Commun. math. Phys. 29, 163—175 (1973)

Thermodynamic Duality for Classical Systems of Arbitrary Spin

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Received June 25, 1972

Abstract. Given a classical spin s system, namely, a set of spin sites of maximum spin s in ν-dimensional space along with a Hamiltonian defined on the possible spin configurations, a general method is described for constructing a large class of dual lattices of the same spin. The method utilizes the commutative group structure with which the configuration space is endowed.

In the classical statistical mechanics of spin $\frac{1}{2}$ lattice models, the set of spin configurations has a natural group structure, a fact that has been used by Sherman, for example, in a generalization of the Griffiths Inequalities [1]. In a similar manner, a group structure can be assigned to the set of spin configurations of a spin s lattice, for any s. Here a lattice refers loosely to any finite collection of spin sites in ν-dimensional space.

Duality, or the connection between high and low temperature properties for appropriately chosen pairs of lattices, has as applied to specific models a long history dating from the early work of Wannier and Onsager [2, 3]. Recently, Wegner has proved that any ferromagnetic spin $\frac{1}{2}$ lattice has a dual [4]. Merlini and Gruber have extended these results to arbitrary spin $\frac{1}{2}$ lattices by a constructive procedure [5].

In this article we generalize the construction of [5] to provide a family of duals for any lattice of arbitrary spin. In the first three sections, a group structure is introduced onto the space of configurations, and the dual groups are defined. For these systems the groups involved are simply products of the cyclic group of order $2s+1$. In fact the results can be extended to general finite abelian groups, corresponding to systems of mixed higher spins.

In the fourth section the dual interaction is derived for groups which are "non-π degenerate". This includes nearly all higher spin models of physical interest. Those groups not satisfying the restriction are dealt with in the following section.

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1. Notation

We use throughout $\mathbb{Z}_n$ for the integers modulo n and θ_i, $0 \leqq i < n$, for the (counter-clockwise) ordered n^{th} roots of unity, $\theta_0 \equiv 1$. For $\mathscr{S}$ an arbitrary set, let $N(\mathscr{S})$ be its cardinality, and let $\mathscr{G} = P_n(\mathscr{S})$ denote the group of functions $f : \mathscr{S} \to \mathbb{Z}_n$ with group multiplication

$$f g(s) = f(s) + g(s) \bmod n, \qquad s \in \mathscr{S}$$

$P_n(\mathscr{S})$ can equally well be thought of as the group of partitions of $\mathscr{S}$ into n subsets, for obviously $f \in P_n(\mathscr{S})$ defines a partition $\{f^{-1}(i)\}_{i \in \mathbb{Z}_n}$ of $\mathscr{S}$. Denote the character group of $\mathscr{G}$ by $\hat{\mathscr{G}}$, and likewise the elements $\hat{g} \in \hat{\mathscr{G}}$ under the canonical isomorphism $\mathscr{G} \leftrightarrow \hat{\mathscr{G}}$.

If $\mathscr{A}$ is a subset of $P_n(\mathscr{S})$, define the product map π and the character projection τ in the following manner:

(i) $\pi : P_n(\mathscr{A}) \to Gp\mathscr{A}$ by $\pi(f) = \prod_{g \in \mathscr{A}} g^{f(g)}$ where $Gp\mathscr{A}$ is the group generated by $\mathscr{A}$; and

(ii) $\tau : P_n(\mathscr{S}) \to P_n(\mathscr{A})$ by $\tau(g) : f \to \sum_{s \in \mathscr{S}} g(s) f(s) \bmod n$.

The maps π and τ are clearly group homomorphisms. They play a fundamental role in the formulation of thermodynamic duality to be developed, as they are associated, respectively, with high and low temperature expansions of the partition function.

In the statistical mechanics of a classical spin $(n-1)/2$ system, or a classical lattice gas of maximum site occupation $n-1$, one is given a set of sites Λ and a complex valued function H on the configurations $P_n(\Lambda)$ of Λ. We call $\mathscr{G} = P_n(\Lambda)$ the group of configurations and $H : \mathscr{G} \to \mathbb{C}$ the energy function. Although in physical situations H will be real valued, it is necessary to consider the more general situation, as the family of duals to be constructed will not necessarily have real energy functions, even if the initial energy function is real. This is nevertheless useful, for the study of the analyticity properties of physical models gives information in general in a complex domain of analyticity.

Expanding H in its Fourier series:

$$H = \sum_{\hat{\alpha} \in \hat{\mathscr{G}}} H_\alpha \hat{\alpha}$$

let $\mathscr{J} \subset \mathscr{G}$ denote the set of non-zero interactions

$$\mathscr{J} = \{\alpha \in \mathscr{G} \mid H_\alpha \neq 0\}$$

and $\mathscr{J}_p$ the set

$$\mathscr{J}_p = \{\alpha' \in \mathscr{G} \mid \alpha' = (\alpha)^j \neq 0 \quad \text{for some} \quad j \in \mathbb{Z}_n \quad \text{and} \quad \alpha \in \mathscr{J}\}.$$

In the usual notation of spin $\frac{1}{2}$ models ($n=2$), $\mathscr{G}=P_2(\Lambda)$ is (isomorphic to) the set of subsets of Λ, H is a sum of products $\sigma_{\lambda_1} \ldots \sigma_{\lambda_n}$ of spin matrices for certain sets $\alpha=\{\lambda_1, \ldots, \lambda_n\}$, $\mathscr{J}$ is the set of these α, or more precisely, of functions assigning 1 to the elements of α and zero to the elements of $\Lambda-\alpha$, and $\mathscr{J}_p=\mathscr{J}$ since $\sigma^2=I$.

The partition function $Z(\beta H)$ is defined by

$$Z(\beta H)=\sum_{g \in \mathscr{G}} e^{-\beta H(g)}=\sum_{g \in \mathscr{G}} \prod_{\alpha \in \mathscr{J}} e^{-\beta H_\alpha \hat{\alpha}(g)}$$

with the temperature taken as inversely proportional to the parameter β, for β real.

Let $\mathscr{K}_\pi$ denote the kernel of the homomorphism

$$\pi: P_n(\mathscr{J}) \rightarrow Gp\,\mathscr{J}\,.$$

By expanding $\exp(-\beta H_\alpha \hat{\alpha}(g))$ in its Fourier series:

$$e^{-\beta H_\alpha \hat{\alpha}(g)}=\sum_{i=0}^{n-1} F_i(\beta H_\alpha)\,(\hat{\alpha}(g))^i$$

and using the orthogonality of the characters, one obtains the high temperature expansion for the partition function. Similarly, let $\mathscr{K}_\tau$ be the kernel of:

$$\tau: \mathscr{G} \rightarrow P_n(\mathscr{J})$$

and $\mathscr{R}_\tau$ its range. Then, in an obvious fashion, one obtains the low temperature expansion for Z.

Theorem 1. (i) *High temperature expansion:*

$$Z(\beta H)=N(\mathscr{G}) \sum_{f \in \mathscr{K}_\pi} \prod_{\alpha \in \mathscr{J}} F_{f(\alpha)}(\beta H_\alpha)\,.$$

(ii) *Low temperature expansion:*

$$Z(\beta H)=N(\mathscr{K}_\tau) \sum_{f \in \mathscr{R}_\tau} \prod_{\alpha \in \mathscr{J}} e^{-\theta_{f(\alpha)} \beta H_\alpha}\,.$$

2. Thermodynamic Duality

The problem in thermodynamic duality, vis-a-vis the expansions of Theorem 2, is to construct a group $\mathscr{G}^*$ whose low (or high) temperature properties can be simply related to the high (low) temperature properties of a given group $\mathscr{G}$, and to specify the corresponding subset $\mathscr{J}^* \subset \mathscr{G}^*$, along with the coefficients H_α^*, $\alpha \in \mathscr{J}^*$. One requires that $\mathscr{G}^*$ be of the same form as $\mathscr{G}$, $\mathscr{G}^*=P_n(\Lambda^*)$, that is, a product of cyclic groups of order n.

In fact, any higher spin model can be converted to an equivalent spin $\frac{1}{2}$ model [6], but then, following the usual spin $\frac{1}{2}$ duality transformation, it is not possible to recover in a sensible way a similar higher spin model.

The solution of the duality problem for $n=2$ is accomplished by establishing a correspondance between the elements of $\mathscr{J} \subset \mathscr{G}$ and the elements of $\mathscr{J}^* \subset \mathscr{G}^*$ in such a way that a bijection is induced between $\mathscr{K}_\pi \subset P_n(\mathscr{J})$ and $\mathscr{R}_\tau^* \subset P_n(\mathscr{J}^*)$. Then by setting $F_i(\beta H_\alpha)$ proportional to $e^{-\theta_i \beta H_\alpha^*}$, $i=1,2$, the terms in the expansions of Theorem 1 are, except for the product of proportionality constants which factor out, pairwise equal. It is clear that such an approach cannot immediately succeed for $n>2$, since the set of equations:

$$F_i(\beta H_\alpha) = K_\alpha e^{-\theta_i \beta H_\alpha^*}, \qquad 0 \leqq i < n$$

cannot be solved for the two variables K_α and H_α^*.

Furthermore, although it is popular in the literature to consider an approximate duality, that is to say, a group $\mathscr{G}^*$ whose partition function can be simply related to that of $\mathscr{G}$, except perhaps for the contribution of boundary terms, it is clearly desirable to solve the exact problem. For while the boundary terms may not affect the partition function, and hence the free energy, in the thermodynamic (infinite volume) limit, nor perhaps the correlation functions above the critical temperature, it is well known that the boundary terms will determine the state, and the correlation functions, below criticality. It may be pointed out here that boundary terms in this formulation are simply additional elements of $\mathscr{J}$. For example, in the spin $\frac{1}{2}$ Ising models, periodic boundary conditions are realized precisely by terms in $\mathscr{J}$ of the form $\sigma_\lambda \sigma_{\lambda'}$, with λ and λ' the corresponding points on the boundary of Λ; and plus or minus boundary conditions are given by field terms in $\mathscr{J}$, terms of the form σ_λ, with λ on the boundary.

As a result of these considerations, we shall solve the problem by two routes. In the case of a non-π degenerate group, defined in the next section and including essentially all models of physical interest, except for certain models with the "wrong" boundary conditions, a simple constructive solution will be obtained. Many of the models with the "wrong" boundary conditions can be made non-π degenerate with a very minor modification of the boundary, and it is probable that in most of these cases, the correlation functions can be rigorously shown to be unaffected.

For those models which are π degenerate, the problem can be reduced to a set of non-linear equations. If the set of equations has a solution, then a set of dual models can be constructed by the method outlined.

3. Construction of Dual Families

In this section we shall generalize the construction of [5] to yield a family of groups of the form $P_n(\Lambda^*)$ for any given group $\mathscr{G} = P_n(\Lambda)$. We suppose as always that $\mathscr{G}$ is given along with a distinguished subset $\mathscr{J} \subset \mathscr{G}$.

Definition 2. Let $\mathscr{H} \subset P_n(\mathscr{J})$ be any set satisfying $Gp\,\mathscr{H} = \mathscr{K}_\pi$. Let $\mathscr{G}^*$ be the group $P_n(\mathscr{H})$ and $\mathscr{J}^*$ the subset of $\mathscr{G}^*$

$$\mathscr{J}^* = \{\gamma \in \mathscr{G}^* \mid \exists \alpha \in \mathscr{J} \text{ s.t. } \gamma(h) = h(\alpha), \forall h \in \mathscr{H}\}$$

and let $\mathscr{K}_\pi^*$ and $\mathscr{R}_\tau^*$ be the indicated subgroups of $P_n(\mathscr{J}^*)$. Define $\varphi : \mathscr{J} \to \mathscr{J}^*$ by $\varphi : \alpha \to \gamma$, where $\gamma(h) = h(\alpha)$, and the maps

$$\Phi_1 : \mathscr{K}_\pi \to P_n(\mathscr{J}^*)$$

$$\Phi_2 : \mathscr{R}_\tau \to P_n(\mathscr{J}^*)$$

by

$$\Phi_i(f) : \gamma \to f(\varphi^{-1}\gamma), \quad i = 1, 2,$$

for $f \in \mathscr{K}_\pi$ (resp. $f \in \mathscr{R}_\tau$) and $\gamma \in \mathscr{J}^*$.

Finally, call $\{\mathscr{G}, \mathscr{J}\}$ non-π degenerate if the elements of $\mathscr{K}_\pi$ separate the elements of $\mathscr{J}$, i.e., given $\alpha_1, \alpha_2 \in \mathscr{J}$, there exists an $f \in \mathscr{K}_\pi$ such that $f(\alpha_1) \neq f(\alpha_2)$.

Lemma 3. *For any group $\mathscr{G}$, Φ_1 is well defined. If $\mathscr{G}$ is non-π degenerate, then Φ_2 is well defined. The maps Φ_1 and Φ_2 are group homomorphisms, and satisfy:*

$$\varphi^{-1}\{(\Phi_i f)^{-1}(j)\} = f^{-1}(j), \quad j \in \mathbb{Z}_n.$$

Proof. If $\varphi(\alpha_1) = \varphi(\alpha_2)$, then $h(\alpha_1) = h(\alpha_2)$ for all $h \in \mathscr{H}$. Let f be any element of $\mathscr{K}_\pi$. Then $f = \prod_{i=1}^{m} h_i^{r_i}$, $r_i \in \mathbb{Z}_n$ and $h_i \in \mathscr{H}$, and

$$f(\alpha) = \sum_{i=1}^{m} r_i h_i(\alpha).$$

Hence $f(\alpha_1) = f(\alpha_2)$.

The restriction to non-π degenerate groups is precisely the restriction that φ be bijective; thus Φ_2 is well defined as well.

The Φ_i are group homomorphisms, as

$$\begin{aligned}\Phi_i(f_1 f_2)(\varphi(\alpha)) &= f_1 f_2(\alpha) = f_1(\alpha) + f_2(\alpha) \bmod n \\ &= \Phi_i(f_1)(\varphi(\alpha)) + \Phi_i(f_2)(\varphi(\alpha)) \bmod n\end{aligned}$$

and the remaining assertion is clear.

To show that the Φ_i are bijections, the following two lemmas will be useful.

Lemma 4. *If* $\mathcal{G} = P_n(\Lambda)$ *is non-*π *degenerate and* $f_\lambda \in P_n(\mathcal{J})$, $f_\lambda : \alpha \to \alpha(\lambda)$, $\lambda \in \Lambda$, *then* $\Phi_2(f_\lambda)$ *is in* $\mathcal{K}_\pi^*$.

Proof. The equivalence

$$h \in \mathcal{K}_\pi \Leftrightarrow \sum_{\alpha \in \mathcal{J}} h(\alpha)\, \alpha(\lambda) = 0 \quad \text{for all} \quad \lambda$$

and the analogous equivalence for $f \in \mathcal{K}_\pi^*$ follows immediately from the definition of π. But then $\Phi_2(f_\lambda) \in \mathcal{K}_\pi^*$ iff

$$\sum_{\varphi(\alpha)} \Phi_2(f_\lambda)\,(\varphi(\alpha))\, \varphi(\alpha)\,(h) = 0$$

or

$$\sum_{\alpha} \alpha(\lambda)\, h(\alpha) = 0$$

for all $h \in \mathcal{K}_\pi$.

Note $f_\lambda = \tau(\delta_\lambda)$, where $\delta_\lambda \in \mathcal{G}$, $\delta_\lambda(s) = 1$ if $s = \lambda$ and zero otherwise.

Lemma 5. *For any* $\mathcal{G}$,

$$N(\mathcal{K}_\pi) = n^{N(\mathcal{J})}\, N(\mathcal{K}_\tau)/N(\mathcal{G})$$
$$N(\mathcal{G}^*) = N(\mathcal{K}_\pi)\, N(\mathcal{K}_\tau^*)$$

Proof. Observe $g \in \mathcal{K}_\tau$ iff

$$\sum_{\lambda \in \Lambda} g(\lambda)\, \alpha(\lambda) = 0, \qquad \forall \alpha \in \mathcal{J},$$

iff

$$\hat{g} : \alpha \to 1, \qquad \forall \alpha \in Gp\,\mathcal{J},$$

using the canonical isomorphism $g \leftrightarrow \hat{g}$,

$$\hat{g} : \alpha \to \exp\left\{\frac{2\pi i}{n} \sum_{\lambda \in \Lambda} g(\lambda)\, \alpha(\lambda)\right\}.$$

But $\{\hat{g} \in \hat{\mathcal{G}} \mid \hat{g}(\alpha) = 1, \forall \alpha \in Gp\,\mathcal{J}\}$ is clearly isomorphic to $(\mathcal{G}/\widehat{Gp\,\mathcal{J}})$, and hence immediately:

$$\mathcal{K}_\tau \cong \mathcal{G}/Gp\,\mathcal{J}.$$

Using $N(P_n(\mathcal{J})) = n^{N(\mathcal{J})}$, we have:

$$n^{N(\mathcal{J})}/N(\mathcal{K}_\pi) = N(Gp\,\mathcal{J}) = N(\mathcal{G})/N(\mathcal{K}_\tau).$$

An element $f \in \mathcal{K}_\tau^*$ iff $\prod_{h \in \mathcal{H}} h^{f(h)} = 0$ in $P_n(\mathcal{J})$ as is easily seen; that is to say, iff f is in the kernel of

$$\pi' : P_n(\mathcal{H}) \to Gp\,\mathcal{H} = \mathcal{K}_\pi.$$

Then $\mathcal{G}^* = P_n(\mathcal{H})$, $\mathcal{K}_\pi = \mathcal{R}_{\pi'}$ completes the proof.

Theorem 6. *Φ_1 is an isomorphism $\Phi_1 : \mathcal{K}_\pi \to \mathcal{R}_\tau^*$. If $\mathcal{G}$ is non-π degenerate, Φ_2 is an isomorphism $\Phi_2 : \mathcal{R}_\tau \to \mathcal{K}_\pi^*$.*

Proof. We assert Φ_i is injective, $i=1,2$. For if $\Phi_i(f_1)=\Phi_i(f_2)\in P_n(\mathcal{J})$ then

$$\Phi_i(f_1)\,(\varphi(\alpha))=\Phi_i(f_2)\,(\varphi(\alpha)), \quad \forall \alpha\in\mathcal{J}$$

and so $f_1(\alpha)=f_2(\alpha)$.

To show Φ_1 is surjective onto $\mathcal{R}_\tau^*$, observe that for $g\in\mathcal{G}^*$ and $\alpha\in\mathcal{J}$,

$$\tau(g)\,(\varphi(\alpha))=\sum_{h\in\mathcal{H}} g(h)\,h(\alpha) \bmod n=\Phi_1\Big(\prod_{h\in\mathcal{H}} h^{g(h)}\Big)\,(\varphi(\alpha))$$

Now $f\in\mathcal{K}_\pi$ implies $f=\prod_{h\in\mathcal{H}} h^{g(h)}$ for some $g\in\mathcal{G}^*$, and then $\Phi_1(f)=\tau(g)$. Moreover, for any $g\in\mathcal{G}^*$, $\Phi_1 : \prod_{h\in\mathcal{H}} h^{g(h)}\to\tau(g)$, which completes the argument for Φ_1.

To demonstrate Φ_2 is in $\mathcal{K}_\pi^*$, it suffices to show $\Phi_2(\tau(\delta_\lambda))\in\mathcal{K}_\pi^*$ for δ_λ as in the proof of Lemma 4, since the δ_λ generate $\mathcal{G}$. But this is precisely Lemma 4. From non-π degeneracy, $N(\mathcal{J})=N(\mathcal{J}^*)$, and from Lemma 5:

$$\begin{aligned} N(\mathcal{R}_\tau)&=N(\mathcal{G})/N(\mathcal{K}_\tau)\,,\\ N(\mathcal{K}_\pi)&=n^{N(\mathcal{J})}N(\mathcal{K}_\tau)/N(\mathcal{G})\,,\\ N(\mathcal{K}_\pi^*)&=n^{N(\mathcal{J})}N(\mathcal{K}_\tau^*)/N(\mathcal{G}^*)\,,\\ N(\mathcal{G}^*)&=N(\mathcal{K}_\pi)N(\mathcal{K}_\tau^*)\,. \end{aligned}$$

Thus

$$N(\mathcal{K}_\pi^*)=n^{N(\mathcal{J})}/N(\mathcal{K}_\pi)=N(\mathcal{R}_\tau)$$

which completes the proof.

4. Non-π Degenerate Groups

We assume in this section that $\mathcal{G}$ is non-π degenerate. From the low temperature expansion, Theorem 1, the bijection Φ_2, and the orthonormality of the characters, we obtain:

$$\begin{aligned} Z(\beta H)&=N(\mathcal{K}_\tau)\sum_{f\in\mathcal{R}_\tau}\prod_{\alpha\in\mathcal{J}} e^{-\beta H_\alpha\theta_{f(\alpha)}}=N(\mathcal{K}_\tau)\sum_{f\in\mathcal{K}_\pi^*}\prod_{\alpha\in\mathcal{J}^*} e^{-\beta\hat{H}_\alpha\theta_{f(\alpha)}}\\ &=\frac{N(\mathcal{K}_\tau)}{N(\mathcal{G}^*)}\sum_{g\in\mathcal{G}^*}\sum_{f\in\mathcal{K}_\pi^*}\prod_{\alpha\in\mathcal{J}^*}\big(e^{-\beta\hat{H}_\alpha\theta_{f(\alpha)}}\hat{\alpha}^{f(\alpha)}(g)\big)\\ &=\frac{N(\mathcal{K}_\tau)}{N(\mathcal{G}^*)}\sum_{g\in\mathcal{G}^*}\prod_{\alpha\in\mathcal{J}^*}\Big(\sum_{s\in\mathbb{Z}_n} e^{-\beta\hat{H}_\alpha\theta_s}\hat{\alpha}^s(g)\Big) \end{aligned}$$

for $\hat{H}_\alpha=H_{\varphi^{-1}(\alpha)}$.

Define the generalized sine functions

$$\operatorname{sing}_i(x) = \sum_{t=0}^{n-1} e^{x\theta_t}\theta_{it}, \qquad i \in \mathbb{Z}_n .$$

Theorem 7. *Given a non-π degenerate group $\mathcal{G}$ with energy function H, assume*

$$\operatorto\operatorname{sing}_i(-\beta H_\alpha) \neq 0 .$$

Note that for a ferromagnetic model, $H_\alpha < 0$ for $\alpha \in \mathcal{J}$, this assumption is necessarily satisfied. Define the constants H_α^* for $\alpha \in \mathcal{J}_p^*$ and $n_\alpha = \mathrm{Order}(\alpha)$ by

$$H_\alpha^* = -\sum_{\substack{\gamma \in \mathcal{J}^* \\ \gamma^j = \alpha}} \frac{1}{n_\gamma \beta} \sum_{m=0}^{n_\gamma - 1} \theta_{-mjn/n_\gamma} \log \operatorname{sing}_{mn/n_\gamma}(-\beta H_{\varphi^{-1}(\gamma)})$$

and $D^0(\alpha)$ for $\alpha \in \mathcal{J}^*$ by:

$$D^0(\alpha) = -\frac{1}{n_\alpha \beta} \sum_{m=0}^{n_\alpha - 1} \log \operatorname{sing}_{mn/n_\alpha}(-\beta H_{\varphi^{-1}(\alpha)}) .$$

Then $\mathcal{G}^*$, $\mathcal{J}_p^*$ is a thermodynamic dual of $\mathcal{G}$, with energy function H^* defined by:

$$H^* = \sum_{\alpha \in \mathcal{J}_p^*} H_\alpha^* \hat{\alpha} .$$

The partition functions $Z_{\mathcal{G}}(\beta H)$ and $Z_{\mathcal{G}^*}(\beta H^*)$ are related by:

$$Z_{\mathcal{G}}(\beta H) = \frac{N(\mathcal{K}_\tau)}{N(\mathcal{G}^*)} \left\{ \prod_{\alpha \in \mathcal{J}^*} e^{-\beta D^0(\alpha)} \right\} Z_{\mathcal{G}^*}(\beta H^*) .$$

Proof. Referring to the expansions for Z above Theorem 7, define the coefficients D_α^j, $\alpha \in \mathcal{J}^*$, $0 \leqq j < n_\alpha = \mathrm{Order}(\alpha)$, by the n_α equations

$$\beta \sum_{j=0}^{n_\alpha - 1} D_\alpha^j \theta_{inj/n_\alpha} = -\log \operatorname{sing}_{in/n_\alpha}(-\beta \hat{H}_\alpha), \qquad i \in \mathbb{Z}_{n_\alpha} .$$

The system is solvable since the $\theta_{in/n_\alpha}(j) \equiv \theta_{inj/n_\alpha}$ are a complete set of characters for $\mathbb{Z}_{n_\alpha}$, and in fact

$$D_\alpha^j = -\frac{1}{n_\alpha \beta} \sum_{m=0}^{n_\alpha - 1} \theta_{-mjn/n_\alpha} \log \operatorname{sing}_{mn/n_\alpha}(-\beta \hat{H}_\alpha) .$$

Furthermore, for $H_\alpha < 0$ the logarithm is necessarily finite, since

$$\operatorname{sing}_i(x) = \sum_{k=1}^{\infty} \frac{x^{kn-i}}{(kn-i)!} + \delta_{i,0} .$$

Now obtain for Z,

$$Z_{\mathscr{G}}(\beta H) = \frac{N(\mathscr{K}_\tau)}{N(\mathscr{G}^*)} \sum_{g \in \mathscr{G}^*} \prod_{\alpha \in \mathscr{I}^*} e^{-\beta \sum_{j=0}^{n_\alpha - 1} D_\alpha^j \hat{\alpha}^j(g)}$$

$$= \frac{N(\mathscr{K}_\tau)}{N(\mathscr{G}^*)} \left\{ \prod_{\alpha \in \mathscr{I}^*} e^{-\beta D_\alpha^0} \right\} \sum_{g \in \mathscr{G}^*} \prod_{\alpha \in \mathscr{I}_p^*} e^{-\hat{\alpha}(g) \beta \sum_{\substack{\gamma \in \mathscr{I}^* \\ \gamma^j = \alpha}} D_\gamma^j}$$

which completes the proof.

For the case $n = 2$,

$$H_\alpha^* = -\frac{1}{2\beta} \{\log[e^{-\beta H_\alpha} + e^{\beta H_\alpha}] - \log[e^{-\beta H_\alpha} - e^{\beta H_\alpha}]\}$$

$$= \frac{1}{2\beta} \log \tanh(-\beta H_\alpha)$$

and

$$e^{-\beta D^0(\alpha)} = \exp\{\tfrac{1}{2} \log[e^{-\beta H_\alpha} + e^{\beta H_\alpha}] + \tfrac{1}{2} \log[e^{-\beta H_\alpha} - e^{\beta H_\alpha}]\}$$

$$= 2\sqrt{\sinh(-\beta H_\alpha) \cosh(-\beta H_\alpha)}$$

yielding the usual expressions.

5. π-Degenerate Groups

Although π-degeneracy does not frequently occur in higher spin models – the same cannot be said about spin $\frac{1}{2}$ models, where commonly used boundary conditions sometimes lead to this degeneracy, at least "near the boundary" – we can deal readily with such higher spin groups at the expense of non-linearity in the equations for the energy coefficients.

Given such a group, with

$$Z_{\mathscr{G}}(\beta H) = \sum_{g \in \mathscr{G}} \prod_{\alpha \in \mathscr{I}} e^{-\beta H_\alpha \hat{\alpha}(g)}$$

let $H_\alpha \equiv 0$ for $\alpha \in \mathscr{I}_p - \mathscr{I}$, and let J_α, K_α^j, $\alpha \in \mathscr{I}_p$ and $0 \leqq j < n_\alpha$, be a solution of the equations:

$$\operatorname{sing}_{in/n_\alpha}(-\beta J_\alpha) = \exp\left\{ -\sum_{j=0}^{n_\alpha - 1} \beta K_\alpha^j \theta_{ijn/n_\alpha} \right\}$$

$$\sum_{\substack{\gamma \in \mathscr{I}_p \\ \gamma^j = \alpha}} K_\gamma^j = H_\alpha \qquad \alpha \in \mathscr{I}_p, i \in \mathbb{Z}_{n_\alpha}.$$

Assuming for the moment that such a solution exists,

$$Z_{\mathscr{G}}(\beta H) = \sum_{g\in\mathscr{G}} \prod_{\alpha\in\mathscr{I}_p} e^{-\beta \sum_{\substack{\gamma\in\mathscr{I}_p\\ \gamma^j=\alpha}} K^j_\gamma \hat{\alpha}(g)}$$

$$= \sum_{g\in\mathscr{G}} \prod_{\alpha\in\mathscr{I}_p} \left\{ e^{-\beta \sum_{j=1}^{n_\alpha-1} K^j_\alpha \hat{\alpha}^j(g)} \right\}$$

since

$$\sum_{\alpha\in\mathscr{I}_p} \sum_{\substack{\gamma\in\mathscr{I}_p\\ \gamma^j=\alpha}} K^j_\gamma \hat{\alpha} = \sum_{j=1}^{n_\alpha-1} \sum_{\alpha\in\mathscr{I}_p} K^j_\alpha \hat{\alpha}^j$$

and so

$$\left\{\prod_{\alpha\in\mathscr{I}_p} e^{-\beta K^0_\alpha}\right\} Z_{\mathscr{G}}(\beta H) = \sum_{g\in\mathscr{G}} \prod_{\alpha\in\mathscr{I}_p} \left\{ \sum_{j=0}^{n-1} e^{-\beta J_\alpha \theta_j} \hat{\alpha}^j(g) \right\}$$

$$= \sum_{g\in\mathscr{G}} \prod_{f\in P_n(\mathscr{I}_p)} \prod_{\alpha\in\mathscr{I}'_p} e^{-\beta J_\alpha \theta_{f(\alpha)}} (\hat{\alpha}(g))^{f(\alpha)}$$

$$= N(\mathscr{G}) \sum_{f\in\mathscr{K}^p_\pi} \prod_{\alpha\in\mathscr{I}'_p} e^{-\beta J_\alpha \theta_{f(\alpha)}} .$$

Here $\mathscr{K}^p_\pi$ is the kernel of $\pi: P_n(\mathscr{I}'_p) \to Gp\,\mathscr{I}'_p$ and $\mathscr{I}'_p = \{\alpha\in\mathscr{I}_p \mid J_\alpha \neq 0\}$.

The connection with $\mathscr{G}^*$, constructed from $\mathscr{G}$, $\mathscr{I}'_p$, can now be made. Denoting the range of $\tau: P_n(\mathscr{I}'^*_p) \to Gp\,\mathscr{I}'^*_p$ by $\mathscr{R}^*_\tau$, use the bijection Φ_1 of Section 3, along with the properties of Lemma 3, to obtain:

$$\left(\prod_{\alpha\in\mathscr{I}_p} e^{\beta K^0_\alpha}\right) Z_{\mathscr{G}}(\beta H) = N(\mathscr{G}) \sum_{f\in\mathscr{R}^*_\tau} \prod_{\alpha\in\mathscr{I}'^*_p} \left\{ \prod_{\gamma\in\varphi^{-1}(\alpha)} e^{-\beta J_\gamma \theta_{f(\gamma)}} \right\}$$

$$= N(\mathscr{G}) \sum_{f\in\mathscr{R}^*_\tau} \prod_{\alpha\in\mathscr{I}'^*_p} e^{-\beta I_\alpha \theta_{f(\alpha)}}$$

for $I_\alpha = \sum_{\gamma\in\varphi^{-1}(\alpha)} J_\gamma$.

We point out finally that the system of equations is necessarily solvable if:

$$\operatorname{sing}_{in/n_\alpha}(-\theta J_\alpha) \neq 0, \qquad \alpha\in\mathscr{I}_p,\ i\in\mathbb{Z}_{n_\alpha},$$

and if the non-linear set of equations

$$-\sum_{\substack{\gamma\in\mathscr{I}_p\\ \gamma^k=\alpha}} \frac{1}{n_\gamma\beta} \sum_{m=0}^{n_\gamma-1} \theta_{-mkn/n_\gamma} \log \operatorname{sing}_{mn/n_\gamma}(-\beta J_\gamma) = H_\alpha,\ \alpha\in\mathscr{I}_p$$

is solvable. For the θ_{it} are a complete set of characters for $\mathbb{Z}_n$ and hence

$$\sum_{t=0}^{n_\alpha-1} \beta K^t_\alpha \theta_{itn/n_\alpha} = -\log \operatorname{sing}_{in/n_\alpha}(-\beta J_\alpha)$$

is solvable. We have proved

Theorem 8. *Suppose $\mathscr{G}=P_n(\Lambda)$ with energy function H is given, and let $\mathscr{G}^*$, $\mathscr{J}_p'^*$ be constructed as above. If J_α satisfies*

$$-\sum_{\substack{\gamma\in\mathscr{J}_p\\ \gamma^k=\alpha}}\frac{1}{n_\gamma\beta}\sum_{m=0}^{n_\gamma-1}\theta_{-mkn/n_\gamma}\log \operatorname{sing}_{mn/n_\gamma}(-\beta J_\gamma)=H_\alpha$$

for $\alpha\in\mathscr{J}_p$, then

$$Z_{\mathscr{G}}(\beta H)=\frac{N(\mathscr{G})}{N(\mathscr{K}_\tau^*)}\left\{\prod_{\alpha\in\mathscr{J}_p}e^{\beta K_\alpha^0}\right\}Z_{\mathscr{G}^*}(\beta H^*)$$

where

$$H_\alpha^*=\sum_{\gamma\in\varphi^{-1}(\alpha)}J_\gamma$$

and

$$K_\alpha^0=-\frac{1}{n_\alpha\beta}\log\left\{\prod_{m=0}^{n_\alpha-1}\operatorname{sing}_{mn/n_\alpha}(-\beta J_\alpha)\right\}.$$

6. Square Ising Lattice, Spin One

Consider a lattice gas consisting of a square array of sites and two species of particles. Assume at each lattice site there may be either no particles or one of the two species, and that the only interactions are an energy contribution of $-J$ between either one of the species at a given site and the other species at a nearest neighboring site. This is the simplest non-trivial example of a spin one ($n=3$) lattice in two dimensions.

If χ_i are used to denote the characters of $\mathbb{Z}_3$, $i\in\mathbb{Z}_3$, $\chi_0\equiv 1$, then the elements of $\mathscr{J}$ may be associated with the

$$\hat{\alpha}_{ab}=\chi_1^a\chi_2^b=(\hat{\alpha}_{ba})^2$$

where a, b indicate pairs of nearest neighbor lattice sites,

$$H_\alpha=H_{\alpha^2}=-J,\qquad \alpha\in\mathscr{J}.$$

To construct $\mathscr{G}^*$ a set $\mathscr{H}\subset\mathscr{K}_\pi$ generating $\mathscr{K}_\pi$ must be chosen. For each choice of $\mathscr{H}$ a different dual $\mathscr{G}^*$ will in general be obtained, and, for example, duals with different ground state degeneracies can thus be arranged. One possibility is the following. For each fixed nearest neighboring pair a, b

$$h_{ab}(\alpha_{ab})=h_{ab}(\alpha_{ba})=2,\qquad h_{ab}=0\quad\text{otherwise}.$$

For every unit square a, b, c, d in the lattice,

$$h_{abcd}(\alpha_{ab})=h_{abcd}(\alpha_{bc})=h_{abcd}(\alpha_{cd})=h_{abcd}(\alpha_{da})=1$$
$$h_{abcd}=0\quad\text{otherwise}.$$

The set of h so defined clearly generate $\mathscr{K}_\pi$.

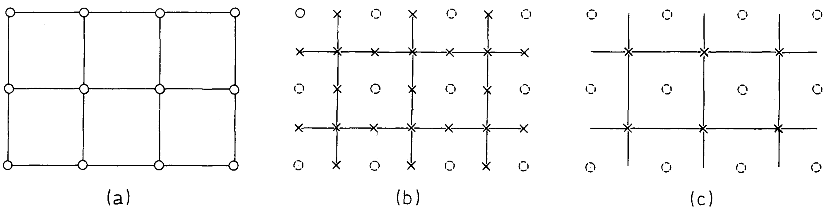

Fig. 1

The resulting dual lattice is illustrated in Fig. 1 (b), with interactions

$$J_i^* = -\frac{1}{3\beta}\log(\mathrm{sing}_1\,\beta J)^{\theta_i}(\mathrm{sing}_2\,\beta J)^{\theta_{2i}}(\mathrm{sing}_0\,\beta J), \qquad i=1 \text{ or } 2 .$$

By taking a partial trace over all the coord. 2 sites in the dual (lattice sites with precisely two nearest neighbors), a square Ising lattice is again recovered, with interactions

$$J' = \frac{1}{3\beta}\log\frac{e^{3\beta J}+2}{e^{3\beta J}-1}\,.$$

In more detail, let $\Lambda^* = \mathscr{H}$ be the set of lattice sites of the new dual lattice, $\Lambda^* = \Lambda_4^* \cup \Lambda_2^*$, Λ_4^* (resp. Λ_2^*) the coord. four (resp. coord. two) sites, and for each $\lambda \in \Lambda_2^*$, let λ^+, λ^- denote its two nearest neighbors. Then:

$$\begin{aligned}
Z^* = &\sum_{g_4 \in P_3(\Lambda_4^*)}\ \sum_{g_2 \in P_3(\Lambda_2^*)} \exp\Big\{\beta \sum_{\lambda \in \Lambda_2^*} (J_1^* \chi_1^{\lambda}(g_2)\, \chi_2^{\lambda^+}(g_4) \\
&+ J_1^* \chi_1^{\lambda}(g_2)\, \chi_2^{\lambda^-}(g_4) + J_2^* \chi_2^{\lambda}(g_2)\, \chi_1^{\lambda^+}(g_4) + J_2^* \chi_2^{\lambda}(g_2)\, \chi_1^{\lambda^-}(g_4))\Big\} \\
= &\sum_{g_4 \in P_3(\Lambda_4^*)}\ \sum_{g_2 \in P_3(\Lambda_2^*)}\ \prod_{\lambda \in \Lambda_2^*} e^{\beta\{J_1^* \theta_{g_2(\lambda)}(\chi_2^{\lambda^+}(g_4)+\chi_2^{\lambda^-}(g_4)) + J_2^* \theta^2_{g_2(\lambda)}(\chi_1^{\lambda^+}(g_4)+\chi_1^{\lambda^-}(g_4))\}} \\
= &\sum_{g_4 \in P_3(\Lambda_4^*)}\ \prod_{\lambda \in \Lambda_2^*}\ \sum_{t=0}^{2} e^{\beta\{J_1^* \theta_t(\chi_2^{\lambda^+}(g_4)+\chi_2^{\lambda^-}(g_4)) + J_2^* \theta_t^2(\chi_1^{\lambda^+}(g_4)+\chi_1^{\lambda^-}(g_4))\}}\,.
\end{aligned}$$

Use

$$\begin{aligned}
&\sum_{t=0}^{2} e^{\beta\{J_1^* \theta_t(\chi_2^{\lambda^+}+\chi_2^{\lambda^-}) + J_2^* \theta_t^2(\chi_1^{\lambda^+}+\chi_1^{\lambda^-})\}} \\
&\quad = 3(e^{2\beta J}+2e^{-\beta J})^{\frac{1}{3}}(e^{2\beta J}-e^{-\beta J})^{\frac{2}{3}}\left(\prod_{m=0}^{2}\mathrm{sing}_m(\beta J)\right)^{-\frac{2}{3}} \\
&\qquad \cdot \exp\left\{\frac{1}{3}\log\frac{e^{2\beta J}+2e^{-\beta J}}{e^{2\beta J}-e^{-\beta J}}\,(\chi_1^{\lambda^+}\chi_2^{\lambda^-}+\chi_2^{\lambda^+}\chi_1^{\lambda^-})\right\}
\end{aligned}$$

to obtain again a square Ising lattice, as in Fig. 1 (c). If the partition function of the square lattice of size $m \times n$ is Z_{mn}, the final result is, except for a boundary term,

$$Z_{mn}(-\beta J) = 3^{-mn}(1 + 2e^{-3\beta J})^{\frac{2}{3}mn}(e^{3\beta J} - 1)^{\frac{4}{3}mn} Z_{mn}(-\beta J').$$

Thus the model is self-dual in the thermodynamic limit, as expected.

This simple model illustrates, we believe, both the strength and the weakness of the proposed duality transformations. The freedom in choosing $\mathscr{H}$ enables the construction of a large class of duals for any given model. The number of lattice sites in the dual lattice will be large, however, when there is no internal symmetry, if the cardinality of $\mathscr{K}_\pi$ is large, as is clear from Lemma 5. This may not necessarily be disadvantageous, and what is more important in any case, the number of interactions in the dual will be no greater than the cardinality of $\mathscr{I}_p^*$.

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