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Trilinear Lorentz Invariant Forms

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Abstract. Trilinear invariant forms are described over spaces transforming under the so-called elementary representations of $SL(2, C)$ obtained from the Gel'fand-Naimark principal series by analytic continuation in the representation parameters (among these are all infinite-dimensional completely irreducible representations). All such forms are described using a manifestly covariant technique. The method is based on a natural one-one correspondence between the invariant forms and invariant separately homogeneous distributions (called kernels of the forms) in three complex two-dimensional non-zero vectors; thus the problem is completely reduced to a problem of distribution theory. The kernels display analyticity properties in the representation parameters; the results on this point are only sketched.

0. Introduction

0.1. Distribution Theoretic Formulation of the Problem

The problem on continuous polylinear invariant forms over spaces transforming under elementary representations[1] of the connected Lorentz group $\mathscr{L}_+^\uparrow$ (or of its universal covering group, $SL(2, C)$, consisted of all complex unimodular 2×2 matrices) has been raised in [2]. The interest in the forms arises from the fact that these provide powerful tools in studying elementary representations. For example, through the use of the bilinear invariant forms (which have been studied thoroughly in [2]) one can determine intertwining operators, equivalence conditions, existence of invariant pre-Hilbert structures, etc. The present paper is devoted to description of continuous trilinear invariant forms (over the spaces mentioned above). The importance of the trilinear forms is in their intimate connection with analysis of tensor product of two elementary (and, in particular, of infinite-dimensional completely irreducible[2]) representations of the Lorentz group (see [3] and Remark

[1] For the definition and general properties of elementary representations of a complex semi-simple Lie group, we refer to [1]. The case of $SL(2, C)$ which is our main concern is treated in great detail in [2].

[2] A representation of a group in a topological vector space is said to be completely irreducible if the weakly closed linear hull of the representation operators contains all continuous operators in the representation space [1]. The complete irreducibility implies the topological and the operator irreducibility.

5.5 below). Another related problem, namely, the description of Lorentz covariant bilinear forms and covariant operators over elementary representation spaces, has been treated in [4]; such forms and operators are basic for infinite – component field theory.

It is a standard convention to take the group $\boldsymbol{SL(2, C)}$ as a substitute for $\mathscr{L}_+^\uparrow$. (This substitution corresponds to treating double-valued representations of $\mathscr{L}_+^\uparrow$ on equal footing with ordinary ones.) According to [1, 2], elementary representations of $\boldsymbol{SL(2, C)}$ are in one-one correspondence with pairs $\chi \equiv (\lambda, \mu)$, called *indices*, where λ and μ are arbitrary complex numbers such that $\lambda - \mu$ is integral. By $\mathfrak{X}$ we denote the set of all indices. The representation T_χ of $\boldsymbol{SL(2, C)}$ corresponding to an index χ can be realized in the complete locally convex nuclear space, $\mathfrak{D}_\chi$, consisted of all (complex-valued) $\mathscr{C}^\infty$ functions of a complex two-dimensional non-zero vector $\zeta \in \mathring{\boldsymbol{C}}_2 = \boldsymbol{C}_2 \backslash \{0\}$ which are homogeneous of index χ. Here and in the following a function (or, more generally, a distribution) f in the complex n-dimensional Euclidean space, $\boldsymbol{C}_n$, or in $\mathring{\boldsymbol{C}}_n := \boldsymbol{C}_n \backslash \{0\}$ is said to be *homogeneous of index* χ [or, equivalently, *homogeneous of bi-degree* $(\lambda - n/2, \mu - n/2)$] provided

$$f(az) = \varphi_\chi^{[n]}(a) f(z), \quad \forall a \in \mathring{\boldsymbol{C}}_1, \tag{0.1}$$

where $\varphi_\chi^{[n]}$ is the following $\mathscr{C}^\infty$ function in $\mathring{\boldsymbol{C}}_1$:

$$\varphi_\chi^{[n]}(a) = |a|^{\lambda + \mu - n} e^{i(\lambda - \mu)\arg a}. \tag{0.2}$$

A topology in $\mathfrak{D}_\chi$ is that induced by the standard topology (of compact convergence of all derivatives) of the space $\mathscr{E}(\mathring{\boldsymbol{C}}_2)$ of all $\mathscr{C}^\infty$ functions in $\mathring{\boldsymbol{C}}_2$ ([5]). Now the action T_χ of $\boldsymbol{SL(2, C)}$ on $\mathfrak{D}_\chi$ is defined in the following manifestly covariant way[3]:

$$(T_\chi(A) f)(\zeta) = f(A^{-1}\zeta) \quad \text{for all} \quad f \in \mathfrak{D}_\chi,\ A \in \boldsymbol{SL(2, C)},\ \zeta \in \mathring{\boldsymbol{C}}_2. \tag{0.3}$$

The elementary representation T_χ is completely irreducible provided $\chi \notin \mathfrak{P}_+^{[2]} \cup \mathfrak{P}_-^{[2]}$ where $\mathfrak{P}_+^{[2]} := \{\chi \in \mathfrak{X} \mid \text{both } \lambda \text{ and } \mu \text{ are positive integers}\}$ and $\mathfrak{P}_-^{[2]} := \{-\chi \equiv (-\lambda, -\mu) \mid \chi \in \mathfrak{P}_+^{[2]}\}$, and two such representations, say T_χ and $T_{\chi'}$ (with $\chi, \chi' \notin \mathfrak{P}_+^{[2]} \cup \mathfrak{P}_-^{[2]}$), are equivalent precisely when $\chi' = \pm\chi$. In case $\chi \in \mathfrak{P}_+^{[2]}$ $\mathfrak{D}_\chi$ contains the finite – dimensional subspace, E_χ, of functions polinomial in ζ and $\bar{\zeta}$, while in case $\chi \in \mathfrak{P}_-^{[2]}$ $\mathfrak{D}_\chi$ contains an invariant (proper closed) infinite-dimensional subspace.

Most of problems on elementary representations (such as the problem on polylinear invariant forms) need an effective realization of the space $\mathfrak{D}'_\chi$, dual of $\mathfrak{D}_\chi$. A manifestly covariant realization of $\mathfrak{D}'_\chi$ can be

[3] We call a group representation [like (0.3)] in a function space manifestly covariant if it is implemented solely via argument transformations of functions.

carried out in terms of distributions[4]. Namely, there exists an $\boldsymbol{SL(2, C)}$ invariant isomorphism of $\mathfrak{D}'_\chi$ onto the subspace $\mathfrak{d}_{-\chi}$ of all distributions in $\overset{\circ}{C}_2$ homogeneous of index $-\chi \equiv (-\lambda, -\mu)$ ([6]). In order to construct explicitly this isomorphism, we define, for every index χ, the following continuous operator I_χ from $D(\overset{\circ}{C}_2)$ into $\mathfrak{D}_\chi$:

$$(I_\chi F)(\zeta) = \int_{\overset{\circ}{C}_1} \varphi_\chi^{[2]}(a^{-1}) F(a\zeta) \frac{|da\, d\bar{a}|}{|a|^2}, \qquad \forall F \in D(\overset{\circ}{C}_2). \tag{0.4}$$

I_χ is $\boldsymbol{SL(2, C)}$ invariant [in the self-evident sense: $I_\chi(F_A) = (I_\chi F)_A$ where $F_A(\zeta) := F(A^{-1}\zeta)$, $\forall A \in \boldsymbol{SL(2, C)}$]. Moreover, it can be proved (see Appendix A) that I_χ is the topological homomorphism of $D(\overset{\circ}{C}_2)$ onto $\mathfrak{D}_\chi$, and its dual operator, $I'_\chi : \mathfrak{D}'_\chi \to D'(\overset{\circ}{C}_2)$, is an isomorphism (topological, provided the dual spaces are endowed e.g. with the weak topologies) of $\mathfrak{D}'_\chi$ onto the subspace $\mathfrak{d}_{-\chi} \subset D'(\overset{\circ}{C}_2)$. Thus we obtain the claimed isomorphism, which sets an element $\varphi \in \mathfrak{D}'_\chi$ into a homogeneous distribution $\Phi \in \mathfrak{d}_{-\chi}$ according as $(\Phi, F) = (\varphi, I_\chi F)$ for all $F \in D(\overset{\circ}{C}_2)$.

We are now in a position to translate the problem on trilinear $\boldsymbol{SL(2, C)}$ invariant forms over elementary representation spaces into the language of distribution theory. (Of course, the construction can be trivially generalized to polylinear invariant forms.) For any triple[5] of indices $X \equiv (\chi_1, \chi_2, \chi_3) \in \mathfrak{X}^3$, we introduce the space $\mathscr{T}(X) \equiv \mathscr{T}(\chi_1, \chi_2, \chi_3)$ of all (separately) continuous trilinear forms φ over $\mathfrak{D}_{\chi_1} \times \mathfrak{D}_{\chi_2} \times \mathfrak{D}_{\chi_3}$ which satisfy the $\boldsymbol{SL(2, C)}$ invariance condition:

$$\varphi(T_{\chi_1}(A) f_1, T_{\chi_2}(A) f_2, T_{\chi_3}(A) f_3) = \varphi(f_1, f_2, f_3), \ \forall A \in \boldsymbol{SL(2, C)}, \ \forall f_j \in \mathfrak{D}_{\chi_j}.$$

On the other hand, let $\mathfrak{T}(X) \equiv \mathfrak{T}(\chi_1, \chi_2, \chi_3)$ be the subspace of all distributions $T \equiv T(\zeta_1, \zeta_2, \zeta_3) \in D'((\overset{\circ}{C}_2)^3) \equiv D'(\overset{\circ}{C}_2 \times \overset{\circ}{C}_2 \times \overset{\circ}{C}_2)$ (in three variables $\zeta_j \in \overset{\circ}{C}_2$) which satisfy the following conditions:

a) $\boldsymbol{SL(2, C)}$ *invariance:*

$$T(A\zeta_1, A\zeta_2, A\zeta_3) = T(\zeta_1, \zeta_2, \zeta_3), \qquad \forall A \in \boldsymbol{SL(2, C)}; \tag{0.5}$$

b) *separate homogeneity* [of triple X]:

$$T(a_1\zeta_1, a_2\zeta_2, a_3\zeta_3) = \left(\prod_{j=1}^{3} \varphi_{\chi_j}^{[2]}(a_j)\right) T(\zeta_1, \zeta_2, \zeta_3), \qquad \forall a_j \in \overset{\circ}{C}_1. \tag{0.6}$$

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[4] Only distributions in complex domains will be used in what follows. Here we remind of the standard notations [5]. For every domain Ω in C_n, $D(\Omega)$ is the Schwartz space of all complex $\mathscr{C}^\infty$ functions with compact supports, and $D'(\Omega)$, the dual of $D(\Omega)$, is the Schwartz space of distributions in Ω. An identification of a complex continuous function f in Ω with a corresponding distribution of $D'(\Omega)$ is performed according as

$$(f, F) = \int_\Omega f(z) F(z) |d^n z\, d^n \bar{z}|, \qquad \forall F \in D(\Omega).$$

[5] In what follows, for a set S and an integer $n > 0$, S^n denotes the Cartesian product of n copies of S.

We claim that there is a one – one correspondence between $\mathscr{T}(\chi_1, \chi_2, \chi_3)$ and $\mathfrak{T}(-\chi_1, -\chi_2, -\chi_3)$ defined as follows: for every $\varphi \in \mathscr{T}(\chi_1, \chi_2, \chi_3)$, a unique distribution $T \in \mathfrak{T}(-\chi_1, -\chi_2, -\chi_3)$, called a *kernel of the trilinear form* φ, exists such that

$$T(F_1 \otimes F_2 \otimes F_3) = \varphi(I_{\chi_1} F_1, I_{\chi_2} F_2, I_{\chi_3} F_3), \quad \forall F_j \in D(\mathring{C}_2). \tag{0.7}$$

Indeed, in view of the Schwartz nuclear theorem, $\mathfrak{T}(-\chi_1, -\chi_2, -\chi_3)$ can be identified with the space of all (separately) continuous trilinear $\boldsymbol{SL(2, C)}$ invariant forms $(F_1, F_2, F_3) \mapsto T(F_1 \otimes F_2 \otimes F_3)$ over $D(\mathring{C}_2) \times D(\mathring{C}_2) \times D(\mathring{C}_2)$ such that, for any $j = 1, 2, 3$, the form $F_j \mapsto T(F_1 \otimes F_2 \otimes F_3)$ considered as a linear functional in F_j is a distribution of $\mathfrak{d}_{-\chi_j}$. Hence, the claimed statement is obtained merely by applying three times $\boldsymbol{SL(2, C)}$ invariant isomorphisms between $\mathfrak{D}'_{\chi_j}$ and $\mathfrak{d}_{-\chi_j}$. Note that φ can be expressed through T by

$$\varphi(f_1, f_2, f_3) = T(R_{\chi_1} f_1 \otimes R_{\chi_2} f_2 \otimes R_{\chi_3} f_3), \quad \forall f_j \in \mathfrak{D}_{\chi_j}, \tag{0.8}$$

where $R_\chi : \mathfrak{D}_\chi \to D(\mathring{C}_2)$ is a right inverse operator for I_χ (which exists but is not unique; however the right-hand side of (0.8) is independent of a choice of R_{χ_j}, according to Lemma A.3 of Appendix A).

0.2. The Main Result. Outline of the Proof

In the present paper we describe the spaces $\mathfrak{T}(X)$ of all distributions of $D'((\mathring{C}_2)^3)$ satisfying conditions (0.5) and (0.6). All continuous trilinear $\boldsymbol{SL(2, C)}$ invariant forms over $\mathfrak{D}_{-\chi_1} \times \mathfrak{D}_{-\chi_2} \times \mathfrak{D}_{-\chi_3}$ are thereby described for an arbitrary triple $X \equiv (\chi_1, \chi_2, \chi_3) \in \mathfrak{X}^3$. It turns out that $\mathfrak{T}(X)$ is non-trivial (i.e. different from zero) if and only if X belongs to

$$\Xi := \left\{ X \equiv (\chi_1, \chi_2, \chi_3) \in \mathfrak{X}^3 \,\middle|\, \tfrac{1}{2} \sum_{j=1}^{3} (\lambda_j - \mu_j) \text{ is integral} \right\}. \tag{0.9}$$

The fact that $\mathfrak{T}(X)$ consists only of zero for $X \in \mathfrak{X}^3 \backslash \Xi$ can be easily justified by comparison between (0.5) for $A = -\mathbf{1}$ and (0.6) for $a_j = -1$ ($\forall j = 1, 2, 3$). This is why the condition $X \in \Xi$ will be assumed in all the following. It is shown in Section 5 that dimension of $\mathfrak{T}(X)$ is at most 4. (In particular, $\dim \mathfrak{T}(X) = 1$ if $\chi_j \notin \mathfrak{P}^{[2]}_{-}$ for all $j = 1, 2, 3$.)

In general an explicit form of distributions of $\mathfrak{T}(X)$ is rather involved. Our classification principle of the kernels originates in the following simple observation. There are three (independent) algebraic $\boldsymbol{SL(2, C)}$ invariant combinations built of the variables $\zeta_1, \zeta_2, \zeta_3$, namely,

$$Z_1 = [\zeta_2, \zeta_3], \; Z_2 = [\zeta_3, \zeta_1], \; Z_3 = [\zeta_1, \zeta_2], \tag{0.10}$$

where, by definition, $[\zeta_i, \zeta_j] \equiv \zeta_i^1 \zeta_j^2 - \zeta_i^2 \zeta_j^1$, and ζ_j^α denotes the αth component of the complex vector $\zeta_j \equiv (\zeta_j^1, \zeta_j^2)$. If we introduce the "dual triple" (χ^1, χ^2, χ^3) of indices $\chi^j \equiv (\lambda^j, \mu^j)$ related to $X \equiv (\chi_1, \chi_2, \chi_3)$ by

$$\lambda^j = \left(\tfrac{1}{2} \sum_{i=1}^{3} \lambda_i\right) - \lambda_j, \quad \mu^j = \left(\tfrac{1}{2} \sum_{i=1}^{3} \mu_i\right) - \mu_j, \tag{0.11}$$

then, for $\mathrm{Re}(\lambda^j + \mu^j) > 1$ ($\forall j = 1, 2, 3$), the expression

$$T(\zeta_1, \zeta_2, \zeta_3) = a \prod_{j=1}^{3} |Z_j|^{\lambda^j + \mu^j - 1} e^{i(\lambda^j - \mu^j) \arg Z_j} \tag{0.12}$$

defines obviously a distribution (indeed, a continuous function) $T \in \mathfrak{T}(X)$. Moreover, in these cases distributions (0.12) exhaust $\mathfrak{T}(X)$. It turns out that the multiple in (0.12) can be picked as a function of X in such a way that the resulting expression be analytic in the representation parameters. Indeed, there exists a family $\Psi \equiv \{\Psi_X \in D'((\mathring{C}_2)^3) \mid X \in \Xi\}$ such that Ψ_X belongs to $\mathfrak{T}(X)$ and is analytic in χ_1, χ_2, χ_3. (These requirements define Ψ_X up to a multiple dependent on X.) For the specification of Ψ we use, the reader is referred to Appendix C where further properties of Ψ are summarized without proof. (The corresponding proof is presented in a separate paper [7].)

The relation of Ψ to the problem on $\mathfrak{T}(X)$ is as follows. $\mathfrak{T}(X)$ is one-dimensional precisely when X is not a zero of Ψ (in the sense that $\Psi_X \neq 0$); in this case $\mathfrak{T}(X)$ consists of multiples of Ψ_X. For almost every zero X of Ψ, $\mathfrak{T}(X)$ is exhausted by distributions, called kernels associated with Ψ at X, which are representable in the form

$$P\left(\frac{\partial}{\partial \chi^1}, \frac{\partial}{\partial \chi^2}, \frac{\partial}{\partial \chi^3}\right) \Psi_X \equiv P\left(\frac{\partial}{\partial X}\right) \Psi_X, \tag{0.13}$$

where P is an appropriate complex polynomial in three variables [indeed, an admissible polynomial in the sense that (0.13) belongs to $\mathfrak{T}(X)$], and $\dfrac{\partial}{\partial \chi^j}$ is the derivative with respect to $\frac{1}{2}(\lambda^j + \mu^j)$ [at fixed integral $\lambda^j - \mu^j$; $j = 1, 2, 3$]. The only exceptions to this rule are triples X, called zeros of type 4, which satisfy the condition: $\chi_j \in \mathfrak{P}_-^{[2]}$ for all $j = 1, 2, 3$ and the finite-dimensional representation of $\boldsymbol{SL(2, C)}$ in $E_{-\chi_1} \otimes E_{-\chi_2} \otimes E_{-\chi_3}$ contains the identity representation. In the exceptional case $\mathfrak{T}(X)$ is spanned by the kernels associated with Ψ at X and a distribution Φ_X which is specified in Section 5. Note that, with respect to permutation of indices 1, 2, 3, the distributions Φ_X possess symmetry properties which are to some extent opposite to those of Ψ_X.

The final description of $\mathfrak{T}(X)$ in terms of distributions Ψ_X and Φ_X is given by Theorem 5.4.

We now turn to some remarks on the proof. Undoubtedly, making the most use of invariant terms has to simplify the consideration. With this in mind, we employ relevant properties of the mapping

$$(\zeta_1, \zeta_2, \zeta_3) \mapsto (Z_1, Z_2, Z_3). \tag{0.14}$$

In the subdomain $\boldsymbol{O}$ of $(\mathring{\boldsymbol{C}}_2)^3$ where at least one of the invariants Z_1, Z_2, Z_3 is non-zero [6]:

$$\boldsymbol{O} := \{(\zeta_1, \zeta_2, \zeta_3) \in (\mathring{\boldsymbol{C}}_2)^3 \mid (Z_1, Z_2) \in \mathring{\boldsymbol{C}}_2\}, \tag{0.15}$$

the mapping (0.14) is regular (i.e. its rank as a rank of a holomorphic mapping is 3), and submanifolds $(Z_1, Z_2, Z_3) = \text{const}$ are $\boldsymbol{SL}(2, \boldsymbol{C})$ orbits. This makes it possible to represent every $\boldsymbol{SL}(2, \boldsymbol{C})$ invariant distribution in $\boldsymbol{O}$ as a distribution of the invariants (cf. Lemma 3.1). On the other hand, the remainder set ω,

$$\omega := (\mathring{\boldsymbol{C}}_2)^3 \backslash \boldsymbol{O} = \{(\zeta_1, \zeta_2, \zeta_3) \in (\mathring{\boldsymbol{C}}_2)^3 \mid Z_1 = Z_2 (= Z_3) = 0\}, \tag{0.16}$$

consists of singular points of (0.14) and is not an $\boldsymbol{SL}(2, \boldsymbol{C})$ orbit. These observations suggest to divide the problem on $\mathfrak{T}(X)$ into the three Steps:

Step One. To describe the space, $\mathfrak{F}(X)$, of all $\boldsymbol{SL}(2, \boldsymbol{C})$ invariant separately homogeneous [of triple X] distributions of $D'(\boldsymbol{O})$.

Step Two. To describe the subspace, $\mathfrak{T}_0(X)$, of all distributions of $\mathfrak{T}(X)$ with supports in ω.

Step Three. To find the subspace, $\hat{\mathfrak{F}}(X)$, of all distributions of $\mathfrak{F}(X)$ which possess extensions to distributions of $\mathfrak{T}(X)$.

While Step One is very easy, treating effectively the other Steps requires the auxiliary technical means developped in Sections 1 and 2 (and utilized also in [7]). The method in Section 1 reduces a space $\mathfrak{T}(X)$ with "too singular" kernels to a space $\mathfrak{T}(X')$ with less singular kernels. For this purpose we use special isomorphisms between $\mathfrak{T}(X)$'s which can be implemented by successive applications of the following (multiplicative and differential) homogeneous $\boldsymbol{SL}(2, \boldsymbol{C})$ invariant operators in $D'((\mathring{\boldsymbol{C}}_2)^3)$: Z_j, Δ_j $(j = 1, 2, 3)$ and their complex conjugates, $\bar{Z}_j$, $\bar{\Delta}_j$, where

$$\begin{gathered} \Delta_1 = \left[\frac{\partial}{\partial \zeta_2}, \frac{\partial}{\partial \zeta_3}\right] \equiv \frac{\partial}{\partial \zeta_2^1} \frac{\partial}{\partial \zeta_3^2} - \frac{\partial}{\partial \zeta_3^2} \frac{\partial}{\partial \zeta_3^1}, \\ \Delta_2 = \left[\frac{\partial}{\partial \zeta_3}, \frac{\partial}{\partial \zeta_1}\right], \quad \Delta_3 = \left[\frac{\partial}{\partial \zeta_1}, \frac{\partial}{\partial \zeta_2}\right]. \end{gathered} \tag{0.17}$$

[6] Note that the vanishing of any two of the invariants Z_1, Z_2, Z_3 implies the vanishing of the third one, since then $\zeta_1, \zeta_2, \zeta_3$ are complex-collinear vectors of $\mathring{\boldsymbol{C}}_2$.

In Section 2 the $\boldsymbol{SL}(2, \boldsymbol{C})$ invariance condition is treated in the infinitesimal form as a system of differential equations in $D'((\overset{\circ}{\boldsymbol{C}}_2)^3)$. It suffices to solve the system in a certain subdomain which generates the whole $(\overset{\circ}{\boldsymbol{C}}_2)^3$ by $\boldsymbol{SL}(2, \boldsymbol{C})$ transformations.

Through accomplishing the three Steps we obtain the complete solution to the problem.

We close this Section with the remark on the use of Appendices. Appendix A has already been exploited in Subsection 0.1; a synopsis in Appendix B on homogeneous distributions in $\overset{\circ}{\boldsymbol{C}}_1$ and in $\boldsymbol{C}_1$ is used in Sections 3 and 5; at last, a use of Appendix C is made in Section 5.

1. Special Isomorphisms between $\mathfrak{T}(X)$'s

The result of this Section is represented by Propositions 1.4 and 1.6 where we point out pairs of triples $X, X' \in \Xi$ which are called equivalent and which possess the property: $\mathfrak{T}(X)$ is an isomorphic image of $\mathfrak{T}(X')$ under an operator in $D'((\overset{\circ}{\boldsymbol{C}}_2)^3)$ obtained by successive applications of operators of the form $Z_j, \bar{Z}_j, \Delta_j, \bar{\Delta}_j$. The next two lemmas underline the construction.

1.1. Lemma. *Let $\mathscr{S}$ be the subspace of all distributions of $D'((\overset{\circ}{\boldsymbol{C}}_2)^3)$ satisfying the $\boldsymbol{SL}(2, \boldsymbol{C})$ invariance condition* (0.5). *Then the operators $Z_j \Delta_j$ and $\Delta_j Z_j$ coincide on $\mathscr{S}$ with $(1+\nabla)(\nabla-\nabla_j)$ and $(2+\nabla)(1+\nabla-\nabla_j)$, respectively, where $\nabla_j = \sum_{\alpha=1}^{2} \zeta_j^\alpha \frac{\partial}{\partial \zeta_j^\alpha}$ and $\nabla = \frac{1}{2} \sum_{j=1}^{3} \nabla_j$. (Of course, the analogous statement for the complex conjugate operators is also valid.)*

Proof. Let $T \in \mathscr{S}$. By differentiating (0.5) with respect to group parameters at the identity, we obtain the infinitesimal form of the $\boldsymbol{SL}(2, \boldsymbol{C})$ invariance condition[7]:

$$\left(\sum_{j=1}^{3} \sum_{\alpha,\beta=1,2} a_\beta^\alpha \zeta_j^\beta \frac{\partial}{\partial \zeta_j^\alpha}\right)^{(*)} T = 0$$

for an arbitrary complex 2×2 matrix (a_β^α) with zero trace; equivalently,

$$\left(\sum_{j=1}^{3} \zeta_j^\beta \frac{\partial}{\partial \zeta_j^\alpha} - \delta_\alpha^\beta \nabla\right)^{(*)} T = 0, \quad \forall \alpha, \beta = 1, 2, \tag{1.1}$$

[7] We use the convention: $\Pi^{(*)} T = 0$ denotes the system of the equations $\Pi T = 0$ and $\bar{\Pi} T = 0$, where T is a distribution, Π is a differential operator, and $\bar{\Pi}$ is its complex conjugate.

where (δ_α^β) is the 2×2 identity matrix. Now elementary manipulations yield for all $j=1,2,3$:

$$\sum_{i\in\{1,2,3\}\setminus\{j\}} Z_i \Delta_i T \equiv \sum_{i=1}^{3} \sum_{\alpha,\beta,\gamma,\varkappa=1,2} \varepsilon_{\alpha\beta} \zeta_i^\alpha \zeta_j^\beta \varepsilon^{\gamma\varkappa} \frac{\partial}{\partial \zeta_i^\gamma} \frac{\partial}{\partial \zeta_j^\varkappa} T$$
$$=(1+\nabla)\nabla_j T, \quad \text{where} \quad \varepsilon_{\alpha\beta}=\varepsilon^{\alpha\beta}=(-1)^\beta \delta_{3-\alpha}^\beta .$$

We have obtained a linear system with respect to $Z_j\Delta_j T$, which implies $Z_j \Delta_j T=(1+\nabla)(\nabla-\nabla_j)T$. At last, using the expression for the commutator in $D'((\mathring{C}_2)^3): \Delta_j Z_j - Z_j \Delta_j = 2+2\nabla-\nabla_j$, we obtain $\Delta_j Z_j T =(2+\nabla)(1+\nabla-\nabla_j)T$. Q.E.D.

Remark that, by virtue of homogeneity condition (0.6), the restrictions of the operators ∇, $\nabla-\nabla_j$ and of their conjugates, $\bar\nabla$, $\bar\nabla-\bar\nabla_j$, to $\mathfrak{T}(X)$ are equal to numbers $\mathfrak{l}-1$, $\lambda^j-\frac{1}{2}$ and $\mathfrak{m}-1$, $\mu^j-\frac{1}{2}$, respectively, where λ^j and μ^j are defined in (0.11) and the index $\mathfrak{x}\equiv(\mathfrak{l},\mathfrak{m})\in\mathfrak{X}$ is defined via

$$\begin{aligned} \mathfrak{l} &= \tfrac{1}{2}\Big(-1+\sum_{j=1}^{3}\lambda_j\Big) = -\tfrac{1}{2}+\sum_{j=1}^{3}\lambda^j, \\ \mathfrak{m} &= \tfrac{1}{2}\Big(-1+\sum_{j=1}^{3}\mu_j\Big) = -\tfrac{1}{2}+\sum_{j=1}^{3}\mu^j . \end{aligned} \tag{1.2}$$

1.2. Lemma. *Let X, X' be a pair of triples of Ξ such that $\mathfrak{l}\neq-1$, $\lambda^j\neq-\frac{1}{2}$ for some $j\in\{1,2,3\}$ and X' is obtained from X by equating all the numbers λ'^i, μ'^i $(i=1,2,3)$, except λ'^j, to the corresponding numbers λ^i, μ^i and the number λ'^j to λ^j+1. Then Δ_j maps isomorphically $\mathfrak{T}(X')$ onto $\mathfrak{T}(X)$ and Z_j maps isomorphically $\mathfrak{T}(X)$ onto $\mathfrak{T}(X')$. Analogous statement holds if all the symbols $\mathfrak{l}$, λ^i, Δ_j, Z_j are replaced by $\mathfrak{m}$, μ^i, $\bar\Delta_j$, $\bar Z_j$, respectively.*

Proof. Lemma 1.1 yields $\Delta_j Z_j T=(\mathfrak{l}+1)(\lambda^j+\frac{1}{2})T$ for all $T\in\mathfrak{T}(X)$ and $Z_j\Delta_j S=(\mathfrak{l}+1)(\lambda^j+\frac{1}{2})S$ for all $S\in\mathfrak{T}(X')$. This is just the assertion of the lemma. Q.E.D.

1.3. Definition. For any two complex numbers a,b such that $a-b$ is integral, we denote by $\mathscr{I}(a;b)$ the following subset of the complex plane:

$$\mathscr{I}(a;b)=\{a+n\,|\,n \text{ is integral and } 0\leqq n<|a-b|\} \text{ if } a-b\leqq 0, \tag{1.3a}$$

$$\mathscr{I}(a;b)=\{b+n\,|\,n \text{ is integral and } 0\leqq n<|a-b|\} \text{ if } a-b\geqq 0. \tag{1.3b}$$

We introduce an *equivalence relation*, $\sim$, on Ξ as follows: $X\sim X'$ if and only if, for all $j=1,2,3$, the following conditions are satisfied:

$$\begin{aligned} &\lambda^j-\lambda'^j \text{ is integral}, \quad -\tfrac{1}{2}\notin\mathscr{I}(\lambda^j;\lambda'^j), \quad -1\notin\mathscr{I}(\mathfrak{l};\mathfrak{l}'), \\ &\mu^j-\mu'^j \text{ is integral}, \quad -\tfrac{1}{2}\notin\mathscr{I}(\mu^j;\mu'^j), \quad -1\notin\mathscr{I}(\mathfrak{m},\mathfrak{m}'). \end{aligned} \tag{1.4}$$

1.4. Proposition. *For a pair of equivalent triples X and X' of Ξ, there exists an operator M in $D'((\mathring{C}_2)^3)$ which is representable as a product of operators of the form Z_j, $\bar{Z}_j$, Δ_j, $\bar{\Delta}_j$ $(j=1,2,3)$ and which maps isomorphically $\mathfrak{T}(X')$ onto $\mathfrak{T}(X)$.*

Proof. By condition, the number $\sum_{j=1}^{3}(|\lambda^j-\lambda'^j|+|\mu^j-\mu'^j|)=n$ is integral. Since case $n=0$ is trivial and case $n=1$ has been actually proved in Lemma 1.2, it suffices to reduce case $n>1$ to case $n=1$. Indeed, there is a finite chain $\{X_{(0)},\ldots,X_{(n)}\}$ of equivalent triples satisfying the conditions: $X_{(0)}=X$, $X_{(n)}=X'$ and

$$\sum_{j=1}^{3}(|\lambda^j_{(r-1)}-\lambda^j_{(r)}|+|\mu^j_{(r-1)}-\mu^j_{(r)}|)=1 \quad \text{for all} \quad r=1,2,\ldots,n.$$

Thus, the general case is obtained by successive applications of case $n=1$. Q.E.D.

1.5. Remark. By replacing a given triple X with an appropriate equivalent one and by using Proposition 1.4, a simplification in an explicit form of distributions of $\mathfrak{T}(X)$ can in general be achieved. We now develop such a reduction procedure for a particular case (which is important in what follows). Define

$$\Xi_0=\{X\in\Xi\,|\,\mathfrak{x}\in\mathfrak{P}^{[2]}_-,\text{ and }\lambda^j\text{ (hence }\mu^j)\text{ is half-integral}^8;\ \forall j=1,2,3\}. \tag{1.5}$$

Given $X\in\Xi_0$, it seems to be reasonable, by replacing X with an equivalent triple (of Ξ_0), to make all the numbers $|\lambda^j|$, $|\mu^j|$ as little as possible. A point $X\in\Xi_0$ for which the number $\sum_{j=1}^{3}|\lambda^j|+|\mu^j|$ cannot be diminished by replacing X with an equivalent triple, is called *extremal*. It is evident that there is only a finite number of extremal triples of Ξ_0. Moreover, for an extremal triple X, $\mathfrak{l}=-2$ if $\lambda^j=-\frac{1}{2}$ for all j, and $\mathfrak{l}=-1$ otherwise; analogous statement holds when $\mathfrak{l}$ and λ^j are replaced by $\mathfrak{m}$ and μ^j; thus $\mathfrak{l}\mathfrak{m}$ is equal to one of the numbers $1,2,4$. Further, it is easy to prove [by induction on the numbers $|\lambda^j|+\frac{1}{2}$, $|\mu^j|+\frac{1}{2}$] that, for every $X\in\Xi_0$, there is a unique extermal triple on Ξ_0 which is equivalent to X; in all the following this triple is denoted by $\dot{X}$. More explicitly, $\dot{X}$ is related to X via

$$\dot{\lambda}^j=\lambda^j+l^j_+-l^j_-, \qquad \dot{\mu}^j=\mu^j+m^j_+-m^j_-\,; \tag{1.6}$$

here $l^j_\pm$ (and analogously $m^j_\pm$) are non-negative integers defined as follows: if λ^j is positive then $l^j_+=\lambda^j-\frac{1}{2}$ and $l^j_-=0$; if λ^j is negative then $l^j_+=0$ and l^j_- equals either $-\lambda^j+\frac{1}{2}$ (provided $\lambda^i<0$ for some $i\neq j$) or

[8] By convention, a half-integer is a number of the form $n+\frac{1}{2}$ with n integral.

$-\lambda^j+\frac{3}{2}$ (provided $\lambda^i>0$ for all $i\neq j$); analogously $m^j_\pm$ are expressed through μ^j. At last, for every $X\in\Xi_0$, we define the number $\nu(X)=\dot{\mathfrak{l}}\dot{\mathfrak{m}}$ [which is evaluated in terms of $\dot{X}$ and which assumes only one of the numbers 1, 2, 4].

Next we can state the following

1.6. Proposition. *Let X be a triple of Ξ_0 (1.5) and let $\dot{X}$ be the extermal triple of Ξ_0 which is equivalent to X. Then every kernel $T\in\mathfrak{T}(X)$ can be represented in the form*

$$T(\zeta_1,\zeta_2,\zeta_3)=\left[\prod_{j=1}^{3}(Z_j)^{l^j_+}(\bar{Z}_j)^{m^j_+}(\Delta_j)^{l^j_-}(\bar{\Delta}_j)^{m^j_-}\right]\dot{T}(\zeta_1,\zeta_2,\zeta_3)\,, \tag{1.7}$$

where $\dot{T}\in\mathfrak{T}(\dot{X})$; $l^j_\pm$ and $m^j_\pm$ are as those in (1.6). *Indeed,* (1.7) *defines a bijection of $\mathfrak{T}(\dot{X})$ onto $\mathfrak{T}(X)$.*

Proposition 1.6 is a special case of Proposition 1.4 and, as the latter, is obtained from Lemma 1.2 by induction [first on l^j_+ and m^j_+ $(j=1,2,3)$, then on l^j_- and m^j_-].

2. Characterization of Kernels of the Forms in Question through Their Restrictions to a Certain Subdomain

The $\boldsymbol{SL(2,C)}$ invariance condition (0.5) in $D'((\mathring{C}_2)^3)$ can be written in the infinitesimal form as system (1.1). Since the complex Lie algebra of $\boldsymbol{SL(2,C)}$ can be generated by two elements, there is a subsystem which is equivalent to (1.1), e.g.

$$\left(\sum_{j=1}^{3}\zeta^1_j\frac{\partial}{\partial\zeta^2_j}\right)^{(*)}T=0,\quad \left(\sum_{j=1}^{3}\zeta^2_j\frac{\partial}{\partial\zeta^1_j}\right)^{(*)}T=0\,. \tag{2.1}$$

It is a general fact (independent of a specific $\mathscr{C}^\infty$ action of a connected Lie group G on a domain Ω) that the infinitesimal form of invariance condition (in other words, the $\mathscr{G}$-invariance condition where $\mathscr{G}$ is the Lie algebra of G) is equivalent to the global form. Moreover, if Ω_1 is a subdomain of Ω such that $\Omega=G\Omega_1$ and, for any $x\in\Omega_1$, the (open) set $G(x):=\{g\in G\,|\,gx\in\Omega_1\}$ is connected, then there is a one-one (canonical) correspondence between the subspace $D'(\Omega)^G$ of all G-invariant distributions of $D'(\Omega)$ and the subspace $D'(\Omega_1)^{\mathscr{G}}$ of all $\mathscr{G}$-invariant distributions of $D'(\Omega_1)$. The statement can be easily proved by the standard argument [9]. It suffices, for every distribution $h_1\in D'(\Omega_1)^{\mathscr{G}}$ to define a distribution $H(g,x):=h_1(gx)$ in two variables (g,x) of the corresponding subdomain of $G\times\Omega$ and to show that H is independent of g and can be written as $H(g,x)=h(x)$ with $h\in D'(\Omega)^G$; this then completes the proof.

By applying this remark to our problem we obtain

[9] Cf. [8], Theorem 2-11.

2.1. Lemma. *Every* $\boldsymbol{SL(2,C)}$ *invariant distribution* $T \in D'((\mathring{\boldsymbol{C}}_2)^3)$ [*which, of course, satisfies* (2.1)] *is characterized uniquely by its restriction,* $T|_{\boldsymbol{Q}}$, *to the subdomain* $\boldsymbol{Q}$,

$$\boldsymbol{Q} := \{(\zeta_1, \zeta_2, \zeta_3) \in (\mathring{\boldsymbol{C}}_2)^3 \,|\, \zeta_j^1 \neq 0, \ \forall j = 1, 2, 3\} = (\mathring{\boldsymbol{C}}_1 \times \boldsymbol{C}_1)^3 . \quad (2.2)$$

On the other hand, any distribution $T_1 \in D'(\boldsymbol{Q})$ *satisfying system* (2.1) *in* $\boldsymbol{Q}$ *possesses a unique* $\boldsymbol{SL(2,C)}$ *invariant extension* $T \in D'((\mathring{\boldsymbol{C}}_2)^3)$. *If, in addition,* T_1 *satisfies* (0.6) *in* $\boldsymbol{Q}$ *then* $T \in \mathfrak{T}(X)$.

Proof. By virtue of the preceding remark, the first two statements of the lemma are implied by the following two facts. First, $(\mathring{\boldsymbol{C}}_2)^3 = \boldsymbol{SL(2,C)Q}$ [i.e. $(\mathring{\boldsymbol{C}}_2)^3$ consists of elements of the form $(A\zeta_1, A\zeta_2, A\zeta_3)$ with $A \in \boldsymbol{SL(2,C)}$ and $(\zeta_1, \zeta_2, \zeta_3) \in \boldsymbol{Q}$]. Second, for any $(\zeta_1, \zeta_2, \zeta_3) \in \boldsymbol{Q}$, the subset $G(\zeta_1, \zeta_2, \zeta_3) := \{A \in \boldsymbol{SL(2,C)} \,|\, (A\zeta_1, A\zeta_2, A\zeta_3) \in \boldsymbol{Q}\}$ is connected; indeed, in the special case, with $\zeta_1 = (1, 0)$, $G(\zeta_1, \zeta_2, \zeta_3)$ is parametrized by the matrix elements A_1^1, A_2^1, A_1^2 which vary in a connected domain of $\boldsymbol{C}_3$ defined by $A_1^1 \neq 0$, $A_1^1 \zeta_2^1 + A_2^1 \zeta_2^2 \neq 0$, $A_1^1 \zeta_3^1 + A_2^1 \zeta_3^2 \neq 0$; in general case $G(\zeta_1, \zeta_2, \zeta_3)$ is also connected since it is obtained from the special one by applying an $A \in \boldsymbol{SL(2,C)}$. At last, the final statement of the lemma is a consequence of the first two ones; in fact, if two $\boldsymbol{SL(2,C)}$ invariant distributions (such as both sides of (0.6)) coincide in $\boldsymbol{Q}$, they coincide throughout $(\mathring{\boldsymbol{C}}_2)^3$. Q.E.D.

Our further results make use of the notion of a change of variables in a distribution and, more generally, the notion of a composite of a regular mapping with a distribution.

2.2. Definition[10]. Let Ω_1 and Ω_2 be domains of $\boldsymbol{C}_m$ and $\boldsymbol{C}_n$, respectively, where $m \geqq n$, and let $u : \Omega_1 \to \Omega_2$ be a regular holomorphic mapping of Ω_1 onto Ω_2. [The regularity of u means that the rank of u as a rank of a holomorphic mapping equals n.] Then, for any fixed $w \in \Omega_2$, $U_w(z) := \delta(w - u(z)) = \prod_{i=1}^{n} \delta(w_i - u_i(z))$ is a well defined distribution of $D'(\Omega_1)$ which depends in $\mathscr{C}^\infty$ way on w as on a parameter. It is easily seen that $U_w(z)$, considered as a "kernel", defines a continuous operator U which maps $D(\Omega_1)$ onto $D(\Omega_2)$ by the formula $(Uf)(w) := (U_w(z), f(z))_z$ for all $f \in D(\Omega_1)$. Now, by definition, *the composite* $F \circ u$ *of the mapping* u *with a distribution* $F \in D'(\Omega_2)$ is the distribution of $D'(\Omega_1)$ obtained by applying the dual operator $U' : D'(\Omega_2) \to D'(\Omega_1)$ [which is injective] to F, i.e. $(F \circ u, f) := (F(w), (U_w(z), f(z))_z)_w$ for all $f \in D(\Omega_1)$. In particular, if $m = n$ and u is a holomorphic differmorphism of $\Omega_1$onto Ω_2, then the mapping $F \mapsto F \circ u$ is an isomorphism of $D'(\Omega_2)$ onto $D'(\Omega_1)$ defined by

[10] For a more detail see e.g. [9], Chapter III.

$(F \circ u, |J|^2 \cdot (f \circ u)) = (F, f)$ for all $f \in D(\Omega_2)$, where $J := \dfrac{D(u_1, \ldots, u_n)}{D(z_1, \ldots, z_n)}$ is the Jacobian of u.

The well known formula of differentiating a composite of two functions is applicable to the composite of a regular mapping with a distribution. There is one another rule which will be tacitly used below: If $u : \Omega_1 \to \Omega_2$ and $v : \Omega_2 \to \Omega_3$ are regular surjections and $F \in D'(\Omega_3)$ then $F \circ (v \circ u) = (F \circ v) \circ u$.

The following lemma, combined with Lemma 2.1, is the next step in constructing global solutions of system (2.1).

2.3. Lemma. *The restriction of an* $\boldsymbol{SL}(2, \boldsymbol{C})$ *invariant distribution* $T \in D'((\mathring{\boldsymbol{C}}_2)^3)$ *to* $\boldsymbol{Q}$ (2.2) *can be represented in the form*

$$T|_{\boldsymbol{Q}}(\zeta_1, \zeta_2, \zeta_3) = B(Z_1, Z_2, p_1, p_2)\,; \tag{2.3}$$

here the right-hand side is the composite of the regular mapping

$$(\zeta_1, \zeta_2, \zeta_3) \mapsto \left(Z_1 = [\zeta_2, \zeta_3], Z_2 = [\zeta_3, \zeta_1], p_1 = \frac{\zeta_1^1}{\zeta_3^1}, p_2 = \frac{\zeta_2^1}{\zeta_3^1} \right) \tag{2.4}$$

of $\boldsymbol{Q}$ *onto* $\boldsymbol{C}_2 \times (\mathring{\boldsymbol{C}}_1)^2$ *with a distribution* $B(z_1, z_2, z_3, z_4) \in D'(\boldsymbol{C}_1 \times \boldsymbol{C}_1 \times \mathring{\boldsymbol{C}}_1 \times \mathring{\boldsymbol{C}}_1)$ *satisfying the system*

$$\left(z_1 \frac{\partial}{\partial z_4} - z_2 \frac{\partial}{\partial z_3} \right)^{(*)} B = 0\,. \tag{2.5}$$

Proof. The mapping $(\zeta_1^1, \zeta_1^2, \zeta_2^1, \zeta_2^2, \zeta_3^1, \zeta_3^2) \mapsto (Z_1, Z_2, p_1, p_2, \zeta_3^1, \zeta_3^2)$ is a holomorphic diffeomorphism of $\boldsymbol{Q}$ onto $\tilde{Q} := \boldsymbol{C}_2 \times (\mathring{\boldsymbol{C}}_1)^2 \times \mathring{\boldsymbol{C}}_2$; indeed, it is linear with respect to the variables subjected to the change and its Jacobian equals -1. Hence $T|_{\boldsymbol{Q}}(\zeta_1, \zeta_2, \zeta_3) = \tilde{T}(Z_1, Z_2, p_1, p_2, \zeta_3^1, \zeta_3^2)$ for some $\tilde{T} \in D'(\tilde{Q})$. In terms of $\tilde{T}$ system (2.1) takes the form

$$\left(\frac{\partial}{\partial z_6} \right)^{(*)} \tilde{T} = 0, \quad \left(-\frac{z_1}{(z_5)^2} \frac{\partial}{\partial z_4} + \frac{z_2}{(z_5)^2} \frac{\partial}{\partial z_3} + z_6 \frac{\partial}{\partial z_5} \right)^{(*)} \tilde{T} = 0\,.$$

After differentiating the second pair of this equation with respect to z_6 and $\bar{z}_6$, we obtain that $\tilde{T}$ satisfies (2.5) and partial derivatives of $\tilde{T}$ with respect to z_5, $\bar{z}_5$, z_6, $\bar{z}_6$ vanish. Since an intersection of any plane $(z_1, z_2, z_3, z_4) =$ const with $\tilde{Q}$ is connected, this implies the existence of $B \in D'(\boldsymbol{C}_2 \times (\mathring{\boldsymbol{C}}_1)^2)$ related to $\tilde{T}$ via $\tilde{T}(z_1, \ldots, z_6) = B(z_1, z_2, z_3, z_4)$, and satisfying (2.5). This then completes the proof. Q.E.D.

By applying Lemmas 3.1 and 3.3 to distributions of $\mathfrak{T}(X)$ we obtain

2.4. Proposition. *The restriction of a distribution* $T \in \mathfrak{T}(X)$ *to* $\boldsymbol{Q}$ (2.2) *is of the form*

$$T|_{\boldsymbol{Q}}(\zeta_1, \zeta_2, \zeta_3) = \varphi_{\chi^1}^{[1]}\left(\frac{1}{p_1} \right) \varphi_{\chi^2}^{[1]}\left(\frac{1}{p_2} \right) b(p_1 Z_1, p_2 Z_2)\,, \tag{2.6}$$

where Z_j *and* p_j $(j=1,2)$ *are defined in* (2.4) *and* $b(z_1,z_2)$ *is a distribution of* $D'(\boldsymbol{C}_2)$ *satisfying the conditions*

$$b(az_1,az_2)=\varphi_{\chi}^{[2]}(a)\,b(z_1,z_2),\qquad \forall a\in \mathring{\boldsymbol{C}}_1\,, \tag{2.7}$$

$$\left[\left(\lambda^2-\tfrac{1}{2}-z_2\frac{\partial}{\partial z_2}\right)z_1-\left(\lambda^1-\tfrac{1}{2}-z_1\frac{\partial}{\partial z_1}\right)z_2\right]b(z_1,z_2)=0\,, \tag{2.8 a}$$

$$\left[\left(\mu^2-\tfrac{1}{2}-\bar{z}_2\frac{\partial}{\partial \bar{z}_2}\right)\bar{z}_1-\left(\mu^1-\tfrac{1}{2}-\bar{z}_1\frac{\partial}{\partial \bar{z}_1}\right)\bar{z}_2\right]b(z_1,z_2)=0\,. \tag{2.8 b}$$

Conversely, for every distribution $b\in D'(\boldsymbol{C}_1)$ *satisfying* (2.7) *and* (2.8), *there exists a unique distribution* $T\in\mathfrak{T}(X)$ *such that* (2.6) *holds.*

Proof amounts to translating the homogeneity condition in terms of the distribution B of the preceding lemma. Q.E.D.

3. Step One: $SL(2, C)$ Invariant Separately Homogeneous Distributions in the Subdomain O of $(\mathring{C}_2)^3$

We now prove that any $\boldsymbol{SL(2,C)}$ invariant distribution in the subdomain $\boldsymbol{O}$ can be represented as a distribution of the invariants Z_j (0.10). Following this result, the homogeneity condition is imposed.

3.1. Lemma. *In domain* $\boldsymbol{O}$ (0.15) *the rank of the holomorphic mapping* (0.14) *is* 3, *hence the* $\boldsymbol{SL(2,C)}$ *invariants* Z_1, Z_2, Z_3 *can be used as local coordinates in* $\boldsymbol{O}$. *Further, any* $\boldsymbol{SL(2,C)}$ *invariant distribution* $F\in D'(\boldsymbol{O})$ *can be represented in the form*

$$F(\zeta_1,\zeta_2,\zeta_3)=\mathscr{F}(Z_1,Z_2,Z_3)\,, \tag{3.1}$$

where $\mathscr{F}\in D'(\mathscr{O})$ *and*

$$\mathscr{O}:=\{(z_1,z_2,z_3)\in \boldsymbol{C}_3\,|\,\text{none or at most one of } z_1,z_2,z_3 \text{ is } 0\}\,. \tag{3.2}$$

Proof. An arguing like that in Lemmas 2.1 and 2.3 shows that it suffices to prove the statement only for the subdomain

$$\Omega:=\{(\zeta_1,\zeta_2,\zeta_3)\in \boldsymbol{O}\,|\,\zeta_3^1\neq 0\}$$

where the representation of type (2.3) is valid:

$$F|_{\Omega}(\zeta_1,\zeta_2,\zeta_3)=F_1(Z_1,Z_2,p_1,p_2)\,; \tag{3.3}$$

here $F_1(z_1,z_2,z_3,z_4)$ is a distribution in the domain

$$\{(z_1,z_2,z_3,z_4)\in \boldsymbol{C}_4\,|\,(z_1,z_4)\in \mathring{\boldsymbol{C}}_2,\ (z_2,z_3)\in \mathring{\boldsymbol{C}}_2\}$$

satisfying (2.5). Next we represent Ω as the union of two subdomains $\Omega_j:=\{(\zeta_1,\zeta_2,\zeta_3)\in\Omega\,|\,Z_j\neq 0\}$, $j=1,2$. In Ω_1 the regular substitution

$p_1 \mapsto Z_3 = -(p_1 Z_1 + p_2 Z_2)$ allows to rewrite (3.3) as $F|_{\Omega_1}(\zeta_1, \zeta_2, \zeta_3) = \mathscr{F}_1(Z_1, Z_2, Z_3)$ with an $\mathscr{F}_1 \in D'(\mathscr{O}_1) := D'(\mathring{\boldsymbol{C}}_1 \times \mathring{\boldsymbol{C}}_2)$. Analogously, there exists a distribution $\mathscr{F}_2(z_1, z_2, z_3)$ in the domain

$$\mathscr{O}_2 := \{(z_1, z_2, z_3) \in \boldsymbol{C}_3 \mid z_2 \in \mathring{\boldsymbol{C}}_1, \, (z_1, z_3) \in \mathring{\boldsymbol{C}}_2\}$$

such that $F|_{\Omega_2}(\zeta_1, \zeta_2, \zeta_3) = \mathscr{F}_2(Z_1, Z_2, Z_3)$. It remains to prove that there exists a distribution $\mathscr{F} \in D'(\mathscr{O}) \equiv D'(\mathscr{O}_1 \cup \mathscr{O}_2)$ which coincides with $\mathscr{F}_j$ in $\mathscr{O}_j$; by the principle of gluing distributions this is equivalent to the following fact: $\mathscr{F}_1 = \mathscr{F}_2$ in $\mathscr{O}_1 \cap \mathscr{O}_2$. This last fact is implied by the equality $\mathscr{F}_1(Z_1, Z_2, Z_3) = \mathscr{F}_2(Z_1, Z_2, Z_3)$ in $\Omega_1 \cap \Omega_2$ and by the fact that the image of $\Omega_1 \cap \Omega_2$ under the mapping (0.15) is $\mathscr{O}_1 \cap \mathscr{O}_2$. This then justifies the validity of (3.1) in Ω and, hence, throughout $\boldsymbol{O}$. Q.E.D.

Next we want to describe $\mathfrak{F}(X)$ for every $X \in \Xi$ and thus to solve Step One.

3.2. Proposition. *The general form of distributions of* $\mathfrak{F}(X)$ [*i.e. of distributions of* $D'(\boldsymbol{O})$ *satisfying* (0.5) *and* (0.6)] *is given by* (3.1) *with a distribution* $\mathscr{F} \in D'(\mathscr{O})$ *of the following form:*

a) *if none or at most one of* χ^1, χ^2, χ^3 *belongs to*

$$\mathfrak{P}_-^{[1]} := \{\chi \equiv (\lambda, \mu) \in \mathfrak{X} \mid \text{both } \lambda \text{ and } \mu \text{ are negative half-integers}\},$$

then

$$\mathscr{F}(z_1, z_2, z_3) = a\left[\prod_{j=1}^{3} \psi_{\chi^j}(z_j)\right]\Bigg|_{\mathscr{O}}, \qquad \exists a \in \boldsymbol{C}_1; \tag{3.4a}$$

b) *if at least two of* χ^1, χ^2, χ^3 *belong to* $\mathfrak{P}_-^{[1]}$, *then*

$$\mathscr{F}(z_1, z_2, z_3) = \sum_{j=1}^{3} \left[a_j \psi_{\chi^j}(z_j) \prod_{i \in \{1,2,3\} \setminus \{j\}} \hat{\varphi}_{\chi^i}(z_i)\right]_{\mathscr{O}}, \tag{3.4b}$$

where a_j $(j = 1, 2, 3)$ *are complex numbers, and* $a_j = 0$ *if* $\chi^j \notin \mathfrak{P}_-^{[1]}$. [Here notations ψ_χ and $\hat{\varphi}_\chi$ of Appendix B are adopted.]

Proof. In terms of $\mathscr{F} \in D'(\mathscr{O})$ the homogeneity condition (0.6) takes the form

$$\mathscr{F}(a^1 z_1, a^2 z_2, a^3 z_3) = \left(\prod_{j=1}^{3} \varphi_{\chi^j}^{[1]}(a^j)\right) \mathscr{F}(z_1, z_2, z_3), \qquad \forall a^j \in \mathring{\boldsymbol{C}}_1.$$

It is obvious that the right-hand sides of (3.4) satisfy this condition. There remains the necessity of Eqs. (3.4) to be verified. It is convenient to represent $\mathscr{O}$ as the union of three subdomains: $\mathscr{O} = \mathscr{O}^{(1)} \cup \mathscr{O}^{(2)} \cup \mathscr{O}^{(3)}$ where by definition

$$\mathscr{O}^{(1)} = \boldsymbol{C}_1 \times \mathring{\boldsymbol{C}}_1 \times \mathring{\boldsymbol{C}}_1, \quad \mathscr{O}^{(2)} = \mathring{\boldsymbol{C}}_1 \times \boldsymbol{C}_1 \times \mathring{\boldsymbol{C}}_1, \quad \mathscr{O}^{(3)} = \mathring{\boldsymbol{C}}_1 \times \mathring{\boldsymbol{C}}_1 \times \boldsymbol{C}_1.$$

In these subdomains results on homogeneous distributions in one complex variable (see [9] or Appendix B below) are applicable; thus we have

$$\mathscr{F}|_{\mathcal{O}^{(j)}}(z_1, z_2, z_3) = a'_j \psi_{\chi^j}(z_j) \prod_{\substack{k \\ k \neq j}} \varphi^{[1]}_{\chi^k}(z_k)\,; \qquad j = 1, 2, 3\,.$$

By requiring the compatibility of these representations in $\mathcal{O}^{(i)} \cap \mathcal{O}^{(j)}$ for $i \neq j$ $(i, j = 1, 2, 3)$, one obtains easily the desired result. Q.E.D.

4. Step Two: $SL(2, C)$ Invariant Separately Homogeneous Distributions in $(\mathring{C}_2)^3$ Vanishing in O

4.1. Proposition. *The dimension of* $\mathfrak{T}_0(X)$, *the subspace of all distributions of* $\mathfrak{T}(X)$ *with supports in* ω (0.16), *is given by*

$$\dim \mathfrak{T}_0(X) = \begin{cases} 0 & \text{if} \quad \mathfrak{x} \notin \mathfrak{P}^{[2]}_{-}\,, \\ 1 & \text{if} \quad \mathfrak{x} \in \mathfrak{P}^{[2]}_{-} \quad \text{and} \quad X \notin \Xi_0\,, \\ \nu(X) & \text{if} \quad X \in \Xi_0\,. \end{cases}$$

[$\nu(X)$ is the function (defined in Remark 1.5) on Ξ_0 (1.5), which assumes values 1, 2, 4.]

Proof. It is fairly obvious that, for a pair X, X' of equivalent triples, the operator M established in Proposition 1.4 maps isomorphically $\mathfrak{T}_0(X')$ onto $\mathfrak{T}_0(X)$, hence we may substitute the triple X by equivalent one. It terms of b (2.6), the support property reads $\operatorname{supp} b \subset \{0\}$. First, if $\mathfrak{x} \notin \mathfrak{P}^{[2]}_{-}$ then (2.7) and the support property of b imply $b = 0$. Second, if $\mathfrak{x} \in \mathfrak{P}^{[2]}_{-}$ and $X \notin \Xi_0$, we may substitute X by an equivalent triple in order to fulfill the condition $\mathfrak{l} = \mathfrak{m} = -1$. Then (2.7) and the support property yield $b(z_1, z_2) = a\delta(z_1)\,\delta(z_2)$, $\exists a \in C_1$. It is clear that every such distribution satisfies (2.8). Thus, in the second case $\dim \mathfrak{T}_0(X) = 1$. At last, let us consider triples of Ξ_0. Proposition 1.6 allows to confine the treatment with extremal triples of Ξ_0. Then we have: $\mathfrak{l} = \mathfrak{m} = -1$ for $\nu(X) = 1$; either $(\mathfrak{l}, \mathfrak{m}) = (-1, -2)$ or $(\mathfrak{l}, \mathfrak{m}) = (-2, -1)$ for $\nu(X) = 2$; and $\mathfrak{l} = \mathfrak{m} = -2$ for $\nu(X) = 4$. Further, (2.7) and the support property imply

$$b(z_1, z_2) = \sum_{\alpha_1 + \alpha_2 = -\mathfrak{l} - 1} \; \sum_{\beta_1 + \beta_2 = -\mathfrak{m} - 1} a_{\alpha_1 \alpha_2 \beta_1 \beta_2} \prod_{j=1}^{2} \left(\frac{\partial}{\partial z_j}\right)^{\alpha_j} \left(\frac{\partial}{\partial \bar{z}_j}\right)^{\beta_j} \delta(z_j)\,. \tag{4.1}$$

It is remarkable that every such distribution satisfies (2.8) automatically. For example, we consider (2.8a), since (2.8b) is analogous. If $\mathfrak{l} = -1$, Eq. (2.8a) is satisfied since $z_j \delta(z_1)\delta(z_2) = 0$ for $j = 1, 2$. If $\mathfrak{l} = -2$ [hence $\lambda^j = -\frac{1}{2}$ for all $j = 1, 2, 3$], Eq. (2.8a) reads

$$\left[\left(1 + z_2 \frac{\partial}{\partial z_2}\right) z_1 - \left(1 + z_1 \frac{\partial}{\partial z_1}\right) z_2\right] b = 0\,;$$

it is justified by the right-hand side of (4.1). Thus, $\dim\mathfrak{T}_0(X)$ is equal to the dimension of the space of distribution (4.1), i.e. to $\mathfrak{l}\mathfrak{m} = \nu(X)$. Q.E.D.

4.2. Remark. In the proof of Proposition 4.1 we have actually put into effect Step Two of Subsection 0.2 and described the space $\mathfrak{T}_0(X)$. Our purpose here is to rewrite distributions of $\mathfrak{T}_0(X)$ in an invariant form. First, we observe that, for a point $(\zeta_1, \zeta_2, \zeta_3) \in \omega$, there are non-zero complex numbers denoted by $\zeta_i : \zeta_j$ such that

$$\zeta_i = (\zeta_i : \zeta_j) \cdot \zeta_j ; \qquad i,j = 1, 2, 3 . \tag{4.2}$$

Further, we claim that an expression like

$$H(\zeta_1, \zeta_2, \zeta_3) = u(\zeta_1 : \zeta_3, \zeta_2 : \zeta_3) \left(\frac{\partial}{\partial Z_i}\right)^m \left(\frac{\partial}{\partial \bar{Z}_j}\right)^n \delta(Z_1)\,\delta(Z_2) \tag{4.3}$$

can be given a meaning of an $\boldsymbol{SL(2, C)}$ invariant distribution of $D'((\mathring{\boldsymbol{C}}_2)^3)$, provided m and n are non-negative integers; $i,j = 1, 2$; $u(z_1, z_2)$ is a $\mathscr{C}^\infty$ function in $(\mathring{\boldsymbol{C}}_1)^2$, which, in addition, must be anti-holomorphic in z_{3-i} if $m > 0$ and holomorphic in z_{3-j} if $n > 0$. Indeed, it is easily verified that the right-hand side of (4.3) becomes a distribution in $\boldsymbol{Q}$ of the form (2.3) [hence satisfying (2.1)] after the substitution of ζ_k^1/ζ_3^1 for $\zeta_k : \zeta_3$ $(k = 1, 2)$; recalling Lemma 2.1, a precise meaning of an $\boldsymbol{SL(2, C)}$ invariant distribution in $(\mathring{\boldsymbol{C}}_2)^3$ is given to (4.3).

We are now in a position to rewrite distributions of $\mathfrak{T}_0(X)$ using invariant symbols. We know that $\mathfrak{T}_0(X)$ is non-trivial if and only if $\mathfrak{x} \in \mathfrak{P}_-^{[2]}$. Due to Proposition 1.4 it suffices to single out at least one point X from every equivalence class on Ξ. In case $\mathfrak{x} \in \mathfrak{P}_-^{[2]}$ and $X \notin \Xi_0$, we may suppose $\mathfrak{l} = \mathfrak{m} = -1$, then the replacement of $b(z_1, z_2)$ in (2.6) by $a\,\delta(z_1)\,\delta(z_2)$ yields

$$\begin{aligned} T(\zeta_1, \zeta_2, \zeta_3) &= a\varphi_{-\chi^1}^{[1]}(\zeta_1 : \zeta_3)\, \varphi_{-\chi^2}^{[1]}(\zeta_2 : \zeta_3)\, \delta(Z_1)\,\delta(Z_2) \\ &= a\varphi_{-\chi^i}^{[1]}(\zeta_i : \zeta_k)\, \varphi_{-\chi^j}^{[1]}(\zeta_j : \zeta_k)\, \delta(Z_i)\,\delta(Z_j), \end{aligned} \tag{4.4}$$

where $a \in \boldsymbol{C}_1$, and $\{i, j, k\} = \{1, 2, 3\}$[11]. In case $X \in \Xi_0$ we may restrict ourselves to extremal points [general case is obtained by applying Proposition 1.6]. If $X \in \Xi_0$ and $\nu(X) = 1$, then $\mathfrak{l} = \mathfrak{m} = -1$, and representation (4.4) is valid. If $X \in \Xi_0$ and $\nu(X) = 2$, then we have two possibilities:

a) $\mathfrak{l} = -1$ and $\mathfrak{m} = -2$, then (2.6) and (4.1) yield

$$T = \sum_{i=1}^{2} a_i \left\{ \prod_{j=1}^{2} (\zeta_j : \zeta_3)^{-\lambda^j - \frac{1}{2}} \right\} (\overline{\zeta_i : \zeta_3})^{-1} \frac{\partial}{\partial \bar{Z}_i} \delta(Z_1)\,\delta(Z_2) ; \tag{4.5a}$$

[11] When this equality of sets holds, we say that (i, j, k) is a permutation of $(1, 2, 3)$.

b) $\mathfrak{l} = -2$ and $\mathfrak{m} = -1$, then

$$T = \sum_{i=1}^{2} a_i' \left\{ \prod_{j=1}^{2} (\overline{\zeta_j : \zeta_3})^{-\mu^j - \frac{1}{2}} \right\} (\zeta_i : \zeta_3)^{-1} \frac{\partial}{\partial Z_i} \delta(Z_1)\,\delta(Z_2)\,; \quad (4.5\,\mathrm{b})$$

here a_i and a_i' are arbitrary complex numbers. At last, if $X \in \Xi_0$ and $\nu(X) = 4$ [hence $\lambda^j = \mu^j = -\frac{1}{2}$ for all $j = 1, 2, 3$] then

$$T = \sum_{i,j=1}^{2} a_{ij} \cdot (\zeta_i : \zeta_3)^{-1} (\overline{\zeta_j : \zeta_3})^{-1} \frac{\partial^2}{\partial Z_i\, \partial \bar{Z}_j} \delta(Z_1)\,\delta(Z_2)\,, \qquad (4.6)$$

where (a_{ij}) is an arbitrary complex 2×2 matrix.

As a consequence of Sections 3 and 4, all the spaces $\mathfrak{T}(X)$, $\mathfrak{T}_0(X)$, $\mathfrak{F}(X)$, $\hat{\mathfrak{F}}(X)$ [defined in Subsection 0.2] are finite-dimensional for every $X \in \Xi$. By the very definition of these spaces, we have

$$\dim \mathfrak{T}(X) = \dim \mathfrak{T}_0(X) + \dim \hat{\mathfrak{F}}(X) \leqq \dim \mathfrak{T}_0(X) + \dim \mathfrak{F}(X)\,. \quad (4.7)$$

5. Step Three: The Extension Problem. The Final Result

The most convenient classification of distributions of $\mathfrak{T}(X)$ for any triple $X \in \Xi$ seems to be that realized almost completely in terms of distributions Ψ_X (see Appendix C). In Appendix C we have described the subspace $\mathfrak{A}(X)$ of all distributions of $\mathfrak{T}(X)$ associated with Ψ at X [i.e. representable in form (0.13)]. Now we will prove that $\dim \mathfrak{T}(X) = \dim \mathfrak{A}(X)$ [hence $\mathfrak{T}(X) = \mathfrak{A}(X)$] for all X, except for a countable series. For the exceptional X, it turns out that $\dim \mathfrak{T}(X) = \dim \mathfrak{T}_0(X)$ [hence $\mathfrak{T}(X) = \mathfrak{T}_0(X)$, $\mathfrak{T}_0(X)$ being already described].

In order to find $\dim \mathfrak{T}(X)$ (and hence to prove the above assertion) we have to accomplish Step Three in our programme, viz., to determine the subspace $\hat{\mathfrak{F}}(X)$ of all distributions of $\mathfrak{F}(X)$ which possess extensions to distributions of $\mathfrak{T}(X)$. It is possible to make it directly. [On the basis of Proposition 2.4, the problem reduces to extension of a distribution in $\mathring{\boldsymbol{C}}_2$ satisfying (2.7) and (2.8) to a distribution in $\boldsymbol{C}_2$ satisfying the same conditions.] Another way is to prove that any distribution of $\hat{\mathfrak{F}}(X)$ coincides with the restriction to $\boldsymbol{O}$ of a distribution of $\mathfrak{A}(X)$. With this is in mind, we use a possibility of translating the extension problem in terms of Ψ as follows. On the basis of Proposition 3.2, it can be easily seen (and is actually seen in the proof of Proposition 5.2 below) that every distribution $F \in \mathfrak{F}(X)$ can be represented in the form $F = P\left(\frac{\partial}{\partial \chi^1}, \frac{\partial}{\partial \chi^2}, \frac{\partial}{\partial \chi^3}\right) \Psi_X|_{\boldsymbol{O}}$. Thus, we obtain the following version of the extension problem: Given a (complex) polynomial $P(z_1, z_2, z_3)$

such that the distribution $P\left(\frac{\partial}{\partial X}\right)\Psi_X|_{\mathbf{O}} \equiv P\left(\frac{\partial}{\partial \chi^1}, \frac{\partial}{\partial \chi^2}, \frac{\partial}{\partial \chi^3}\right)\Psi_X|_{\mathbf{O}}$ belongs to $\mathfrak{F}(X)$; whether $P\left(\frac{\partial}{\partial X}\right)\Psi_X|_{\mathbf{O}}$ possesses an extension to a distribution $T \in \mathfrak{T}(X)$?

5.1. Lemma. *Let $X \in \Xi$, and $P(z_1, z_2, z_3)$ be a polynomial in three variables such that $P\left(\frac{\partial}{\partial X}\right)\Psi_X|_{\mathbf{O}} = T|_{\mathbf{O}}$ for some $T \in \mathfrak{T}(X)$. Then $Q\left(\frac{\partial}{\partial X}\right)\Psi_X = 0$ where $Q(z_1, z_2, z_3) = \sum_{i=1}^{3} \frac{\partial}{\partial z_i} P(z_1, z_2, z_3)$.*

Proof. We denote $T_1 = P\left(\frac{\partial}{\partial X}\right)\Psi_X - T$ and represent $T_1|_{\mathbf{Q}}$ in form (2.3): $T_1|_{\mathbf{Q}}(\zeta_1, \zeta_2, \zeta_3) = B_1(Z_1, Z_2, p_1, p_2)$. The support property of T_1 [that is, $\operatorname{supp} T_1 \subset \omega$] implies the corresponding support property of B_1: $\operatorname{supp} B_1 \subset \{(z_1, z_2, z_3, z_4) \in \boldsymbol{C}_2 \times (\mathring{\boldsymbol{C}}_1)^2 \mid z_1 = z_2 = 0\}$; consequently,

$$B_1(z_1, z_2, z_3, z_4) = \sum_{\alpha_j, \beta_j} f_{\alpha_1\alpha_2\beta_1\beta_2}(z_3, z_4) \prod_{j=1}^{2} \left(\frac{\partial}{\partial z_j}\right)^{\alpha_j} \left(\frac{\partial}{\partial \bar{z}_j}\right)^{\beta_j} \delta(z_j).$$

Therefore $T_1|_{\mathbf{Q}}$ can be represented as the finite sum

$$T_1|_{\mathbf{Q}}(\zeta_1, \zeta_2, \zeta_3) = \sum_{\chi \in \mathfrak{P}^{[2]}_{-}} t_\chi(\zeta_1, \zeta_2, \zeta_3), \tag{5.1}$$

where t_χ is a distribution of $D'(\mathbf{Q})$ which satisfies the following homogeneity condition

$$t_\chi(a\zeta_1, a\zeta_2, a\zeta_3) = \varphi_\chi^{[2]}(a^2)\, t_\chi(\zeta_1, \zeta_2, \zeta_3), \quad \forall a \in \mathring{\boldsymbol{C}}_1. \tag{5.2}$$

On the other hand, since both T and Ψ_X satisfy (0.6) we have

$$\begin{aligned} &T_1|_{\mathbf{Q}}(a\zeta_1, a\zeta_2, a\zeta_3) \\ &\quad = \varphi_\chi^{[2]}(a^2)\left\{T_1|_{\mathbf{Q}}(\zeta_1, \zeta_2, \zeta_3) + \left[P\left(r + \frac{\partial}{\partial \chi^1}, r + \frac{\partial}{\partial \chi^2}, r + \frac{\partial}{\partial \chi^3}\right)\right.\right. \\ &\quad\quad \left.\left. - P\left(\frac{\partial}{\partial X}\right)\right]\Psi_X|_{\mathbf{Q}}(\zeta_1, \zeta_2, \zeta_3)\right\}, \quad \forall a \in \mathring{\boldsymbol{C}}_1, \end{aligned} \tag{5.3}$$

where $r = 4\ln|a|$. Next we express the left-hand side of (5.3) through t_χ according to (5.1) and (5.2). We observe that expressions at $\varphi_\chi^{[2]}(a^2) r^n$ $(n = 1, 2, \ldots)$ in the resulting equation need equal zero; thus

$$\left[P\left(r + \frac{\partial}{\partial \chi^1}, r + \frac{\partial}{\partial \chi^2}, r + \frac{\partial}{\partial \chi^3}\right) - P\left(\frac{\partial}{\partial X}\right)\right]\Psi_X = 0 \quad \text{for all real } r.$$

Differenting this equation with respect to r at $r=0$ provides the desired result. Q.E.D.

5.2. Proposition. *In notations of Appendix C, dimension of* $\mathfrak{T}(X)$ *is given by*

$$\dim\mathfrak{T}(X)=\begin{cases}1 & \text{if} \quad X\in\Xi\backslash\mathfrak{Z},\\ 3 & \text{if} \quad X\in\mathfrak{Z}_1,\\ 2 & \text{if} \quad X\in\mathfrak{Z}_2\cup\mathfrak{Z}_3,\\ 4 & \text{if} \quad X\in\mathfrak{Z}_4.\end{cases}$$

Proof. Note that $\dim\mathfrak{A}(X)\leqq\dim\mathfrak{T}(X)$ for all X, therefore if $\dim\mathfrak{F}(X)+\dim\mathfrak{T}_0(X)\leqq\dim\mathfrak{A}(X)$ for some X then (4.7) implies $\dim\mathfrak{T}(X)=\dim\mathfrak{A}(X)$.

Let us first consider the case $X\in\Xi\backslash\mathfrak{Z}$ (when $\Psi_X\neq 0$ and hence $\dim\mathfrak{A}(X)\geqq 1$). There are three possibilities:

α) $\mathfrak{x}\notin\mathfrak{P}_-^{[2]}$, and at most one of χ^1,χ^2,χ^3 belongs to $\mathfrak{P}_-^{[1]}$;

β) $\mathfrak{x}\in\mathfrak{P}_-^{[2]}$, $X\notin\Xi_0$, and $\chi^j\notin\mathfrak{P}_+^{[1]}$ for all j;

γ) $X\in\Xi_0$, $\nu(X)=1$, and $\chi^j\notin\mathfrak{P}_+^{[1]}$ for all j.

In case α) $\dim\mathfrak{F}(X)=1$ (according to Proposition 3.2) and $\dim\mathfrak{T}_0(X)=0$ (according to Proposition 4.1), hence $\dim\mathfrak{F}(X)+\dim\mathfrak{T}_0(X)\leqq\dim\mathfrak{A}(X)$. By the preceding note this means that $\dim\mathfrak{T}(X)=\dim\mathfrak{A}(X)=1$. In cases β) and γ) we have $\Psi_X\neq 0$ and $\dim\mathfrak{T}_0(X)=1$. For an arbitrary distribution $T\in\mathfrak{T}(X)$, comparison between Proposition 3.2 [part a)] and formula (C.3) gives $T|_{\boldsymbol{O}}=a\dfrac{\partial}{\partial\chi^j}\Psi_X|_{\boldsymbol{O}}$ for some $a\in\boldsymbol{C}_1$. Now Lemma 5.1 implies $a=0$, i.e. $supp\, T\subset\omega$. Thus, $\dim\mathfrak{T}(X)=\dim\mathfrak{T}_0(X)=1$.

Next we direct our attention to zeros of Ψ. We will treat separately each case of Classification C.2 in Appendix C and make use of the information (in Theorem C.3) on $\mathfrak{A}(X)$.

1) Let $X\in\mathfrak{Z}_1$, hence $\dim\mathfrak{A}(X)=3$. By the note at the beginning of the proof, it suffices to observe that $\dim\mathfrak{F}(X)=1$ [by Proposition 3.2, part a)] and $\dim\mathfrak{T}_0(X)=2$ [by Proposition 4.1 in case $X\in\Xi_0$ and $\nu(X)=2$].

2a) In this case $\dim\mathfrak{F}(X)=2$, $\dim\mathfrak{T}_0(X)=0$ and $\dim\mathfrak{A}(X)=2$; by the same note, $\dim\mathfrak{T}(X)=\dim\mathfrak{A}(X)=2$.

2b) In this case $\left(\dfrac{\partial}{\partial\chi^k}\right)^n\Psi_X=0$ for all $n>0$ [due to Classification C.2] and $\dfrac{\partial}{\partial\chi^i}\Psi_X,\dfrac{\partial}{\partial\chi^j}\Psi_X$ are two linearly independent distributions of $\mathfrak{A}(X)$ [due to Theorem C.3]. Further, comparison between Proposition 3.2 [part b)] and formula (C.3) gives:

For any $T \in \mathfrak{T}(X)$ there are $a_1, a_2 \in \boldsymbol{C}_1$ such that

$$T|_{\boldsymbol{o}} = \left(a_1 \frac{\partial^2}{\partial \chi^i \partial \chi^k} + a_2 \frac{\partial^2}{\partial \chi^j \partial \chi^k}\right) \Psi_X|_{\boldsymbol{o}}. \tag{5.4}$$

Now Lemma 5.1 implies

$$\left[a_1 \frac{\partial}{\partial \chi^i} + a_2 \frac{\partial}{\partial \chi^j} + (a_1 + a_2) \frac{\partial}{\partial \chi^k}\right] \Psi_X = 0\,; \tag{5.5}$$

consequently, $a_1 = a_2 = 0$. Therefore, *supp* $T \subset \omega$ for all $T \in \mathfrak{T}(X)$, and $\dim \mathfrak{T}(X) = \dim \mathfrak{T}_0(X) = 2$.

2c) In this case $\dim \mathfrak{F}(X) = \dim \mathfrak{T}_0(X) = 1$ and $\dim \mathfrak{A}(X) = 2$; hence $\dim \mathfrak{T}(X) = \dim \mathfrak{A}(X) = 2$.

3a) As in case 2b), for an arbitrary $T \in \mathfrak{T}(X)$, there are two complex numbers such that (5.4) and (5.5) hold. Since

$$\left(\frac{\partial}{\partial \chi^k}\right)^n \Psi_X = \left(\frac{\partial}{\partial \chi^i} - \frac{\partial}{\partial \chi^j}\right)^n \Psi_X = 0 \quad \text{for all } n$$

[due to Classification C.2] and $\frac{\partial}{\partial \chi^i} \Psi_X \neq 0$, we have $a_1 + a_2 = 0$. Therefore $\dim \hat{\mathfrak{F}}(X) \leqq 1$. Moreover, $\dim \mathfrak{T}_0(X) = 1$ and $\dim \mathfrak{A}(X) = 2$. Hence, $\dim \mathfrak{T}(X) = \dim \mathfrak{A}(X) = 2$.

3b) This case is similar to 2c).

4) For an arbitrary $T \in \mathfrak{T}(X)$, a representation analogous to (5.4) is valid; indeed, there are complex numbers a_1, a_2, a_3 such that

$$T|_{\boldsymbol{o}} = \left[a_1 \frac{\partial^3}{(\partial \chi^1)^2 \partial \chi^2} + a_2 \frac{\partial^3}{(\partial \chi^1)^2 \partial \chi^3} + a_3 \frac{\partial^3}{(\partial \chi^2)^2 \partial \chi^3}\right] \Psi_X|_{\boldsymbol{o}}.$$

Lemma 5.1 implies $\left[2 a_1 \frac{\partial^2}{\partial \chi^1 \partial \chi^2} + 2 a_2 \frac{\partial^2}{\partial \chi^1 \partial \chi^3} + 2 a_3 \frac{\partial^2}{\partial \chi^2 \partial \chi^3} + (a_1 + a_2)\left(\frac{\partial}{\partial \chi^1}\right)^2 + a_3 \left(\frac{\partial}{\partial \chi^2}\right)^2\right] \Psi_X = 0$. Since $\left(\frac{\partial}{\partial \chi^i}\right)^n \Psi_X = 0$ for all $n > 0$ and $i = 1, 2, 3$ [according to Classification C.2] and $\frac{\partial^2}{\partial \chi^i \partial \chi^j} \Psi_X$ $(1 \leqq i < j \leqq 3)$ are linearly independent [according to Theorem C.3], we have $a_1 = a_2 = a_3 = 0$ Consequently, *supp* $T \subset \omega$, and $\dim \mathfrak{T}(X) = \dim \mathfrak{T}_0(X) = 4$.

Q.E.D.

We obtain the following important corollary: If X is not a zero of Ψ or if X belongs to $\mathfrak{Z}_1 \cup \mathfrak{Z}_2 \cup \mathfrak{Z}_3$, then $\mathfrak{T}(X) = \mathfrak{A}(X)$. Indeed, by definition, $\mathfrak{A}(X)$ always is a subspace of $\mathfrak{T}(X)$; moreover, if $X \in \Xi \backslash \mathfrak{Z}$ then both $\mathfrak{A}(X)$ and $\mathfrak{T}(X)$ are one-dimensional, and if $X \in \mathfrak{Z}_1 \cup \mathfrak{Z}_2 \cup \mathfrak{Z}_3$ then comparison between Proposition 5.2 and Theorem C.3 gives $\dim \mathfrak{A}(X) = \dim \mathfrak{T}(X)$.

The only triples for which $\dim \mathfrak{A}(X) < \dim \mathfrak{T}(X)$ are those of $\mathfrak{Z}_4$, that is, triples X satisfying the condition: $\chi^j \in \mathfrak{P}_{-}^{[1]}$ for all $j=1,2,3$ or, equivalently, $X \in \Xi_0$ and $\nu(X)=4$. In this case $\mathfrak{T}(X)$ is spanned by $\mathfrak{A}(X)$ and some one distribution, $\Phi_X \in \mathfrak{T}(X)$. Although a choice of Φ_X seems to be immaterial, we point out one natural specification of Φ_X with an interesting symmetry property.

5.3. Definition. Let $\dot{X}$ be the unique extremal triple among the triples X of Ξ_0 with $\nu(X)=4$; this means that $\dot{\lambda}_j = \dot{\mu}_j = -1$ [and $\dot{\lambda}^j = \dot{\mu}^j = -\frac{1}{2}$] for all $j=1,2,3$. In accordance with (4.6), the distribution $\Phi_{\dot{X}} \in \mathfrak{T}(\dot{X})$ is defined by

$$\Phi_{\dot{X}} = i\left\{(\zeta_1:\zeta_3)^{-1}(\overline{\zeta_2:\zeta_3})^{-1}\frac{\partial^2}{\partial Z_1\,\partial \bar{Z}_2} - (\overline{\zeta_1:\zeta_3})^{-1}(\zeta_2:\zeta_3)^{-1}\frac{\partial^2}{\partial \bar{Z}_1\,\partial Z_2}\right\}$$
$$\times\, \delta(Z_1)\,\delta(Z_2)\,. \tag{5.6a}$$

For an arbitrary triple X of Ξ_0 with $\nu(X)=4$, let $\mu(X)\,M(X)$ by the operator defined by (C.7), then $\Phi_X \in \mathfrak{T}(X)$ is defined by setting

$$\Phi_X = \mu(X)\,M(X)\,\Phi_{\dot{X}}\,. \tag{5.6b}$$

We claim that Φ_X does not belong to $\mathfrak{A}(X)$ and hence $\mathfrak{T}(X)$ is spanned by $\mathfrak{A}(X)$ and Φ_X. Indeed, in the special case $X=\dot{X}$, $\mathfrak{A}(\dot{X})$ consists of distributions (4.6) with (a_{ij}) an arbitrary symmetric complex 2×2 matrix [see Appendix C], while $\Phi_{\dot{X}}$ is a non-zero distribution of the form (4.6) with a skew-symmetric matrix (a_{ij}); hence $\Phi_{\dot{X}} \notin A(\dot{X})$. In general case, the operator $\mu(X)\,M(X)$ maps isomorphically $\mathfrak{T}(\dot{X})$ onto $\mathfrak{T}(X)$ [according to Proposition 1.6] and $\mathfrak{A}(\dot{X})$ onto $\mathfrak{A}(X)$ [according to Appendix C]; therefore $\Phi_X \notin \mathfrak{A}(X)$ which justifies the assertion.

It is of interest to note that the distribution $\Phi_{\dot{X}}$ possesses the following property whose verification is straightforward: for an arbitrary odd permutation (i,j,k) of $(1,2,3)$, $\Phi_{\dot{X}}(\zeta_i,\zeta_j,\zeta_k) = -\Phi_{\dot{X}}(\zeta_1,\zeta_2,\zeta_3)$; this property characterizes up to a multiple $\Phi_{\dot{X}}$ among distributions of $\mathfrak{T}(\dot{X})$. More generally, the distributions Φ_X possess a symmetry property which is to some extent opposite to that of Ψ_X. Namely, under an odd permutation (i,j,k) of $(1,2,3)$ Ψ_X behaves according to (C.2), while

$$\Phi_{(\chi_i,\chi_j,\chi_k)}(\zeta_i,\zeta_j,\zeta_k) = -(-1)^{2\mathfrak{k}}\,\Phi_{(\chi_1,\chi_2,\chi_3)}(\zeta_1,\zeta_2,\zeta_3)\,. \tag{5.7}$$

We now summarize the results on $\mathfrak{T}(X)$, the space of all kernels of trilinear $\boldsymbol{SL(2,C)}$ invariant forms over $\mathfrak{D}_{-\chi_1}\times\mathfrak{D}_{-\chi_2}\times\mathfrak{D}_{-\chi_3}$. Remind that only the triples of Ξ (i.e. the triples of $\mathfrak{X}^3$ satisfying the condition: $\frac{1}{2}\sum_{j=1}^{3}(\lambda_j-\mu_j)$ is integral) are to be treated; otherwise $\mathfrak{T}(X)=\{0\}$.

5.4. Theorem. *For an arbitrary triple* $X \equiv (\chi_1,\chi_2,\chi_3) \in \Xi$ *the space* $\mathfrak{T}(X)$ *of all* $\boldsymbol{SL(2,C)}$ *invariant distributions* $T(\zeta_1,\zeta_2,\zeta_3) \in D'((\mathring{C}_2)^3)$

separately homogeneous of indices χ_1, χ_2, χ_3 *in* $\zeta_1, \zeta_2, \zeta_3$, *respectively, is non-trivial, and its dimension is given by Proposition 5.2. Distributions of* $\mathfrak{T}(X)$ *can be described in terms of the distribution-valued analytic function* Ψ *on* Ξ *[introduced in Appendix C] and the countable family of distributions* Φ_X *[introduced in Definition 5.3] as follows.*

I. *If* X *is not a zero of* Ψ, *i.e. if* X *satisfies none of the conditions* (i), (ii), (iii) *of Theorem C.1 [in Appendix C], then* $\mathfrak{T}(X)$ *consists of multiples of* Ψ_X.

II. *If* X *is a zero of* Ψ *of type* $t=1,2,3$, *i.e. if* X *satisfies one of the conditions* (i), (ii) *or* (iii) *of Theorem C.1 and, in addition, at most two of* χ^1, χ^2, χ^3 *belong to* $\mathfrak{P}_-^{[1]}$, *then* $\mathfrak{T}(X)$ *coincides with the space* $\mathfrak{A}(X)$ *[of the kernels associated with* Ψ *at* X*].*

III. *If* X *is a zero of* Ψ *of type* 4, *i.e. if* $\chi^j \in \mathfrak{P}_-^{[1]}$ *for all* $j=1,2,3$, *then an arbitrary distribution of* $\mathfrak{T}(X)$ *can be represented uniquely as sum of a distribution of* $\mathfrak{A}(X)$ *and a multiple of* Φ_X.

For a description of the space $\mathfrak{A}(X)$ of the kernels associated with Ψ at X, the reader is referred to Appendix C. (Further information involving explicit forms of kernels of $\mathfrak{A}(X)$ can be found in [7].)

5.5. *Remark.* We point out the connection between trilinear $\boldsymbol{SL(2, C)}$ invariant forms over elementary representation spaces and the problem on $\boldsymbol{SL(2, C)}$ analysis of tensor product of two elementary representations, say T_{χ_1} and T_{χ_2}. Let $\mathfrak{D}_{\chi_1} \otimes \mathfrak{D}_{\chi_2}$ be the complete tensor product of $\mathfrak{D}_{\chi_1}$ and $\mathfrak{D}_{\chi_2}$ (for definiteness, in the projective topology); it can be identified with the subspace of $\mathscr{E}(\mathring{C}_2 \times \mathring{C}_2)$ of all $\mathscr{C}^\infty$ functions $f(\zeta_1, \zeta_2)$ separately homogeneous of indices χ_1 and χ_2 in ζ_1 and ζ_2, respectively. The problem consists of two parts. First, to describe the space $\mathfrak{t}(\chi_1, \chi_2; \chi_3)$ of all continuous $\boldsymbol{SL(2, C)}$ invariant operators from $\mathfrak{D}_{\chi_1} \otimes \mathfrak{D}_{\chi_2}$ into $\mathfrak{D}_{\chi_3}$ where (χ_1, χ_2, χ_3) is an arbitrary triple of Ξ. Second, to find a characterization of an arbitrary element $f \in \mathfrak{D}_{\chi_1} \otimes \mathfrak{D}_{\chi_2}$ in terms of functions $\tilde{f}_{\chi_3, t} \equiv t f \in \mathfrak{D}_{\chi_3}$ dependent on χ_3 and $t \in \mathfrak{t}(\chi_1, \chi_2; \chi_3)$ as on parameters. The solution to the first part of the problem is obtained by constructing a natural one-one correspondence between $\mathfrak{t}(\chi_1, \chi_2; \chi_3)$ and $\mathfrak{T}(-\chi_1, -\chi_2, -\chi_3)$ [while the second part requires further investigations]. More explicitly, reasoning like that in Subsection 0.1 shows that the formula

$$(T, F_1 \otimes F_2 \otimes F_3) = \int t(I_{\chi_1} F_1 \otimes I_{\chi_2} F_2)(\zeta_3)\, F_3(\zeta_3)\, |d^2\zeta_3\, d^2\overline{\zeta_3}|, \quad \forall F_j \in D(\mathring{C}_2),$$

defines the correspondence $\mathfrak{t}(\chi_1, \chi_2; \chi_3) \ni t \leftrightarrow T \in \mathfrak{T}(-\chi_1, -\chi_2, \chi_3)$. Indeed, for a given t, the corresponding distribution $T \in D'((\mathring{C}_2)^3)$ is separately homogeneous of indices $-\chi_1, -\chi_2, \chi_3$ [by virtue of Lemma A.3]. On the other hand, an arbitrary $T(\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{T}(-\chi_1, -\chi_2, \chi_3)$ can be considered as a distribution in ζ_1, ζ_2 dependent on ζ_3 in $\mathscr{C}^\infty$ way [by virtue of Lemma 2.3]; consequently, there exists a continuous $\boldsymbol{SL(2, C)}$

invariant operator $b: D(\overset{\circ}{C}_2 \times \overset{\circ}{C}_2) \to \mathfrak{D}_{\chi_3}$ vanishing on the kernel of $I_{\chi_1} \otimes I_{\chi_2}$ and such that $(T, F \otimes F_3) = \int (bF)(\zeta_3)\, F_3(\zeta_3)\, |d^2 \zeta_3\, d^2 \bar{\zeta}_3|$, $\forall F \in D(\overset{\circ}{C}_2 \times \overset{\circ}{C}_2)$, $\forall F_3 \in D(\overset{\circ}{C}_2)$; now applying an analogue of Lemma A.1 [which reads that $\mathfrak{D}_{\chi_1} \otimes \mathfrak{D}_{\chi_2}$ is isomorphic with $D(\overset{\circ}{C}_2 \times \overset{\circ}{C}_2)/\ker(I_{\chi_1} \otimes I_{\chi_2})$] and using the canonical decomposition for b show $b = t(I_{\chi_1} \otimes I_{\chi_2})$ for some $t \in \mathfrak{t}(\chi_1, \chi_2; \chi_3)$, which proves the assertion.

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Appendix *A* [12]. Isomorphism between $\mathfrak{D}'_\chi$ and $\mathfrak{d}_{-\chi}$

Our purpose here is to establish the properties (claimed in Subsection 0.1) of the continuous $SL(2, C)$ invariant operator $I_\chi : D(\overset{\circ}{C}_2) \to \mathfrak{D}_\chi$ [defined by (0.4) for any $\chi \in \mathfrak{X}$] and of its dual $I'_\chi : \mathfrak{D}'_\chi \to D'(\overset{\circ}{C}_2)$.

Let $\mathfrak{M}_\chi := I_\chi^{-1}(0)$ be the kernel of I_χ and $P_\chi : D(\overset{\circ}{C}_2) \to D(\overset{\circ}{C}_2)/\mathfrak{M}_\chi$ be the canonical mapping. Due to the universality property of quotient spaces, there exists a unique continuous injection $Q_\chi : D(\overset{\circ}{C}_2)/\mathfrak{M}_\chi \to \mathfrak{D}_\chi$ related to I_χ by the canonical decomposition, $I_\chi = Q_\chi P_\chi$. Moreover, we have

A.1. Lemma. *$Q_\chi : D(\overset{\circ}{C}_2)/\mathfrak{M}_\chi \to \mathfrak{D}_\chi$ is an isomorphism of topological vector spaces. (In other words, I_χ is a topological homomorphism of $D(\overset{\circ}{C}_2)$ onto $\mathfrak{D}_\chi$.)*

Proof. I_χ possesses at least one continuous right inverse operator, i.e. an operator $R_\chi : \mathfrak{D}_\chi \to D(\overset{\circ}{C}_2)$ such that $I_\chi R_\chi = \mathbf{1}_{\mathfrak{D}_\chi}$. For example, a straightforward verification shows that R_χ can be chosen in the form: $(R_\chi f)(\zeta) = H(\zeta) \cdot f(\zeta)$, $\forall f \in \mathfrak{D}_\chi$, where H is a picked function of $D(\overset{\circ}{C}_2)$ such that $\int\limits_{\overset{\circ}{C}_1} H(a\zeta)\, |a|^{-2}\, |da\, d\bar{a}| \equiv 1$. Then $P_\chi R_\chi$ is a continuous right inverse operator for Q_χ. Since Q_χ is injective, this implies that $P_\chi R_\chi$ is the two-sided inverse operator for Q_χ. Q.E.D.

Let $D'(\overset{\circ}{C}_2)$ and $\mathfrak{D}'_\chi$ be the (topological) dual spaces of $D(\overset{\circ}{C}_2)$ and $\mathfrak{D}_\chi$, respectively; for definiteness, they are endowed with the weak topologies. By the very definition of P_χ, the dual operator P'_χ maps isomorphically $(D(\overset{\circ}{C}_2)/\mathfrak{M}_\chi)'$ onto the subspace $\mathfrak{M}_\chi^\perp$ of all functionals of $D'(\overset{\circ}{C}_2)$ which vanish on $\mathfrak{M}_\chi$. Now Lemma A.1 and the equation $I'_\chi = P'_\chi Q'_\chi$ (dual of $I_\chi = Q_\chi P_\chi$) imply that I'_χ is actually the composite of two isomorphisms.

A.2. Corollary. *The operator I'_χ maps isomorphically $\mathfrak{D}'_\chi$ onto the subspace $\mathfrak{M}_\chi^\perp$ of all functionals of $D'(\overset{\circ}{C}_2)$ which vanish on $\mathfrak{M}_\chi$.*

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[12] This Appendix represents another exposition of the result of Appendix A.2 in [6]. Note that the consideration can be trivially extended to arbitrary dimension n (while here only the case $n = 2$ is dealt with).

A.3. Lemma. $\mathfrak{M}_\chi^\perp = \mathfrak{d}_{-\chi}$, *where* $\mathfrak{d}_{-\chi}$ *is the subspace of all distributions in* $\mathring{C}_2$ *homogeneous of index* $-\chi$.

Proof. If $\Phi \in \mathfrak{M}_\chi^\perp$ then, by Corollary A.2, $\Phi = I'_\chi \varphi$ for some $\varphi \in \mathfrak{D}'_\chi$. For any $a \in \mathring{C}_1$ and $F \in D(\mathring{C}_2)$, we have

$$\begin{aligned}(\Phi(a\zeta), F(\zeta)) &= |a|^{-4}(\Phi, F_a) = |a|^{-4}(\varphi, I_\chi F_a) = |a|^{-4}(\varphi, (I_\chi F)_a) \\ &= |a|^{-4}\varphi_\chi^{[2]}(a^{-1})\,(\varphi, I_\chi F) = \varphi_{-\chi}^{[2]}(a)\;(\Phi, F),\end{aligned}$$

where $F_a(\zeta) := F(a^{-1}\zeta)$. This implies $\Phi \in \mathfrak{d}_{-\chi}$.

Conversely, let $\Phi \in \mathfrak{d}_{-\chi}$. We claim that $(\Phi, F) = (\Phi, H \cdot (I_\chi F))$ for all $F \in D(\mathring{C}_2)$ and a picked function H as in the proof of Lemma A.1. This implies $\Phi \in \mathfrak{M}_\chi^\perp$, hence there remains this formula to be proved. To this end we integrate with $|a|^{-2}\,|da\,d\bar{a}|$ both sides of the equation

$$(\Phi, H_a \cdot F)\,[\equiv |a|^4\,(\Phi_{a^{-1}}, H \cdot F_{a^{-1}})] = \varphi_\chi^{[2]}(a^{-1})\,(\Phi, H \cdot F_{a^{-1}})\,; \qquad \forall a \in \mathring{C}_1\,.$$

Then the left-hand side can be construed as the value of the distribution $\Phi(\zeta) \otimes |a|^{-2} \in D'(\mathring{C}_2 \times \mathring{C}_1)$ on the test function $H(a^{-1}\zeta) \cdot F(\zeta) \in D(\mathring{C}_2 \times \mathring{C}_1)$; an analogous meaning can be given to the right-hand side. By virtue of commutativity of tensor product of two distributions, we obtain

$$(\Phi(\zeta), F(\zeta) \cdot \textstyle\int H_a(\zeta)\,|a|^{-2}\,|da\,d\bar{a}|) = (\Phi, H \cdot (I_\chi F))\,,$$

which is just the required formula. Q.E.D.

By combining Corollary A.2 with Lemma A.3 we obtain

A.4. Theorem. *The operator* $I'_\chi : \mathfrak{D}'_\chi \to D'(\mathring{C}_2)$ *maps isomorphically* $\mathfrak{D}'_\chi$ *onto the subspace* $\mathfrak{d}_{-\chi} \subset D'(\mathring{C}_2)$.

The isomorphism sets an element $\varphi \in \mathfrak{D}'_\chi$ into the distribution $\Phi = I'_\chi \varphi$ by $(\Phi, F) = (\varphi, I_\chi F)$, $\forall F \in D(\mathring{C}_2)$. Conversely, for a given $\Phi \in \mathfrak{d}_{-\chi}$, the corresponding element $\varphi \in \mathfrak{D}'_\chi$ is related to Φ by $(\varphi, f) = (\Phi, R_\chi f)$, $\forall f \in \mathfrak{D}_\chi$.

Note that I'_χ is $\boldsymbol{SL(2, C)}$ invariant in the natural sense. Indeed, let T_χ^* denote the representation of $\boldsymbol{SL(2, C)}$ in $\mathfrak{D}'_\chi$ which is adjoint to T_χ, i.e. $(T_\chi^*(A)\,\varphi, f) = (\varphi, T_\chi(A^{-1})\,f)$ for all $\varphi \in \mathfrak{D}'_\chi$, $f \in \mathfrak{D}_\chi$. Then $(I'_\chi T_\chi^*(A)\varphi)(\zeta) = (I'_\chi \varphi)(A^{-1}\zeta)$ for all $\varphi \in \mathfrak{D}'_\chi$ and $A \in \boldsymbol{SL(2, C)}$.

Appendix B. Homogeneous Distributions in One Complex Variable

We remind the reader of basic facts on homogeneous distributions in $\mathring{C}_1$ and in C_1 ([9], Appendix B).

For an arbitrary index $\chi \equiv (\lambda, \mu) \in \mathfrak{X}$, the spaces of distributions in $\mathring{C}_1$ and in C_1, respectively, homogeneous of index χ are one-dimensional and consist, respectively, of multiples of the $\mathscr{C}^\infty$ function $\varphi_\chi^{[1]}$ defined in

(0.2) and of multiples of the distribution ψ_χ defined as follows. For $\mathrm{Re}(\lambda+\mu)>1$, ψ_χ is a continuous function,

$$\psi_\chi(z) := (\Gamma(\lambda \vee \mu) + \tfrac{1}{2}))^{-1}\, |z|^{\lambda+\mu-1}\, e^{i(\lambda-\mu)\arg z}\,;$$

here and in the following $\lambda \vee \mu = \frac{1}{2}(\lambda+\mu+|\lambda-\mu|)$, $\lambda \wedge \mu = \frac{1}{2}(\lambda+\mu-|\lambda-\mu|)$. It is clear that [for $\mathrm{Re}(\lambda+\mu)>1$] ψ_χ is a distribution-valued analytic function in χ [in the sense of analyticity in $\frac{1}{2}(\lambda+\mu)$ at fixed integral $\lambda-\mu$] and satisfies the identities: $\psi_\chi(z) = \dfrac{\partial}{\partial z}\psi_{(\lambda+1,\mu)}(z)$ for $\lambda-\mu \geqq 0$, and $\psi_\chi(z) = \dfrac{\partial}{\partial \bar z}\psi_{(\lambda,\mu+1)}(z)$ for $\lambda-\mu \leqq 0$. By induction on n, these identities provide a (unique) analytic continuation of ψ_χ into $\{\chi \in \mathfrak{X} \mid \mathrm{Re}(\lambda+\mu) > 1-n\}$ and hence into the whole $\mathfrak{X}$. In this way the homogeneous distribution ψ_χ is constructed for all $\chi \in \mathfrak{X}$. In particular, if $\chi \in \mathfrak{P}_-^{[1]}$, i.e. if both λ and μ are negative half-integers, then a straightforward calculation gives

$$\begin{aligned}\psi_\chi(z) &= \frac{(-1)^{-(\lambda\wedge\mu)-\frac{1}{2}}}{\Gamma(-(\lambda\wedge\mu)+\frac{1}{2})}\left(\frac{\partial}{\partial z}\right)^{-\lambda-\frac{1}{2}}\left(\frac{\partial}{\partial \bar z}\right)^{-\mu-\frac{1}{2}}[2\pi\delta(z)]\\ &= (-1)^{-(\lambda\wedge\mu)-\frac{1}{2}}\,\psi_{-\chi}\left(\frac{\partial}{\partial z}\right)[2\pi\delta(z)]\,,\end{aligned}$$

where δ is a distribution in $\boldsymbol{C}_1$ defined by $(\delta, F) = F(0)$ for all $F \in D(\boldsymbol{C}_1)$, and $\psi_{-\chi}(z)$ is a polynomial in z and $\bar z$. It is easy to see that, for any $\chi \in \mathfrak{X}$, the restriction of ψ_χ to $\mathring{\boldsymbol{C}}_1$ coincides with $(\Gamma((\lambda \vee \mu)+\frac{1}{2}))^{-1}\,\varphi_\chi^{[1]}$; therefore $\mathit{supp}\,\psi_\chi = \boldsymbol{C}_1$ for $\chi \notin \mathfrak{P}_-^{[1]}$ and $\mathit{supp}\,\psi_\chi = \{0\}$ for $\chi \in \mathfrak{P}_-^{[1]}$.

At last, for any χ, define $\hat\varphi_\chi \in D'(\boldsymbol{C}_1)$ by setting $\hat\varphi_\chi = \Gamma((\lambda \vee \mu)+\frac{1}{2})\psi_\chi$ if $\chi \notin \mathfrak{P}_-^{[1]}$, and $\hat\varphi_\chi = (-1)^{-(\lambda\vee\mu)-\frac{1}{2}} \times (\Gamma(-(\lambda\vee\mu)-\frac{1}{2}))^{-1}\dfrac{\partial}{\partial\chi}\psi_\chi$ if $\chi \in \mathfrak{P}_-^{[1]}$, where $\dfrac{\partial}{\partial\chi}$ is the derivative with respect to $\frac{1}{2}(\lambda+\mu)$ (at fixed $\lambda-\mu$). It is easy to see that $\hat\varphi_\chi$ is an extension of the distribution $\varphi_\chi^{[1]}$ from $\mathring{\boldsymbol{C}}_1$ into $\boldsymbol{C}_1$. For $\chi \notin \mathfrak{P}_-^{[1]}$ this extension is homogeneous, while for $\chi \in \mathfrak{P}_-^{[1]}$ it is the so-called associated homogeneous distribution of the first order, since

$$\hat\varphi_\chi(az) = \varphi_\chi^{[1]}(a)\left\{\hat\varphi_\chi(z) + \ln|a|\,\frac{2(-1)^{-(\lambda\vee\mu)-\frac{1}{2}}}{\Gamma(-(\lambda\vee\mu)+\frac{1}{2})}\psi_\chi(z)\right\}, \qquad \forall a \in \mathring{\boldsymbol{C}}_1\,.$$

Appendix C. The Family of Kernels with Analytic Dependence on the Representation Parameters

We now construct the family $\Psi \equiv \{\Psi_X \mid X \in \Xi\}$ mentioned in Introduction. First of all, we note that Ξ can be considered naturally as a complex analytic manifold (which consists of countably many disjoint copies of $\boldsymbol{C}_3$). Below the following parameterization on Ξ is used. Every triple $X \equiv (\chi_1, \chi_2, \chi_3) \in \Xi$ characterized by its "dual triple" $(\chi^1, \chi^2, \chi^3) \in \mathfrak{X}^3$ [where $\chi^j \equiv (\lambda^j, \mu^j)$ are defined in (0.12)] or, equivalently, by two triples (c^1, c^2, c^3) and (k^1, k^2, k^3) where $c^j \equiv \frac{1}{2}(\lambda^j + \mu^j)$ ranges in the complex plane and $k^j \equiv \frac{1}{2}(\lambda^j - \mu^j)$ is an integer or a half-integer. The numbers c^1, c^2, c^3 will be considered as local coordinates on Ξ and, by definition, $\dfrac{\partial}{\partial \chi^j} \equiv \dfrac{\partial}{\partial c^j}$. At last, for every triple $X \in \Xi$, we introduce the index $\mathfrak{x} \equiv (\mathfrak{l}, \mathfrak{m})$ by (1.2) and the numbers $\mathfrak{c}, \mathfrak{k}$ by $\mathfrak{c} = -\frac{1}{2} + \sum_{j=1}^{3} c^j$ and $\mathfrak{k} = \sum_{j=1}^{3} k^j$.

C.1. Theorem. *Let Ψ_X be the distribution (indeed a continuous function) of $D'((\mathring{C}_2)^3)$ defined for* $\operatorname{Re} c^j > \frac{1}{2}$ $(\forall j = 1, 2, 3)$ *by*

$$\Psi_X(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{\Gamma(\mathfrak{c} + |\mathfrak{k}| + 1)} \prod_{j=1}^{3} \psi_{\chi^j}(Z_j). \tag{C.1}$$

Ψ_X can be considered as a distribution valued analytic function in X which possesses a unique analytic continuation into the whole Ξ, Ψ_X belonging to $\mathfrak{T}(X)$ for all $X \in \Xi$. Furthermore, a triple $X \in \Xi$ is a zero of Ψ (i.e. X satisfies the condition $\Psi_X = 0$) if and only if (at least) one of the following conditions (i), (ii), (iii) *is fulfilled:*

(i) *at least two of χ^1, χ^2, χ^3 belong to $\mathfrak{P}_-^{[1]}$;*

(ii) *$\mathfrak{x} \in \mathfrak{P}_-^{[2]}$, and $\chi^j \in \mathfrak{P}_+^{[1]}$ for some $j \in \{1, 2, 3\}$;*

(iii) *$\mathfrak{x} \in \mathfrak{P}_-^{[2]}$, and one of the triples $(\lambda^1, \lambda^2, \lambda^3)$, (μ^1, μ^2, μ^3) consists of negative half-integers while the other consists of negative half-integer and two positive half-integers.*

Here, by definition,

$$\mathfrak{P}_+^{[n]} = \{\chi \equiv (\lambda, \mu) \in \mathfrak{X} \mid \text{both } \lambda - n/2 \text{ and } \mu - n/2 \text{ are non-negative integers}\},$$
$$\mathfrak{P}_-^{[n]} = \{-\chi \equiv (-\lambda, -\mu) \mid \chi \in \mathfrak{P}_+^{[n]}\}.$$

The proof of this theorem (as well as of all the other statements in this Appendix) is given in [7]. Of the properties of Ψ_X the next two ones deserve mentioning. First, Ψ_X behaves under permutations of indices as follows; for an odd permutation (i, j, k) of $(1, 2, 3)$, we have

$$\Psi_{(\chi_i, \chi_j, \chi_k)}(\zeta_i, \zeta_j, \zeta_k) = (-1)^{2\mathfrak{k}} \Psi_{(\chi_1, \chi_2, \chi_3)}(\zeta_1, \zeta_2, \zeta_3). \tag{C.2}$$

Second, we remind that mapping (0.14) is regular in $\boldsymbol{O}$; this implies

$$\Psi_X|_{\boldsymbol{O}}(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{\Gamma(\mathfrak{c} + |\mathfrak{k}| + 1)} \left(\prod_{j=1}^{3} \psi_{\chi^j}(Z_j) \right) \Bigg|_{\boldsymbol{O}} \quad \text{for all} \quad X \in \Xi. \tag{C.3}$$

We intend to adapt Ψ to description of the spaces $\mathfrak{T}(X)$. For this purpose we introduce the subspace $\mathfrak{A}(X)$ of all distributions $T \in \mathfrak{T}(X)$ which are said to be *associated with Ψ at X* and which can be represented in form (0.13) where $P(z_1, z_2, z_3)$ is a complex polynomial in 3 variables. [Note that in general $P\left(\frac{\partial}{\partial X}\right) \Psi_X$ need not belong to $\mathfrak{T}(X)$.] It turns out that, for X a non-zero, $\mathfrak{A}(X)$ consists of multiples of Ψ_X. Thus it suffices to confine our attention to zeros of Ψ. At first we classify zeros. It is convenient to devide the set $\mathfrak{Z}$ of all zeros of Ψ into four parts $\mathfrak{Z}_t$ $(t=1,2,3,4)$ according to character of location of zeros near points of Z_t. More precisely, by definition, $X \in \mathfrak{Z}_t$ provided $X \in \mathfrak{Z}$ and there are exactly $t-1$ non-collinear vectors $\mathfrak{a} \equiv (\mathfrak{a}^1, \mathfrak{a}^2, \mathfrak{a}^3) \in \mathring{C}_3$, called zero direction vectors at X, such that $\left(\sum_{i=1}^{3} \mathfrak{a}^i \frac{\partial}{\partial \chi^i}\right)^n \Psi_X = 0$ for all integers $n \geqq 0$.

C.2. Classification of Zeros. The subsets $\mathfrak{Z}_t \subset \mathfrak{Z}$ $(t=1,2,3,4)$ exhaust zeros of Ψ, i.e. $\mathfrak{Z} = \bigcup_{t=1}^{4} \mathfrak{Z}_t$. The subsets $\mathfrak{Z}_t$ can be described in the following way.

1) $\mathfrak{Z}_1$ is the set of isolated zeros of Ψ, i.e. the set of all triples X satisfying condition (iii) of Theorem C.1.

2) $X \in \mathfrak{Z}_2$ if and only if one of the following three conditions 2a), 2b), 2c) is satisfied:

2a) $\mathfrak{x} \notin \mathfrak{P}_-^{[2]}$, and $\chi^j \in \mathfrak{P}_-^{[1]}$ for exactly two values of j;

2b) $\mathfrak{x} \in \mathfrak{P}_-^{[2]}$, $\chi^i \in \mathfrak{P}_-^{[1]}$, $\chi^j \in \mathfrak{P}_-^{[1]}$ and $\chi^k \in \{\chi \equiv (\lambda, \mu) \mid \lambda$ and μ are half-integers of different signs$\}$ for some permutation (i,j,k) of $(1,2,3)$;

2c) $\mathfrak{x} \in \mathfrak{P}_-^{[2]}$, $\chi^j \in \mathfrak{P}_+^{[1]}$ for exactly one value of j, and $\chi^i \notin \mathfrak{P}_+^{[1]} \cup \mathfrak{P}_-^{[1]}$ for at least one value of i.

3) $X \in \mathfrak{Z}_3$ if and only if one of the following two conditions is satisfied:

3a) $\mathfrak{x} \in \mathfrak{P}_-^{[2]}$ and $\chi^i \in \mathfrak{P}_-^{[1]}$, $\chi^j \in \mathfrak{P}_-^{[1]}$, $\chi^k \in \mathfrak{P}_+^{[1]}$ where $\{i,j,k\} = \{1,2,3\}$;

3b) $\mathfrak{x} \in \mathfrak{P}_-^{[2]}$ and $\chi^i \in \mathfrak{P}_+^{[1]}$, $\chi^j \in \mathfrak{P}_+^{[1]}$, $\chi^k \in \mathfrak{P}_-^{[1]}$ where $\{i,j,k\} = \{1,2,3\}$.

4) $\mathfrak{Z}_4 = \{X \in \Xi \mid \chi^j \in \mathfrak{P}_-^{[1]}$ for all $j = 1,2,3\}$.

Zero direction vectors at $X \in \mathfrak{Z}$ can be described (up to multiples) in each case as follows.

1) No zero direction vectors.

2) The corresponding zero direction vector $\mathfrak{a} \equiv (\mathfrak{a}^1, \mathfrak{a}^2, \mathfrak{a}^3)$ is defined by the following conditions, respectively:

2a)–2b) $\mathfrak{a}^l = 0$ if $\chi^l \in \mathfrak{P}_-^{[1]}$, $l \in \{1,2,3\}$;

2c) $\sum_{r=0}^{3} \mathfrak{a}^r = 0$, and $\mathfrak{a}^l = 0$ if $\chi^l \in \mathfrak{P}_+^{[1]}$, $l \in \{1,2,3\}$.

3) The corresponding zero direction vectors $\mathfrak{a}_{(1)}$ and $\mathfrak{a}_{(2)}$ are determined by the following conditions, respectively:

3a) $\mathfrak{a}_{(1)}^i = \mathfrak{a}_{(1)}^j = 0$, $\quad \mathfrak{a}_{(2)}^i + \mathfrak{a}_{(2)}^j = \mathfrak{a}_{(2)}^k = 0$;

3b) $\mathfrak{a}_{(1)}^i = \mathfrak{a}_{(1)}^j + \mathfrak{a}_{(1)}^k = 0$, $\quad \mathfrak{a}_{(2)}^i + \mathfrak{a}_{(2)}^k = \mathfrak{a}_{(2)}^j = 0$.

4) The three zero direction vectors $\mathfrak{a}_{(1)}, \mathfrak{a}_{(2)}, \mathfrak{a}_{(3)}$ are defined by $\mathfrak{a}_{(j)}^i = 0$ for $i \neq j$ $(i,j = 1,2,3)$.

C.3. Theorem. *Let X be a zero of Ψ of type t ($t=1,2,3,4$). Further, let $\mathfrak{a}_{(1)}, \ldots, \mathfrak{a}_{(t-1)}$ be zero direction vectors at X (as in Classification C.2) and $\{\mathfrak{a}_{(1)}, \ldots, \mathfrak{a}_{(t-1)}, \mathfrak{b}_{(1)}, \ldots, \mathfrak{b}_{(4-t)}\}$ be a basis in $\boldsymbol{C}_3$. Define the following polynomials in 3 variables:*

$$P_{(r)}(z_1, z_2, z_3) = (\mathfrak{b}_{(r)} \cdot z), \qquad P_{(l,m)}(z_1, z_2, z_3) = (\mathfrak{a}_{(l)} \cdot z)\,(\mathfrak{a}_{(m)} \cdot z), \quad \text{(C.4)}$$

where $r=1, \ldots, 4-t$; $1 \leqq l < m \leqq t-1$, and, by definition, $\mathfrak{a} \cdot z \equiv \sum_{i=1}^{3} \mathfrak{a}^i z_i$.

Then the set of distributions $P_{(r)}\left(\frac{\partial}{\partial X}\right) \Psi_X$, $P_{(l,m)}\left(\frac{\partial}{\partial X}\right) \Psi_X$ forms a basis in the subspace $\mathfrak{A}(X)$ of the kernels associated with Ψ at X. The dimension of $\mathfrak{A}(X)$ coincides with the number of polynomials (C.4), *i.e.*

$$\dim \mathfrak{A}(X) = 2 \ \textit{if} \ X \in \mathfrak{Z}_2 \cup \mathfrak{Z}_3, \quad \dim \mathfrak{A}(X) = 3 \ \textit{if} \ X \in \mathfrak{Z}_1 \cup \mathfrak{Z}_4. \quad \text{(C.5)}$$

C.4. Special Case. We close this exposition by considering in more detail zeros of type 4. Note that X belongs to $\mathfrak{Z}_4$ precisely when $X \in \Xi_0$ and $v(X) = 4$. Among such triples is a unique extremal one equivalent to all the others. This triple, denoted by $\dot{X}$, is defined by setting $\dot{\lambda}^j = \dot{\mu}^j = -\frac{1}{2}$ for all $j = 1, 2, 3$. The corresponding space $\mathfrak{A}(\dot{X})$ consists of distributions (4.6) where (a_{ij}) is an arbitrary symmetric complex 2×2 matrix. For an arbitrary $X \in \mathfrak{Z}_4$, elements of $\mathfrak{A}(X)$ can be expressed through those of $\mathfrak{A}(\dot{X})$ in analogy with Proposition 1.6. Indeed, an element

$$T = P\left(\frac{\partial}{\partial X}\right) \Psi_X \in \mathfrak{A}(X)$$

is related to the corresponding element

$$\dot{T} = P\left(\frac{\partial}{\partial X}\right) \Psi_X|_{X=\dot{X}} \in \mathfrak{A}(\dot{X})$$

by the formula

$$T = \mu(X)\, M(X)\, \dot{T}, \quad \text{(C.6)}$$

where $\mu(X)$ is an appropriate multiple and $M(X) = \prod_{j=1}^{3} (\Delta_j)^{-\lambda^j - \frac{1}{2}} (\bar{\Delta}_j)^{-\mu^j - \frac{1}{2}}$.
A straightforward calculation yields

$$\begin{aligned} \mu(X)\, M(X) &= \frac{(-1)^{-\mathfrak{c}+|\mathfrak{k}|}}{\Gamma(-\mathfrak{c}+|\mathfrak{k}|)} \prod_{j=1}^{3} (-1)^{-c^j+|k^j|-\frac{1}{2}} \psi_{-\chi^j}(\Delta_j) \\ &= (-1)^{-\mathfrak{c}+|\mathfrak{k}|-1} \left(\prod_{j=1}^{3} (-1)^{-c^j+|k^j|-\frac{1}{2}} \right) \Psi_{(-\chi_1, -\chi_2, -\chi_3)}\left(\frac{\partial}{\partial \zeta_1}, \frac{\partial}{\partial \zeta_2}, \frac{\partial}{\partial \zeta_3}\right). \end{aligned} \quad \text{(C.7)}$$

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