfrak{H}[\mathfrak{H}^n]$ instead of $\mathfrak{H}(\mathbb{R}^\nu)$ $[\mathfrak{H}^n(\mathbb{R}^\nu)]$. When a subspace $\mathfrak{M}$ of an Hilbert space $\mathfrak{H}$ is stable under the linear operator A acting on $\mathfrak{H}$, we denote by the same A the restriction to $\mathfrak{M}$ of the operator.

**Definition 2.** Let $\Lambda_1, \Lambda_2 \subset \mathbb{R}^\nu$, $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\Lambda_1 \cup \Lambda_2 = \Lambda$ then there exists an isomorphism between the Hilbert spaces $\mathfrak{H}(\Lambda)$ and $\mathfrak{H}(\Lambda_1) \otimes \mathfrak{H}(\Lambda_2)$, namely

$$\varphi(X) \in \mathfrak{H}(\Lambda_1),\ \psi(X) \in \mathfrak{H}(\Lambda_2) \rightarrow \chi(X) \in \mathfrak{H}(\Lambda) \quad \chi(X \cup Y) = \varphi(X)\, \psi(Y).$$

Further let $A(\Lambda_1)$ be a linear operator on $\mathfrak{H}(\Lambda_1)$ and $\mathbf{1}(\Lambda - \Lambda_1)$ be the unit operator on $\mathfrak{H}(\Lambda - \Lambda_1)$ then we define on $\mathfrak{H}(\Lambda)$ via the above isomorphism the operator

$$\hat{A}(\Lambda_1, \Lambda) = A(\Lambda_1) \otimes \mathbf{1}(\Lambda - \Lambda_1).$$

If $\Lambda = \mathbb{R}^\nu$ we simply write $\hat{A}(\Lambda_1)$ instead of $\hat{A}(\Lambda_1, \mathbb{R}^\nu)$.

The physical observables of the system are then represented by operators acting on $\mathfrak{H}$ (or $\mathfrak{H}(\Lambda)$ if the system is confined in the region $\Lambda$). We need to specify a self-adjoint extension of the symmetric operator which is generally used to represent the hamiltonian. The extension will be studied introducing linear forms[3] and using the following lemma.

**Lemma 2.1.** *Let $t$ be a densely defined, closed, positive form on $\mathfrak{H}$. There exists a positive self-adjoint operator* T *with domain $\mathscr{D}(T)$ dense in $\mathfrak{H}$ and such that*

1) *$\mathscr{D}(T) \subset \mathscr{D}(t)$ and $t(\varphi, \psi) = (\varphi, T\psi)$ for every $\varphi \in \mathscr{D}(T)$ and $\psi \in \mathscr{D}(t)$. The operator* T *is uniquely determined by this condition.*

2) *$\mathscr{D}(T)$ is a core of $t$.*

3) *If $\psi \in \mathscr{D}(T)$, $\chi \in \mathfrak{H}$ and $t(\varphi, \psi) = (\varphi, \chi)$ holds for every $\varphi$ in a core of $t$ then $\psi \in \mathscr{D}(T)$ and*

$$T\psi = \chi.$$

[3] The general approach to this problem can be found in the mathematical literature [7]; for its application to statistical mechanics see Robinson [8]; to its formalism we refer in this paper. For instance Lemma 2.1 and 2.4 are respectively Proposition A.1 and A.3 of Ref. [8a].

4) $\mathscr{D}(t)=\mathscr{D}(\mathrm{T}^{1/2})$ *and* $t(\varphi,\psi)=(\mathrm{T}^{1/2}\varphi,\mathrm{T}^{1/2}\psi)$ $\varphi,\psi\in\mathscr{D}(t)$ $\mathscr{D}'\subset\mathscr{D}(t)$ *is a core of* $t$ *if and only if it is a core of* $\mathrm{T}^{1/2}$.

*Analogous notations will be used for the operators, see Definition 1* and Definition 2, and for the corresponding forms.

We now define the operators $\mathrm{N}(\Lambda)$, $\Delta[\mathrm{N}(\Lambda),m]$, $\mathrm{V}_\gamma$ and $\mathrm{U}(\Lambda)$ as follows:

**Definition 3**

$$\mathscr{D}[\mathrm{N}(\Lambda)]=\{\psi\in\mathfrak{H}(\Lambda):\exists p\in\mathbb{Z}^+ \quad \text{such that} \quad \psi(\mathrm{X})=0 \quad \text{if} \quad \mathit{Card}\,\mathrm{X}>p\},$$

$$\mathscr{D}\{\Delta[\mathrm{N}(\Lambda),m]\}=\mathfrak{H}(\Lambda),$$

$$[\mathrm{N}(\Lambda)\psi](\mathrm{X})=\sum_{x\in\mathrm{X}}\psi(\mathrm{X}) \quad \text{for} \quad \psi\in\mathscr{D}[\mathrm{N}(\Lambda)],$$

$$\{\Delta[\mathrm{N}(\Lambda),m]\psi\}(\mathrm{X})=\delta(\mathit{Card}\,\mathrm{X}-m)\,\psi(\mathrm{X}),$$

where $\delta$ denotes the usual Kronecker symbol and $m$ is a positive integer.

**Definition 4**

$$\mathscr{D}(\mathrm{V}_\gamma)=\left\{\psi\in\mathfrak{H}(\Lambda):\psi\in\mathscr{D}[\mathrm{N}(\Lambda)],\ \int d\mathrm{X}\left|\sum_{x\in\mathrm{X}}v(\gamma x)\,\psi(\mathrm{X})\right|^2<+\infty\right\}$$

$$[\mathrm{V}_\gamma\psi](\mathrm{X})=\sum_{x\in\mathrm{X}}v(\gamma x)\,\psi(\mathrm{X}) \quad \text{for} \quad \psi\in\mathscr{D}(\mathrm{V}_\gamma)$$

where $v:\mathbb{R}^\nu\to[0,\infty]$ satisfies

(i) $\mathrm{N}_\mathrm{V}=\{x\in\mathbb{R}^\nu:v(x)=\infty\}$ consists of a finite number of points in each bounded region.

(ii) $\exp[-v(x)]$ is Riemann integrable in $\mathbb{R}^\nu$.

(iii) $\exists x_0,\ \alpha>\nu$ so that $\exp[-v(x)]\geqq\|x\|^{-\alpha}$ for $\|x\|>x_0$.

**Definition 5**

$$\mathscr{D}[\mathrm{U}(\Lambda)]$$
$$=\left\{\psi\in\mathfrak{H}(\Lambda):\psi\in\mathscr{D}\{\Delta[\mathrm{N}(\Lambda),m]\},\ \int_\Lambda d\mathrm{X}\left|\tfrac{1}{2}\sum_{\substack{x\neq y\\ x,y\in\mathrm{X}}}\Phi(\|x-y\|)\,\psi(\mathrm{X})\right|^2<\infty\right\}$$

$$[\mathrm{U}(\Lambda)\psi](\mathrm{X})=\tfrac{1}{2}\sum_{\substack{x\neq y\\ x,y\in\mathrm{X}}}\Phi(\|x-y\|)\,\psi(\mathrm{X}) \quad \text{for} \quad \psi\in\mathscr{D}[\mathrm{U}(\Lambda)]$$

where $\Phi:[0,\infty]\to\mathbb{R}^\nu$ satisfies

(i) $\Phi$ is Lebesgue measurable

(ii) the set $\mathrm{N}_\Phi=\{x\in\mathbb{R}^\nu:\Phi(x)=\infty\}$ consists of a finite number of points in each bounded region.

$\Delta[\mathrm{N}(\Lambda),m]$ is a projector. $\mathrm{N}(\Lambda)$, $\mathrm{V}_\gamma$ and $\mathrm{U}(\Lambda)$ are symmetric operators. This property will be used in the proof of Lemma 2.3.

In correspondence to the above operators we introduce the densely defined forms $\Delta[n(\Lambda),m]$, $n(\Lambda)$, $v_\gamma$ and $u(\Lambda)$.

**Definition 6.** $\mathscr{A}(\Lambda)$ [$\mathscr{B}(\Lambda)$] is the set of infinitely differentiable [continuously differentiable] wave functions. Further the functions in $\mathscr{A}(\Lambda)$ vanish outside of a compact region strictly contained in $\Lambda$.

We now define the kinetic energy forms $t(\Lambda)$, $t^0(\Lambda)$, $t$.

**Definition 7**

$$\mathscr{D}[t(\Lambda)] = \mathscr{B}(\Lambda)$$

$$\mathscr{D}[t^0(\Lambda)] = \mathscr{A}(\Lambda)$$

$$\mathscr{D}[t] = \mathscr{A}(\mathbb{R}^\nu)$$

$$t(\psi) \equiv t(\psi, \psi) = \int_{\mathbb{R}^\nu} dX \sum_{x \in X} |\nabla_x \psi(X)|^2 \quad \text{for} \quad \psi \in \mathscr{D}[t]$$

and $t(\Lambda)$ and $t^0(\Lambda)$ act analogously.

Since the above forms are positive, closable and densely defined [8], they determine the self-adjoint operators T, T($\Lambda$), $T^0(\Lambda)$ through Lemma 2.1. $T^0(\Lambda)$ [T($\Lambda$)] represents the kinetic energy of particles contained in infinitely repulsive [perfectly elastic] walls.

We have now defined the kinetic and the potential energy (interaction + external energies) for our system. In order to write down the total energy operator we would have to sum the above self-adjoint operators. However this requires some care. In fact a sum of self-adjoint operators is generally neither self-adjoint nor essentially self-adjoint. On the other hand in order to completely specify the statistics of our system we need a self-adjoint extension of the hamiltonian. In the following Lemma 2.3 we will show that under general assumptions on the configurational energy, Definition 11, the sums of the forms corresponding to the potential and kinetic energies are still densely defined, bounded below and closable, so that they define an essentially self-adjoint operator which will be taken as hamiltonian. It will be denoted as the sum of the operators T, U and $V_\gamma$, even if this sum has now a generalized meaning.

Before proving Lemma 2.3 we state some definitions and we recall a Lemma (Lemma 2.2) which will be often used in the sequel.

**Definition 8.** Given two forms, $a$ and $b$, we say that $a \geqq b$ if

$$\mathscr{D}(a) \subset \mathscr{D}(b); \qquad a(\psi, \psi) \geqq b(\psi, \psi) \qquad \forall \psi \in \mathscr{D}(a).$$

**Lemma 2.2.** *Let* $\Lambda_1, \Lambda_2 \subset \mathbb{R}^\nu$, $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$. *Then*

$$\hat{t}^0(\Lambda_1, \Lambda) + \hat{t}^0(\Lambda_2, \Lambda) \geqq t^0(\Lambda) \geqq t(\Lambda) \geqq \hat{t}(\Lambda_1, \Lambda) + \hat{t}(\Lambda_2, \Lambda).$$

The meaning of $\hat{t}^0(\Lambda_1, \Lambda)$, $t(\Lambda_1, \Lambda)$ is given in Definition 2. The proof of the lemma is then a direct consequence of the definitions of the kinetic energies.

**Definition 9a.** For $\lambda \in \mathbb{R}^+$ and $w \in \mathbb{R}^\nu$ let

$$\Gamma_\lambda(w) = \{x \in \mathbb{R}^\nu : \lambda(w^i - \tfrac{1}{2}) \leqq x^i < \lambda(w^i + \tfrac{1}{2}) \qquad i = 1, \ldots, \nu\}.$$

**Definition 9b.** For every $x \in \mathbb{R}^\nu$ we define $|x| = \max_{1 \leqq i \leqq \nu} |x^i|$.

**Definition 9c.** For every configuration X and region $\Lambda \subset \mathbb{R}^\nu$ we denote by $X \hat{\cap} \Lambda$ the subconfiguration of X contained in $\Lambda$.

**Definition 10.** Stability. There exists $B_s \geqq 0$ such that

$$u \geqq -B_s n$$

where $u$ and $n$ are defined in Definition 5 and Definition 3. The meaning of the inequality is explicited in Definition 8.

**Definition 11.** Superstability. $\Phi$, defined in Definition 5, can be written as $\Phi = \Phi' + \Phi''$ where $\Phi'$ is stable and $\Phi''$ is a continuous non negative function such that $\Phi''(0) > 0$. We will often use the following property deriving from the superstability: there exist $A > 0$ and $B \geqq 0$ such that for every $\mathcal{R} \subset \mathbb{Z}^\nu$, $\lambda > 0$ the following holds

$$u \geqq \sum_{r \in \mathcal{R}} [A \hat{n}^2(\Gamma_\lambda(r)) - B \hat{n}(\Gamma_\lambda(r))].$$

**Lemma 2.3.** *The form $h_\gamma - \mu n$ ($\mu \in \mathbb{R}$) defined as*

$$h_\gamma - \mu n = t + u + v_\gamma - \mu n; \qquad \mathcal{D}(h_\gamma - \mu n) = \mathcal{D}(t) \cap \mathcal{D}(u) \cap \mathcal{D}(v_\gamma) \cap \mathcal{D}(n)$$

*is densely defined, bounded below and closable.*

*Proof.* Domain of definition. The form is densely defined in the Hilbert space of functions with support in regions obtained by subtracting neighbourhoods of the singularities of the potential energy and containing a finite number of particles. The arbitrariness of this number and of those neighbourhoods (together with Definition 4 (i) and Definition 5 (ii)) proves the thesis. Boundedness. By Definition 11 $u$ can be written as $u = u' + u''$ where $u'$ arises from the stable interaction $\Phi'$ and $u''$ from the positive interaction $\Phi''$. Therefore there exist $\hat{B}$ so that the following is true. For every bounded region $\Lambda$, $\tilde{A} > 0$ and $\tilde{B} \geqq 0$ can be found in such a way that

$$u \geqq [\tilde{A} \hat{n}^2(\Lambda) - \tilde{B} \hat{n}(\Lambda)] - \hat{B} n(\mathbb{R}^\nu). \tag{2.2}$$

By (ii) of Definition 4 we can choose $\Lambda$ in Eq. (2.2) so that

$$v_\gamma(\Lambda^c) \geqq (\mu + \hat{B})\, n(\Lambda^c) \qquad \Lambda^c = \mathbb{R}^\nu - \Lambda. \tag{2.3}$$

Therefore

$$t+u+v_\gamma-\mu n \geqq u+v_\gamma-\mu n$$
$$\geqq \tilde{A}\hat{n}^2(\Lambda)-(\tilde{B}+\hat{B}+\mu)\,\hat{n}(\Lambda)+[\hat{v}_\gamma(\Lambda^c)-(\mu+\hat{B})\,n(\Lambda^c)]$$
$$\geqq \tilde{A}\hat{n}^2(\Lambda)-(\tilde{B}+B+\mu)\,\hat{n}(\Lambda)$$

which is bounded below.

Closability. $u+v_\gamma-\mu n$ is bounded below and it is implemented by the symmetric operator $\mathrm{U}+\mathrm{V}_\gamma-\mu\mathrm{N}$ so that it is closable (see for instance 1.2.8 of Ref. [8 b]). Since $t$ is also closable and so is the sum of closable forms, the thesis is proved. Q.E.D.

In the sequel we shall assume the validity of the hypotheses Definition 1–Definition 11 without mentioning them explicitely.

In order to introduce the pressure both for bounded and unbounded regions we need the following three Lemmata.

**Lemma 2.4.** [8]. *Let $t$ and $t'$ be densely defined, closed, lower semibounded forms on $\mathfrak{H}$ and let* $\mathrm{T}$ *and* $\mathrm{T}'$ *be the associated self-adjoint operators.*

1) *Let $\mathscr{D}$ be a core of $t$ and $\mathscr{F}$ a finite family of orthonormal vectors $\varphi\in\mathscr{D}$. The following conditions are equivalent*

a) $\sup_{\mathscr{F}} \sum_{\varphi\in\mathscr{F}} \exp[-t(\varphi,\varphi)]$.

b) $\mathrm{Tr}_{\mathfrak{H}}\exp(-\mathrm{T})<\infty$

*and if they are satisfied then*

$$\sup_{\mathscr{F}} \sum_{\varphi\in\mathscr{F}} \exp[-t(\varphi,\varphi)]=\mathrm{Tr}_{\mathfrak{H}}[\exp(-\mathrm{T})]$$

2) *Consequently if $t'\geqq t$ and* $\exp[-\mathrm{T}]$ *is of trace class then*

$$\mathrm{Tr}_{\mathfrak{H}}\exp(-\mathrm{T}')\leqq \mathrm{Tr}_{\mathfrak{H}}\exp(-\mathrm{T})\,.$$

**Lemma 2.5.** *There exist $c_1>0$, $c_2>0$ such that for every $l$ and $m$ positive integers the following holds*

$$\begin{aligned}\mathrm{Tr}_{\mathfrak{H}^m(\Gamma_l(0))}\exp\{-\beta[\mathrm{T}(\Gamma_l(0))+\mathrm{U}(\Gamma_l(0))-\mu\mathrm{N}(\Gamma_l(0))]\}\\ \leqq \exp[\beta c_1 m+c_2 l^\nu-\beta(Am^2 l^{-\nu}-Bm)]\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\mathrm{Tr}_{\mathfrak{H}(\Gamma_l(0))}\exp\{-\beta[\mathrm{T}(\Gamma_l(0))+\mathrm{U}(\Gamma_l(0))-\mu\mathrm{N}(\Gamma_l(0))]\}\\ &\leqq\left[1+\sum_{n=1}^{\infty}\exp\{\beta c_1 n+c_2-\beta(An^2-(B+\mu)n)\}\right]^{l^\nu}\end{aligned} \tag{2.5}$$

*where $A$ and $B$ are defined in Definition 11: $m\geqq 1$ in Eq.* (2.4).

*Proof.* We have by Definition 11 and Lemma 2.2 in

$$\mathfrak{H}^m(\Gamma_l(0)): t+u \geqq \sum_{|r|\leqq l} [\hat{t}(\Gamma_1(r), \Gamma_l(0)) + c_1 \hat{n}(\Gamma_1(r), \Gamma_l^1(0))]$$

$$+ Am^2 l^{-\nu} - Bm - c_1 m.$$

By use of Lemma 2.4 we obtain Eq. (2.4) with the notation

$$\exp c_2 = \mathrm{Tr}_{\mathfrak{H}(\Gamma_1(0))} \exp\{-\beta[\mathrm{T}(\Gamma_1(0)) + c_1 \mathrm{N}(\Gamma_1(0))]\}$$

this is finite if $c_1$ is positive. In the same way Eq. (2.5) can be checked. Q.E.D.

**Lemma 2.6.** *The operator* $\exp\{-\beta[\mathrm{H}_\gamma - \mu \mathrm{N}]\}$ *is a trace class operator in* $\mathfrak{H}$.

*Proof.* We note that by Definition 11

$$t+u+v_\gamma - \mu n \geqq \sum_{r\in\mathbb{Z}^\nu} \{\hat{t}[\Gamma_1(r)] + A\hat{n}^2(\Gamma_1(r)) + (v_{1,r} - \mu - B)\,\hat{n}(\Gamma_1(r))\} \quad (2.6)$$

where

$$v_{1,r} = \inf_{x\in\Gamma_1(r)} v(\gamma x).$$

The form in the r.h.s. is defined in the domain

$$\bigcap_r \mathscr{D}[\hat{t}(\Gamma_1(r)) + A\hat{n}^2(\Gamma_1(r)) + (v_{1,r} - \mu - B)\,\hat{n}(\Gamma_1(r))]$$

such that the sum in Eq. (2.6) converges. The form is closable ([8], 1.2.9). Its domain is dense since it contains the wave functions infinitely often differentiable with support in bounded regions and with finite number of particles. To prove the lemma we have then to check the convergence of the infinite product

$$\prod_{r\in\mathbb{Z}^\nu} \mathrm{Tr}_{\mathfrak{H}(\Gamma_1(r))} \exp\{-\beta[\mathrm{T} + A\mathrm{N}^2 + (v_{1,r} - B - \mu)\mathrm{N}]\} \leqq \prod_{r\in\mathbb{Z}^\nu} \{1 + Ke^{-\beta v_{1,r}}\}$$

$$K = \sum_{n=1}^{\infty} \exp[-\beta(An^2 - \mu - Bc)n]\, \mathrm{Tr}_{\mathfrak{H}^n(\Gamma_1(0))} \exp(-\beta\mathrm{T}).$$

The positiveness of $v(x)$, Definition 4, and condition Definition 4 (iii) prove the lemma. Q.E.D.

Using Lemma 2.6 we define the grand partition functions:

**Definition 12.** $Z_\gamma(\mu, \beta) = \mathrm{Tr}_{\mathfrak{H}} \exp[-\beta(\mathrm{H}_\gamma - \mu\mathrm{N})]$

$$Z(\mu, \beta, \Lambda) = \mathrm{Tr}_{\mathfrak{H}(\Lambda)} \exp[-\beta\{\mathrm{T}(\Lambda) + \mathrm{U}(\Lambda) - \mu\mathrm{N}(\Lambda)\}]$$

$$Z^0(\mu, \beta, \Lambda) = \mathrm{Tr}_{\mathfrak{H}(\Lambda)} \exp[-\beta\{\mathrm{T}^0(\Lambda) + \mathrm{U}(\Lambda) - \mu\mathrm{N}(\Lambda)\}]$$

and the corresponding pressures

$$P_\gamma(\mu,\beta)=\gamma^\nu\beta^{-1}\ln Z_\gamma(\mu,\beta),$$
$$P(\mu,\beta,\Lambda)=|\Lambda|^{-1}\beta^{-1}\ln Z(\mu,\beta,\Lambda),$$
$$P^0(\mu,\beta,\Lambda)=|\Lambda|^{-1}\beta^{-1}\ln Z^0(\mu,\beta,\Lambda).$$

*Note.* The existence of the grand canonical partition function for every value of the chemical potential $\mu$ both for fermions and bosons lies on the assumption of superstability for the interaction. We used extensively this condition in the proof of Lemmata 2.3, 2.5 and 2.6.

To perform the thermodynamic limit for the above pressure we need conditions on the asymptotic behaviour of the interaction $\Phi$.

**Definition 13.** Weak-tempering. There exist $\alpha>\nu$, $k>0$ and $R_1>0$ such that

$$\Phi(\|x_i-x_j\|)\leqq k\|x_i-x_j\|^{-\alpha} \quad \text{for} \quad \|x_i-x_j\|\geqq R_1.$$

**Definition 14.** Lower regularity. There exist $k>0$, $\alpha>\nu$ and $R>0$ such that

$$\Phi(\|x_i-x_j\|)>-k\|x_i-x_j\|^{-\alpha} \quad \text{for} \quad \|x_i-x_j\|>R.$$

In the sequel we shall use this condition written as follows. We define $w$ as:

$$w(X,Y)=\sum_{x\in X}\sum_{y\in Y}\Phi(\|x-y\|)$$

where X and Y are two configurations, $X\cap Y=\emptyset$. Then there exists a decreasing positive function $\Psi$ on the positive integers for which

$$\sum_{r\in\mathbb{Z}^\nu}\Psi(|r|)<\infty$$

so that if $\mathcal{R}$ and $\mathcal{S}$ are finite subsets of $\mathbb{Z}^\nu$ and

$$X=(x_1,\ldots,x_m)\quad x_i\in\bigcup_{r\in\mathcal{R}}\Gamma_1(r)\qquad i=1,\ldots,m,$$
$$Y=(y_1,\ldots,y_n)\quad y_i\in\bigcup_{r\in\mathcal{S}}\Gamma_1(r)\qquad i=1,\ldots,n$$

then

$$w(X,Y)\geqq-\sum_{r\in\mathcal{R}}\sum_{s\in\mathcal{S}}\Psi(|s-r|)\left[\tfrac{1}{2}n^2(X\cap\Gamma_1(r))+\tfrac{1}{2}n^2(Y\cap\Gamma_1(s))\right].$$

**Definition 15.** We introduce the pressure in the thermodynamic limit as

$$P^0(\mu,\beta)=\lim_{|\Lambda|\to\infty}P^0(\mu,\beta,\Lambda) \tag{2.7}$$

where $\Lambda$ invades $\mathbb{R}^\nu$ in the Fisher sense [9]. Its existence depends on the assumptions Definition 10 and Definition 13 on $\Phi$.

We then define

$$\mathrm{P}(\mu, \beta) = \lim_{|\Lambda| \to \infty} \mathrm{P}(\mu, \beta, \Lambda) \tag{2.8}$$

where the limit is over a net of increasing cubes.

The existence of the limit (2.8) has not been proved in the general case; so it has to be considered as an assumption. This assumption has been shown to hold in some cases (see Proposition 1, Section 3). We further assume that the limit in Eq. (2.8) is the same of Eq. (2.7). This independence of the thermodynamic limit of the boundary conditions will be needed in the proof of the barometric formula.

## 3. Results

In this section we find first a lower and an upper bound for the non uniform pressure defined in Definition 12 (Theorems 3.1 and 3.2). Then we find a link between this pressure and the usual one (barometric formula) (Theorem 3.3).

**Theorem 3.1.** *Lower bound. Let $\Phi$ satisfy Definition 11 and Definition 13 then*

$$\liminf_{\gamma \to 0} \beta^{-1} \gamma^{\nu} \ln \mathrm{Z}_{\gamma}(\mu, \beta) \geqq \int_{\mathbb{R}^{\nu}} dx\, \mathrm{P}^{0}(\mu - v(x), \beta).$$

*Proof.* We now sketch the main lines of the proof. We follow the physical ideas discussed in the introduction. We confine the particles into cells separated by corridors. Therefore we require that the wave functions vanish in the corridors. By the indeterminacy principle, or Lemma 2.2, this gives rise to an increase of the kinetic energy. The other steps are similar to the classical ones. We limit the density in each region so that the interaction between different cells can be evaluated. We perform the macroscopic limit and then we let go to infinity both the cutoff on the density and the size of the cells.

We introduce the set of cubes $\Gamma_{l,R}(r)$ as follows

$$\Gamma_{l,R}(r) = \{x \in \mathbb{R}^{\nu} : (r^i - \tfrac{1}{2})\, l + R \leqq x^i < (r^i + \tfrac{1}{2})\, l - R \qquad i = 1, \ldots, \nu\} \tag{3.1}$$

$$\frac{l}{2} > R \geqq R_1$$

where $R_1$ is defined in Definition 13. We define the subspace in $\mathfrak{H}$, $\mathfrak{H}_M$, as follows

$$\begin{aligned} \mathfrak{H}_M = \{\psi \in \mathfrak{H} : \psi(\mathrm{X}) = 0 \quad &\text{if for} \quad r \in \mathbb{Z}^{\nu}\ \mathit{Card}(\mathrm{X} \cap \Gamma_{l,R}(r)) \geqq M + 1 \\ &\text{and} \quad \mathit{Card}[\mathrm{X} \cap (\Gamma_l(r) - \Gamma_{l,R}(r))] \geqq 1\}. \end{aligned} \tag{3.2}$$

We obviously have

$$Z_\gamma(\mu, \beta) \geqq \mathrm{Tr}_{\mathfrak{H}_M}\{\exp[-\beta(\mathrm{H}_\gamma - \mu \mathrm{N})]\}.$$

It is possible to bound the restriction of the form $u$ to $\mathfrak{H}_M$: there exist $k>0$ such that

$$\text{in } \mathfrak{H}_M : u \leqq \sum_{r\in\mathbb{Z}^\nu} \{\hat{u}(\Gamma_{l,R}(r)) + kM^2R^{-\alpha}[1-\Delta(\hat{u}\{\Gamma_{l,R}(r), 0\})]\} \quad (3.4)$$

where $\Delta$ is defined in Definition 3. Eq. (3.4) is obtained directly from the analogous classical estimate Eq. (3.2) in I. By use of Lemma 2.2 we obtain a bound analogous the the one of Eq. (3.4) for the form $h_\gamma - \mu n$. By Lemma 2.4 Eq. (3.4) gives

$$Z_\gamma(\mu, \beta) \geqq \prod_{r\in\mathbb{Z}^\nu} \Big\{1 + \sum_{n=1}^{M} \exp[-\beta kM^2R^{-\alpha} + \beta n(\mu - \mathrm{V}_{r,l})] \cdot \mathrm{Tr}_{\mathfrak{H}^n(\Gamma_{l,R}(0))} \exp[-\beta \mathrm{H}(\Gamma_{l,R}(0))]\Big\} \quad (3.5)$$

where

$$\mathrm{V}_{l,r} = \sup_{x\in\Gamma_l(r)} v(\gamma x). \quad (3.6)$$

Eq. (3.5) is the quantal analogue of Eq. (3.3) of I. Proceeding as in I we can then write

$$\begin{aligned}
&\liminf_{\gamma\to 0} \gamma^\nu \ln Z_\gamma(\mu, \beta) \\
&\geqq |\Gamma_l(0)|^{-1} \int_{\mathscr{D}} dx \ln \mathrm{Tr}_{\mathfrak{H}(\Gamma_{l,R}(0))} \exp\{-\beta[\mathrm{H}(\Gamma_{l,R}(0)) - (\mu - v(x))\mathrm{N}(\Gamma_{l,R}(0))]\} \\
&- \beta kM^2R^{-\alpha} \frac{|\mathscr{D}|}{|\Gamma_l(0)|} - \frac{|\mathscr{D}|}{|\Gamma_l(0)|} \\
&\cdot \ln\Big\{1 + \sum_{n=M+1}^{\infty} \mathrm{Tr}_{\mathfrak{H}^n(\Gamma_{l,R}(0))} \exp[-\beta(\mathrm{H}(\Gamma_{l,R}(0)) - \mu \mathrm{N}(\Gamma_{l,R}(0)))]\Big\}
\end{aligned} \quad (3.7)$$

where $\mathscr{D} \subset \mathbb{R}^\nu$ has a finite Lebesgue measure. We now let $M$, $l$ and $R$ go to infinity, so that $M \gg l \gg R$, namely, defined $\eta$ by

$$0 < \eta < (\alpha - \nu)\nu^{-1}(\nu + 2\alpha)^{-1} < (2\nu)^{-1} \quad (3.8)$$

we choose for every $M$

$$l = M^{\nu^{-1}-\eta}; \qquad R = M^{\nu^{-1}-2\eta}. \quad (3.9)$$

Then the second and the third terms in Eq. (3.7) are vanishing in the limit $M\to\infty$. We only note that the argument in the logarithm is bounded: this follows directly from Lemma 2.5.

From Eq. (3.7) we then obtain

$$\liminf_{\gamma\to 0} \beta^{-1}\gamma^\nu \ln Z_\gamma(\mu, \beta) \geqq \int_{\mathscr{D}} dx\, \mathrm{P}^0(\mu - v(x), \beta). \quad (3.10)$$

Eq. (3.10) has been obtained by use of the Lebesgue theorem. We have

$$\mathrm{P}^0(\mu - v(x), \beta, \Lambda) \leqq \beta^{-1} \ln\{1 + K \exp[-\beta v(x)]\} \tag{3.11}$$

$$K = \sum_{n=1}^{\infty} \exp\{-\beta[An^2 - n(\mu + B - c_1) + c_2]\}$$

which is summable in $\mathbb{R}^\nu$. Furthermore Eq. (3.11) allows us to apply the Lebesgue theorem and to perform the limit of $\mathscr{D}$ invading the whole $\mathbb{R}^\nu$. Then the proof in completed. Q.E.D.

For the upper bound, as in the classical case, we need an estimate for the fluctuations of density in bounded regions. The one we state in Lemma 3.3 is sufficient for our purposes and to obtain it we follow essentially the same line of the classical case. As it is reported in the sequel we adapt the results of Ruelle to our framework. The main difficulty is that the partition function is not ensured to be an increasing function of the volume of the region in which it is defined if its walls are perfectly elastic.

**Definition 3.1.** Let $\mathscr{S}$ be the set of all sequences of integers $\{l_j\}$ such that

Sa) $j \geqq P_0 > 0$.

Sb) If $\{l_j\} \in \mathscr{S}$ then $l_{P_0} = \mathscr{I}[1 + (2\alpha)^{-1}]$ where $0 < \alpha < 1$ and $\{(1+3\alpha)^{2\nu+2} - 1\} \sum_{r \in \mathbb{Z}_\nu} \Psi(|r|) \leqq \frac{A}{4}$. $A$ is defined in Definition 11 and $\Psi$ in Definition 14.

Sc) $\left|\frac{l_{j+1}}{l_j} - (1 + 2\alpha)\right| < \alpha$.

**Lemma 3.1.** *There exists $0 < \delta < \infty$ such that for every $1 \leqq n \in \mathbb{Z}^\nu$ a sequence $\{l_j\} \in \mathscr{S}$ and an element $l_{j_0} \in \{l_j\}$ can be selected so that*

(i) $(2l_{j_0} + 1) < \delta(2n + 1)$.

(ii) $(2n+1)^{-1}(l_j - n) \in \mathbb{Z}^+$ for $j \geqq j_0$.

*Proof.* Let $\{l_j\} \in \mathscr{S}$; we modify in a way that it still belongs to $\mathscr{S}$ but verifies i) and ii). If $l_q \in \{l_j\}$ and

$$l_q > (2\alpha)^{-1}(2n+1) \tag{3.12}$$

then $l_{q+1}$ belongs to the interval

$$l_q(1+\alpha) < l_{q+1} < l_q(1+3\alpha) \tag{3.13}$$

which is $2\alpha l_q$ large. By Eq. (3.12) therefore $l_{q+1}$ can be changed so that (ii) is verified. As a consequence every $l_j$ with $j > q$ can be chosen so that (ii) holds. By Eq. (3.12)

$$l_{j_0 - 1} > (1+3\alpha)^{-1} l_{j_0} \quad \text{(see Eq. (3.13))}$$

and

$$(1+3\alpha)^{-1} l_{j_0} > (2\alpha)^{-1}(2n+1), \quad l_{j_0} > (2\alpha)^{-1}(1+3\alpha)(2n+1). \tag{3.14}$$

We then choose $l_{j_0}$ as the first number of the sequence for which Eq. (3.14) holds. This will not exceed the number $(2\alpha)^{-1}(2n+1)(1+3\alpha)^2$.

Then (i) holds with $\delta = 2\alpha^{-1}(1+3\alpha)^2$. Q.E.D.

*Note.* The geometrical meaning of Lemma 3.1 is that for every integer $n$ a sequence $\{l_j\} \in \mathscr{S}$ can be found so that from a certain $l_{j_0}$ on, the cubes $\Gamma_{2l_j+1}(0)$ can be filled exactly by cubes $\Gamma_{2n+1}(r)$. Further it is proved that $l_{j_0}$ can be made to grow not faster than $n$.

**Definition 3.2.** Let $\psi$ be an increasing positive function on the integer for which

$$\psi \geqq 1 \quad \lim_{l\to\infty} \psi(l) = +\infty\,, \tag{3.15a}$$

$$\frac{\psi(l+1)}{\psi(l)} \leqq \frac{l+1}{l}\,, \tag{3.15b}$$

$$\sum_{r\in\mathbb{Z}^\nu} \psi(|r|)\,\Psi(|r|) < +\infty\,. \tag{3.15c}$$

To every $\{l_j\} \in \mathscr{S}$ there corresponds a sequence $\{\psi_j\}$, $\psi_j = \psi(l_j)$.

With these definitions we can state Proposition 2.5a of Ref. [6] as follows

**Lemma 3.2.** *For every fixed $\beta$, $\mu$ there exists an integer $P > P_0$ such that the following is true*

i) For every $\{l_j\} \in \mathscr{S}$ and $q \geqq P$, defined $c = (1+3\alpha)^{-\nu-1}\dfrac{A}{4}$, we have

$$\beta c \psi_{q+1} > \ln \mathrm{Tr}_{\mathfrak{H}(\Gamma_1(0))} \exp\left\{-\beta\left[\mathrm{T}(\Gamma_1(0)) - \mu \mathrm{N}(\Gamma_1(0)) + \frac{A}{8}\mathrm{N}^2(\Gamma_2(0))\right]\right\}. \tag{3.16}$$

ii) Given $\mathrm{X} = (x_1, \ldots, x_n)$, $x_i \in \mathbb{R}^\nu$ and given a sequence $\{l_j\} \in \mathscr{S}$ suppose that there exist $q$ such that $q \geqq P$ and $q$ is the largest integer for which

$$\sum_{|r|\leqq l_p} n^2(\mathrm{X}, r) \geqq \psi_q |\Gamma_{2l_p+1}(0)| \tag{3.17}$$

where $n(\mathrm{X}, r) = \mathit{Card}(\mathrm{X} \cap \Gamma_1(r))$.

Let $\mathrm{X}' = \mathrm{X} \hat{\cap} \Gamma_{2l_p+1}(0)$ and $\mathrm{X}'' = \mathrm{X} \hat{\cap} \Gamma^c_{2l_p+1}(0)$ then we have

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{x \neq y \\ x, y \in \mathrm{X}'}} \Phi(\|x-y\|) - w(\mathrm{X}', \mathrm{X}'') \\ &\leqq -\frac{A}{4} \sum_{|r|\leqq l_{p+1}} n^2(\mathrm{X}, r) - c\psi_{q+1} |\Gamma_{2l_{p+1}+1}(0)|\,. \end{aligned} \tag{3.18}$$

*Proof*. Since $\psi_j$ is a function increasing to infinity and the r.h.s. member of the inequality (3.16) is finite (Lemma 2.5) (i) trivially holds. (ii) is just Proposition 2.5a of Ruelle. Q.E.D.

**Definition 3.3.** For $\{l_j\} \in \mathscr{S}$ and $r \in \mathbb{Z}^\nu$ we define the set of regions $\Gamma^c_{2l_j+1}(r)$ which are the complements in $\mathbb{R}^\nu$ of the cubes with center $r$ and side $2l_j+1$. We then write

$$Z^+_\gamma(\mu, \beta, \{l_j\}) = \max \left\{ Z_\gamma(\mu, \beta), \sup_{\substack{r \in \mathbb{Z}^\nu \\ l_j \in \{l_j\}}} Z_\gamma(\mu, \beta, \Gamma^c_{2l_j+1}(r)) \right\}. \tag{3.19}$$

**Lemma 3.3.** *Let $\Phi$ satisfy Definition 11 and Definition 14; given one of the cubes $\Gamma_1(s)$, then there exist $\eta > 0$, $\zeta > 0$ so that for every sequence $\{l_j\} \in \mathscr{S}$ and every $m \geqq m_0$*

$$m_0 = \mathscr{I}\{[(2(1+3\alpha)^P l_{P_0} + 1)^\nu \psi((1+3\alpha)^P l_{P_0})]^{1/2}\}$$

*($P$, $P_0$ are defined in Lemma 3.2), then*

$$\begin{aligned} &\mathrm{Tr}_{\mathfrak{H}} \Delta[\hat{N}(\Gamma_1(s)), m] \exp[-\beta(H_\gamma - \mu N)] \\ &\leqq \zeta \exp(-\eta m^2 - \beta \tilde{V}_{1,s} m)\, Z^+_\gamma(\mu, \beta, \{l_j\}) \end{aligned}$$

*where*

$$\tilde{V}_{1,s} = \inf_{x \in \Gamma_1(s)} v(\gamma x). \tag{3.20}$$

*Proof*. Let us prove the lemma in the case $s=0$; the proof for a different $s$ is completely analogous. For any $q \geqq P$ (Lemma 3.2) let $\mathfrak{H}_q$ be the closed subspace of $\mathfrak{H}$ of the wave functions with support on the configurations satisfying Eq. (3.17). The restriction of the form $u$ to this subspace, is, in the wave function representation, implemented by the function $u(X)$ for which we have the bound

$$-u(X') - w(X', X'') \leqq -\frac{A}{8} \sum_{|r| \leqq l_{q+1}} n^2(X, r) - \frac{A}{8} m^2 - c\psi_{q+1} |\Gamma_{2l_{q+1}+1}(0)| \tag{3.21}$$

(see Eq. (3.18)).

In $\mathfrak{H}_q$ we then have, using Lemma 2.2 and Eq. (3.21)

$$\begin{aligned} t + u + v_\gamma - \mu n \geqq &\sum_{|r| \leqq l_{q+1}} \left\{ \hat{t}(\Gamma_1(r)) - \frac{A}{8} \hat{n}^2(\Gamma_1(r)) - \mu \hat{n}(\Gamma_1(r)) \right\} \\ &+ \Big\{ \hat{t}(\Gamma^c_{2l_{q+1}+1}(0)) + \hat{u}(\Gamma^c_{2l_{q+1}+1}(0)) + \hat{v}_\gamma(\Gamma^c_{2l_{q+1}+1}(0)) \\ &- \mu \hat{n}(\Gamma^c_{2l_{q+1}+1}(0)) + -\frac{A}{8} m^2 - \tilde{V}_{1,0} m - c\psi_{q+1} |\Gamma_{2l_{q+1}+1}(0)| \Big\}. \end{aligned}$$

Because $m^2 > \psi_P |\Gamma_{2l_P+1}(0)|$ we have

$$\Delta[\hat{\mathrm{N}}(\Gamma_1(0)), m]\,\mathfrak{H} = \Delta(\hat{\mathrm{N}}(\Gamma_1(0)), m) \sum_{q \geqq P}^{\oplus} \mathfrak{H}_q .$$

Then applying Lemma 2.4

$$\mathrm{Tr}_{\mathfrak{H}} \Delta[\hat{\mathrm{N}}(\Gamma_1(0)), m] \exp[-\beta(\mathrm{H}_\gamma - \mu \mathrm{N})] \leqq \sum_{q \geqq P} \exp\left[-\beta\left(\tilde{\mathrm{V}}_{1,0} m + -\frac{A}{8} m^2\right)\right]$$

$$\cdot \left\{ \prod_{|r| \leqq l_q+1} \mathrm{Tr}_{\mathfrak{H}(\Gamma_1(r))} \exp\left[-\beta\left(\mathrm{T}(\Gamma_1(r)) - \mu \mathrm{N}(\Gamma_1(r)) + \frac{A}{8} \mathrm{N}^2(\Gamma_1(r))\right)\right]\right.$$

$$\left. \cdot \exp[-\beta c \psi_{q+1} |\Gamma_{2l_{q+1}+1}(0)|] \right\}$$

$$\cdot \mathrm{Tr}_{\mathfrak{H}(\Gamma^c_{2l_{q+1}+1}(0))} \exp\{-\beta[\mathrm{H}_\gamma(\Gamma^c_{2l_{q+1}+1}(0)) - \mu \mathrm{N}(\Gamma^c_{2l_{q+1}+1}(0))]\}$$

$$\leqq \exp\left[-\beta\left(\tilde{\mathrm{V}}_{1,0} m - \frac{A}{8} m^2\right)\right]$$

$$\cdot \mathrm{Z}^+_\gamma(\mu, \beta, \{l_j\}) \sum_{q \geqq P} \exp\{|\Gamma_{2l_{q+1}+1}(0)|\, F(q)\}$$

$$F(q) = -\beta c \psi_{q+1} + \ln \mathrm{Tr}_{\mathfrak{H}(\Gamma_1(0))} \exp\{-\beta[\mathrm{T}(\Gamma_1(0)) - \mu \mathrm{N}(\Gamma_1(0))]\} .$$

Therefore the Lemma is proved if

$$\eta = \frac{A}{8} \beta$$

$$\zeta = \sup_{\{l_j\} \in \mathscr{S}} \sum_{q \geqq P} \exp\{|\Gamma_{2l_{q+1}+1}(0)|\, F(q)\}$$

which is finite (as can be seen by use of Definition 3.1).

The proof is similar to that of Lemma 2.5. Q.E.D.

We can now prove the following upper bound.

**Theorem 3.2.** *Let $\Phi$ satisfy Definition 11 and Definition 14 then*

$$\limsup_{\gamma \to 0} \beta^{-1} \gamma^\nu \ln \mathrm{Z}_\gamma(\mu, \beta) \leqq \int_{\mathbb{R}^\nu} dx\, \mathrm{P}(\mu - v(x), \beta) .$$

*Proof.* The procedure is similar to the classical one. We again divide the space in cells. Confining the particles in cells by elastic conditions (free boundary conditions), the kinetic energy decreases (Lemma 2.2). To evaluate the interaction between particles in different zones, we consider two partitions of $\mathbb{R}^\nu$ made up by the cubes $\{\Gamma_1(r)\}$ and $\{\Gamma_{2n+1}(r)\}$. We separate the contribution to $\mathrm{Z}_\gamma(\mu, \beta)$ arising from the subspace $\mathfrak{H}_M$ in which no more than $M$ particles are in the cubes $\Gamma_1(r)$ and the subspace of the remaining configurations. In the former we bound the interaction between particles in different "large" cubes, $\Gamma_{2n+1}(r)$, and in the latter we use the estimate of Lemma 3.3 relatively to the particles in the "small" cubes, $\{\Gamma_1(r)\}$.

For every $\gamma \in (0, 1]$ we choose the following values for $n$ and $M$

$$M = \mathscr{I}(m_0 + \ln \gamma^{-1}), \tag{3.22}$$

$$n = \mathscr{I}(\gamma^{\varepsilon - 1}) \qquad 0 < \varepsilon < 1 \tag{3.23}$$

where $m_0$ is defined in Lemma 3.3. For every $\gamma$ and with $n$ fixed by Eq. (3.23), we select a sequence $\{l_j\} \in \mathscr{S}$ so that Lemma 3.1 holds. We will find an upper bound for the partition functions $Z_\gamma(\mu, \beta, \Gamma^c_{2l_j+1}(r))$ for every $r \in \mathbb{Z}^\nu$ and $\{l_j\} \in \mathscr{S}$. Therefore the same bound is also valid for the supremum of the above partition functions and this will reconstruct the quantum analogue of the inequality (3.13) of I, which was the basis of the proof of the classical case.

We consider first the regions $\Gamma_{2l_j+1}(s)$ where $l_j \in \{l_j\}$ and $s (\in \mathbb{Z}^\nu)$ satisfies the condition

$$\exists p \in \mathbb{Z} \text{ such that } (2p+1)^{-1} s \in \mathbb{Z}^\nu . \tag{3.24}$$

Let

$$\tilde{l}_j = \max(l_j, l_{j_0}) \quad \text{for} \quad l_j \in \{l_j\} \tag{3.25}$$

($l_{j_0}$ is chosen as in Lemma 3.1) and

$$\mathbb{R}_1 \subset \mathbb{R}^\nu, \ \mathbb{R}_1 = \Gamma^c_{2\tilde{l}_j+1}(s) \cap \Big\{ \bigcup_{|r| \leqq r_\gamma} \Gamma_{2n+1}(r) \Big\} , \tag{3.26}$$

$$r_\gamma = \mathscr{I}[(2n+1)^{-1} \gamma^{-1} q_0], \qquad (q_0 > 0) , \tag{3.27}$$

$$\mathbb{R}_2 \subset \mathbb{R}^\nu \ \ \mathbb{R}_2 = \Gamma^c_{2\tilde{l}_j+1}(s) \cap \Big\{ \bigcup_{|r| \geqq r_\gamma} \Gamma_{2n+1}(r) \Big\} , \tag{3.28}$$

$$\mathbb{R}_3 \subset \mathbb{R}^\nu \ \ \mathbb{R}_3 = \Gamma^c_{2l_j+1}(s) \cap \{\mathbb{R}^\nu - \mathbb{R}_1 - \mathbb{R}_2\} . \tag{3.29}$$

We note that by the choice of $\{l_j\}$ and Eqs. (3.24), (3.25) the set $\mathbb{R}_1$ can be filled up with cubes $\Gamma_{2n+1}(r)$. We define in $\mathfrak{H}_M$ the forms

$$W_{M,n}(r) = -\eta M^2 (n^{2\nu - \alpha} + k') \{1 - \Delta[\hat{n}(\Gamma_{2n+1}(r), 0)]\} , \tag{3.30}$$

$$W(r) = -\eta' M^2 \{1 - \Delta[\hat{n}(\Gamma_1(r)), 0]\} . \tag{3.31}$$

There exist values of $\eta$, $k'$ and $\eta'$ in Eqs. (3.30), (3.31) so that $W_{M,n}(r)$ $[W(r)]$ bounds from below the interaction between particles in $\Gamma_{2n+1}(r) \subset \mathbb{R}_1 [\Gamma_1(r) \subset \mathbb{R}_3]$ with all the others. Therefore

$$\begin{aligned} \text{in } \mathfrak{H}_M : h_\gamma - \mu n \geqq & \sum_{\Gamma_{2n+1}(r) \subset \mathbb{R}_1} \{\hat{t}(\Gamma_{2n+1}(r)) + (\tilde{V}_{n,r} - \mu)\, \hat{n}(\Gamma_{2n+1}(r)) + W_{M,n}(r)\} \\ & + \sum_{\Gamma_1(r) \subset \mathbb{R}_3} \{\hat{t}(\Gamma_1(r)) - (\mu + B)\, \hat{n}(\Gamma_1(r)) + A \hat{n}^2(\Gamma_1(r)) + W(r)\} \\ & + \hat{h}_\gamma(\mathbb{R}_2) - \mu \hat{n}(\mathbb{R}_2) \end{aligned} \tag{3.32}$$

where $\tilde{V}_{r,n}$ is defined in Eq. (3.20). Using the ẽstimate Eq. (3.32) in Lemma 2.4 and applying Lemma 3.3 we have

$$\begin{aligned} &Z_\gamma(\mu, \beta, \Gamma^c_{2l_j+1}(s)) \\ &\leqq \prod_{\Gamma_{2n+1}(r) \subset \mathbb{R}_1} \Bigg\{ 1 + \sum_{l=1}^{\infty} \exp[\beta\eta M^2(n^{2\nu-\alpha} + k') + \beta l(\mu - \tilde{V}_{2n+1,r})] \\ &\cdot \mathrm{Tr}_{\mathfrak{H}^l(\Gamma_{2n+1}(s))} \exp[-\beta \mathrm{H}(\Gamma_{2n+1}(s))] \Bigg\} \\ &\cdot \{\mathrm{Tr}_{\mathfrak{H}(\mathbb{R}_2)} \exp[-\beta(\mathrm{H}_\gamma(\mathbb{R}_2)) - \mu \mathrm{N}(\mathbb{R}_2)]\} \cdot \{Z_S(\mu, \beta)\}^{|\mathbb{R}_3|} \\ &+ Z^+_\gamma(\mu, \beta) \Bigg\{ \sum_{r \in \mathbb{Z}^\nu} \zeta \sum_{m=M+1}^{\infty} \exp[-\beta m \tilde{V}_{1,r} - \eta m^2] \Bigg\} \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} Z_S(\mu, \beta) = \mathrm{Tr}_{\mathfrak{H}(\Gamma_1(0))} \exp\{-\beta[\mathrm{T}(\Gamma_1(0)) + A\,\mathrm{N}^2(\Gamma_1(0)) \\ -(B+\mu)\,\mathrm{N}(\Gamma_1(0)) + \mathrm{W}(0)]\} \end{aligned} \tag{3.34}$$

We want to extend Eq. (3.33) to the case in which Eq. (3.24) does not hold. Then $a \in \mathbb{Z}^\nu$ exists so that

$$(s+a)(2n+1)^{-1} \in \mathbb{Z}^\nu, \qquad |a| < 2n+1 \tag{3.35}$$

Eq. (3.33) holds for the new case if one reads there $r+a$ instead of $r$. Therefore a common upper bound for both cases is obtained if one puts $V_{l,r}$ in place of $\tilde{V}_{l,r}$:

$$V_{l,r} = \tilde{V}_{2l,r/2} \tag{3.36}$$

and if one bounds the trace on $\mathfrak{H}(\mathbb{R}_2)$ using superstability by means of products of $Z_S(\mu - V_{1,r}, \beta)$ and extends the products to values of $r$ for which $|r| \geqq r_\gamma$.

Observing that

$$\begin{aligned} &\sum_{r \in \mathbb{Z}^\nu} \zeta \sum_{m=M+1}^{\infty} \exp(-\beta m V_{1,r} - \eta m^2) \\ &\leqq \zeta \exp(-\eta M^2) \sum_{r \in \mathbb{Z}^\nu} \sum_{m=M+1}^{\infty} \exp(-\beta m V_{1,r}) \xrightarrow[\gamma \to 0]{} 0 \end{aligned} \tag{3.37}$$

(see Eq. (3.22)), for $m$ sufficiently large the l.h.s. of Eq. (3.37) is $<1$.

We take the supremum in the l.h.s. of Eq. (3.36) and remembering that

$$Z^+_\gamma(\mu, \beta, \{l_j\}) \geqq Z_\gamma(\mu, \beta)$$

we finally have

$$\gamma^\nu \ln Z_\gamma(\mu,\beta) + \gamma^\nu \ln\left\{1 - \zeta \sum_{r\in\mathbb{Z}^\nu} \sum_{m=M+1}^{\infty} \exp[-\beta m V_{1,r} - \eta m^2]\right\}$$
$$\leqq \gamma^\nu \sum_{|r|\leqq r_\gamma} \ln\left\{1 + \sum_{l=1}^{\infty} \exp[\beta\eta M^2(n^{2\nu-\alpha}+k') - \beta(V_{2n+1,r}-\mu)]\right.$$
$$\left.\cdot [\mathrm{Tr}_{\mathfrak{H}^l(\Gamma_{2n+1}(0))} \exp(-\beta H(\Gamma_{2n+1}(0)))]\right\}$$
$$+ |\Gamma_{2l_{j_0}+1}(0)| \gamma^\nu \ln\left\{1 + \sum_{l=1}^{\infty} \exp[-\beta(\eta' M^2 + A l^2 - l(B+\mu)]\right. \tag{3.38}$$
$$\left.\cdot \mathrm{Tr}_{\mathfrak{H}^l(\Gamma_1(0))} \exp(-\beta T(\Gamma_1(0)))\right\}$$
$$+ \gamma^\nu \sum_{r\geqq r_\gamma} \ln\{\mathrm{Tr}_{\mathfrak{H}(\Gamma_1(0))} \exp[-\beta(T(\Gamma_1(0)) + A N^2(\Gamma_1(0))$$
$$-(B+\mu-V_{1,r}) N(\Gamma_1(0)))]\}\,.$$

We now perform the limit for $\gamma\to 0$, $M\to\infty$, $n\to\infty$ according to Eqs. (3.22) and (3.23). The procedure is quite analogous to the one used in the classical case. By means of Eq. (3.37) the second term in the l.h.s. of Eq. (3.38) vanishes. Since $l_{j_0}$ does not increase faster than $n$ (Lemma 3.1 (i)) also the second term of the r.h.s. vanishes. The first term in the r.h.s. reconstructs the integral of the free boundary condition pressure in the thermodynamic limit extended to a region $|x|\leqq q_0$. The Lebesgue theorem has been taken into account, by Eq. (2.5) of Lemma 2.5. Use of the same bound (2.5) shows that the last term in the l.h.s. of Eq. (3.38) reproduces the integral of a summable function in the region $|x|\geqq q_0$. Finally we let $q_0\to\infty$ and the proof is completed. Q.E.D.

**Proposition 1.** *The limit* (2.8) *exists and*

$$P^0(\mu,\beta) = P(\mu,\beta)$$

(see Definition 15).

The validity of Proposition 1 has been proved for free particles and for positive decreasing interactions [8]. Furthermore a similar problem has been treated by the Wiener integral technique and independence of the boundary conditions has been proved in certain cases for hard core interactions and positive interactions [10, 11][4]. For a larger class of interactions studies are in progress.

[4] The results given in the above papers are in fact proved only in certain regions of the space $(\beta,\mu)$. In Ref. [10] the region is specified, for boson, by the condition $\exp[\beta(\mu+B)] < +1$. In Ref. [11] the thesis is proved if one of the two conditions is fulfilled

$$\text{either } \exp[\beta(\mu+B)]<1 \quad \text{or} \quad \exp\left[\beta(B+\mu)-\frac{a^2}{2\beta}\right] < \frac{2}{3\nu}$$

where $a\geqq 0$ is the hard core radius (when $a=0$ the interaction is assumed to be non negative). Further in Maxwell-Boltzmann statistic the result is obtained for all $\beta,\mu$.

**Theorem 3.3.** *Let $\Phi$ satisfy Definition 11, Definition 13 and Definition 14 and Proposition 1 hold then*

$$\lim_{\gamma\to 0}\gamma^{\nu}\beta^{-1}\ln Z_{\gamma}(\mu,\beta)=\int_{\mathbb{R}^{\nu}} dx\, P(\mu-v(x),\beta)\,.$$

The proof follows trivially from Theorems 3.1 and 3.2.

## 4. An Application

We show now a possible application of Theorem 3.3 (the main results of this section are already contained in Ref. [12, 13]). Let us consider a one dimensional gas of $N$ identical quantal particles of mass $m$ interacting via a pair inverse square potential $V_{ij}=g(x_i-x_j)^{-2}$ and an harmonic oscillator well $V_i=\frac{1}{2}m\gamma^2\omega^2x_i^2$. In this case $\Phi$ satisfies Definition 5, Definition 11, Definition 13, and Definition 14 and $V$ Definition 4; moreover the interaction is positive and decreasing so that Proposition 1 holds (Robinson) and Theorem 3.3 applies. Further the self-adjoint extension of the hamiltonian that we consider in Section 2 is the most general one since the operator is essentially self-adjoint [14–16]. The energy spectrum is [14]

$$E_{n_1,\ldots,n_N}=\hbar\gamma\omega\left\{\sum_{k=1}^{N}kn_k+\frac{N^2}{2}+\frac{1}{2}\left(a-\frac{1}{2}\right)N(N-1)\right\}$$

$$n_k=0,1,2,\ldots$$

$$a=\tfrac{1}{2}[1+4mg\hbar^{-2}]^{1/2}$$

where $g>-\hbar^2/4m$.

In the limit $\gamma\to 0$ we have

$$\lim_{\gamma\to 0}\gamma\ln Z_{\gamma}=(\hbar\beta\omega)^{-1}\int_0^{y^*}dx\ln\frac{\exp[\beta\mu-x(a+\frac{1}{2})]}{1-\exp(-x)} \tag{4.1}$$

where $y^*$ is implicitly defined by

$$\exp y^*-\exp(\beta\mu)\exp[-y^*(a-\tfrac{1}{2})]=1\,.$$

We transform Eq. (5.1) by a change of variables

$$\lim_{\gamma\to 0}\gamma\ln Z_{\gamma}=(\hbar\beta\omega)^{-1}\int_0^{\infty}dy\,u(y) \tag{4.2}$$

where $u$ is defined by

$$[1-\exp(-u)]=\exp[\beta\mu-y-(a+\tfrac{1}{2})u]\,.$$

It is easily proved that Eq. (4.2) is compatible with Theorem 3.3 if we assume the pressure with rigid walls:

$$P(\mu,\beta)=\frac{m}{\pi\hbar\beta}\int_0^{\infty}dt\ln\{1+\exp[-\beta\varepsilon(t)]\}$$

where

$$\varepsilon(t) = -\mu + \frac{m}{2} t^2 + \beta^{-1} \left(a - \frac{1}{2}\right) \ln\{1 + \exp[-\beta \varepsilon(t)]\} .$$

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Carlo Marchioro
Istituto Matematico
Università di Roma
I-00100 Roma, Italia

Errico Presutti
Istituto di Fisica
Università dell'Aquila
l'Aquila, Italia