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The Ising Problem

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Abstract. A method to solve Ising problems is developed giving all correlation functions. As an example the one-dimensional nearest and next-nearest neighbour models have been calculated explicitly.

Introduction

One of the basic problems in statistical mechanics is to understand the phenomenon of a phase transition, i.e. why short range interactions can act like long range interactions or to put it otherwise: why is the cluster property destroyed? A quite logical approach to this problem is to solve models, which exhibit a phase transition. The conventional way is to calculate the partition function for a finite system, from which all thermodynamical properties are derived. The phase transition only occurs after taking the thermodynamic limit, because all thermodynamic functions are analytic in the temperature for a finite system [1], whereas the definition of a phase transition is that they should contain singular points. If one wants a better understanding of the nature of the phase transition it is necessary to know more about the equilibrium state of the system (i.e. all correlation functions) than only its thermodynamics. This knowledge can not be obtained from the partition function.

Up to now there is, except for the free gas, no complete description of any model in statistical mechanics. In the following we will present a method to calculate all equilibrium correlation functions of the simplest models in statistical mechanics, namely the one-dimensional Ising models with finite range interactions. The results for the nearest-neighbour and the next nearest-neighbour model are given. This technique is based on the topological and algebraic structure of the system. The system is taken to be infinite right from the beginning and its equilibrium state is obtained from the K.M.S. boundary condition [2]. We expect this method to be independent of the dimension of the lattice; it gives a finite closed set of algebraic equations for all finite-dimensional Ising systems with finite range interactions with or without magnetic field. For the two-dimensional nearest neighbour model, however, the number of equations is of the order 100.

The reason for discussing the one-dimensional next nearest neighbour model is the fact that its algebraic structure is identical to that of the two-dimensional nearest neighbour case.

Section I

The Hamiltonian of the n-Dimensional Ising Model Reads

$$H = -\tfrac{1}{2}\sum_{i,a} v_{i,i+a}\sigma_i^z\sigma_{i+a}^z - h\sum_i \sigma_i^z . \quad (1.1)$$

σ_i^z are variables assuming the values ± 1. The summation indices $\boldsymbol{i}$ and $\boldsymbol{a}$ run through the whole n-dimensional lattice; h denotes the applied magnetic field. The interaction strength $v_{i,i+a}$ satisfies the following conditions

1) $v_{i,i}=0$,
2) $v_{i,i+a}=v_{i+a,i}$,
3) $\sum_a v_{i,i+a}=V<\infty$, V independent of $\boldsymbol{i}$.

Changing to the lattice-gas representation

$$\sigma_i^z = 2(\tfrac{1}{2} - c_i^+ c_i) \quad (1.2)$$

the following equivalent Hamiltonian is obtained

$$H = 2(V+h)\sum_i n_i - 2\sum_{i,a} v_{i,i+a} n_i n_{i+a} - (\tfrac{1}{2} V + h)\sum_i , \quad (1.3)$$

where $n_i = c_i^+ c_i$. The operators c_i^+ and c_i are the well-known paulion operators satisfying

$$[c_i^+, c_i^+]_+ = [c_i, c_i]_+ = 0,\ [c_i, c_i^+]_+ = 1 ,$$
$$[c_i, c_j^+]_- = [c_i, c_j]_- = [c_i^+ c_j^+]_- = 0, \quad i \neq j .$$

The internal energy per particle u is given by

$$u = 2(V+h)\,\omega(n_i) - 2\sum_a v_{i,i+a}\,\omega(n_i n_{i+a}) - (\tfrac{1}{2} V + h) , \quad (1.4)$$

where the thermodynamic limit state ω is assumed to be invariant for space translations. The advantages of the lattice-gas representation over the original one are

i) all correlation functions are positive,

ii) a time translation is defined, i.e. the K.M.S. boundary condition can be used [3].

The time dependent operator $c_i^+(t)$ is defined as

$$\begin{aligned} c_i^+(t) &= e^{itH} c_i^+ e^{-itH} = \sum_{h=0}^{\infty} \frac{(it)^h}{h!} [\mathrm{ad}\, H]^h c_i^+ \\ &= \exp\left[it\left(2V + 2h - 4\sum_{\boldsymbol{a}} v_{i,i+\boldsymbol{a}} n_{i+\boldsymbol{a}}\right)\right] c_i^+ \end{aligned} \tag{1.5}$$

with $[\mathrm{ad}\, H]\, x = [H, x]_-$.

Using the K.M.S. boundary condition

$$\omega(A(t - i\beta)\, B) = \omega(B A(t))$$

with $A = c_j^+$ and $B = c_j N$, where $N = N^+$ and $[N, c_j^+] = [N, n_{j+\boldsymbol{a}}] = 0$ for all $\boldsymbol{a} \neq 0$, it follows that

$$\boxed{e^{2\beta(V+h)}\, \omega\left(\exp\left[-4\beta \sum_{\boldsymbol{a}} v_{j,j+\boldsymbol{a}}\, n_{j+\boldsymbol{a}}\right] n_j N\right) = \omega((1 - n_j)\, N)\,.} \tag{1.6}$$

Putting $N = \prod_{k=1}^{n} n_{i_k}(i_k \neq j)$ all correlation functions can be obtained. These correlation functions are coupled to each other through an infinite number of Eqs. (1.6). The problem consists, as in Green function theory, of solving this infinite hierarchy.

In Section II this is done for the one-dimensional nearest neighbour model, while the one-dimensional next nearest neighbour model is treated in Section III.

Section II

The One-dimensional Ising Model with Nearest Neighbour Interaction

In the case of the one-dimensional Ising model with nearest neighbour interaction we have

$$v_{i,i+a} = J(\delta_{a,1} + \delta_{a,-1})\,. \tag{2.1}$$

Eq. (1.6) reads now

$$e^{2\beta(2J+h)}\, \omega(\exp\left[-4\beta J(n_{j+1} + n_{j-1})\right] n_j N) = \omega((1 - n_j)\, N)\,. \tag{2.2}$$

Defining $H = e^{2\beta h}$, $X = e^{-4\beta J}$, $Y = X - 1$ and using the identity

$$e^{x n_j} = 1 + n_j(e^x - 1)\,,$$

Eq. (2.2) can be rewritten as

$$\begin{aligned} &\left(\frac{H}{X} + 1\right) \omega(n_j N) + \frac{HY}{X} \left[\omega(n_{j-1} n_j N) + \omega(n_j n_{j+1} N)\right] \\ &\qquad + \frac{HY^2}{X}\, \omega(n_{j-1} n_j n_{j+1} N) = \omega(N)\,. \end{aligned} \tag{2.3}$$

Substitution of $N=n_{j-1}n_{j+1}N'$, $N=n_{j-1}N'$, $N=n_{j+1}N'$, $N=N'$ results respectively in the following four equations

$$\omega(n_{j-1}n_{j+1}N')=(HX+1)\,\omega(n_{j-1}n_jn_{j+1}N')\,, \tag{2.4a}$$

$$\omega(n_{j-1}N')=(H+1)\,\omega(n_{j-1}n_jN')+HY\omega(n_{j-1}n_jn_{j+1}N')\,, \tag{2.4b}$$

$$\omega(n_{j+1}N')=(H+1)\,\omega(n_jn_{j+1}N')+HY\omega(n_{j-1}n_jn_{j+1}N')\,, \tag{2.4c}$$

$$\begin{aligned}\omega(N')=&\left(\frac{H}{X}+1\right)\omega(n_jN')+\frac{HY}{X}\,\omega(n_{j-1}n_jN')\\ &+\frac{HY}{X}\,\omega(n_jn_{j+1}N')+\frac{HY^2}{X}\,\omega(n_{j-1}n_jn_{j+1}N')\,.\end{aligned} \tag{2.4d}$$

Define

$$\phi_j(i_1\,\ldots\,i_k)=\begin{bmatrix}\omega(n_{j-1}n_jn_{j+1}N') & \omega(n_{j-1}n_{j+1}N')\\ \omega(n_{j-1}n_jN') & \omega(n_{j-1}N')\\ \omega(n_jn_{j+1}N') & \omega(n_{j+1}N')\\ \omega(n_jN') & \omega(N')\end{bmatrix}$$

with $N'=\prod_{l=1}^{k} n_{i_l}(i_l\neq j)$ and ϕ_j as the same vector with $N'=1$.

Define next

$$\psi(i_1,\ldots,i_k)=(\phi_0(2,i_1,\ldots,i_k),\,\phi_0(i_1,\ldots,i_k))$$

$$\equiv\begin{bmatrix}\psi_{(1,1)}(i_1,\ldots,i_k) & \psi_{(1,5)}(i_1,\ldots,i_k) & \psi_{(2,1)}(i_1,\ldots,i_k) & \psi_{(2,5)}(i_1,\ldots,i_k)\\ \psi_{(1,2)}(i_1,\ldots,i_k) & \psi_{(1,6)}(i_1,\ldots,i_k) & \psi_{(2,2)}(i_1,\ldots,i_k) & \psi_{(2,6)}(i_1,\ldots,i_k)\\ \psi_{(1,3)}(i_1,\ldots,i_k) & \psi_{(1,7)}(i_1,\ldots,i_k) & \psi_{(2,3)}(i_1,\ldots,i_k) & \psi_{(2,7)}(i_1,\ldots,i_k)\\ \psi_{(1,4)}(i_1,\ldots,i_k) & \psi_{(1,8)}(i_1,\ldots,i_k) & \psi_{(2,4)}(i_1,\ldots,i_k) & \psi_{(2,8)}(i_1,\ldots,i_k)\end{bmatrix}$$

$$=\begin{bmatrix}\omega(n_{-1}n_0n_1n_2N') & \omega(n_{-1}n_1n_2N') & \omega(n_{-1}n_0n_1N') & \omega(n_{-1}n_1N')\\ \omega(n_{-1}n_0n_2N') & \omega(n_{-1}n_2N') & \omega(n_{-1}n_0N') & \omega(n_{-1}N')\\ \omega(n_0n_1n_2N') & \omega(n_1n_2N') & \omega(n_0n_1N') & \omega(n_1N')\\ \omega(n_0n_2N') & \omega(n_2N') & \omega(n_0N') & \omega(N')\end{bmatrix}$$

with $N'=\prod_{l=1}^{k} n_{i_l},\quad i_l\neq 0,1,2\,.$

and

$$\psi=(\phi_0(2),\,\phi_0)\,.$$

As can easily be seen the following properties, which are independent of N', now hold:

1) ϕ_j and $\phi_j(\mathrm{i}_1, \ldots, \mathrm{i}_k)$ satisfy Eqs. (2.4)

2) $$\begin{bmatrix} \psi_{(1,1)}(i_1, \ldots, i_k) & \psi_{(1,2)}(i_1, \ldots, i_k) \\ \psi_{(2,1)}(i_1, \ldots, i_k) & \psi_{(2,2)}(i_1, \ldots, i_k) \\ \psi_{(1,5)}(i_1, \ldots, i_k) & \psi_{(1,6)}(i_1, \ldots, i_k) \\ \psi_{(2,5)}(i_1, \ldots, i_k) & \psi_{(2,6)}(i_1, \ldots, i_k) \end{bmatrix} = \phi_1(-1, i_1, \ldots, i_k)\,. \tag{2.5}$$

3) $$\begin{bmatrix} \psi_{(1,3)}(i_1, \ldots, i_k) & \psi_{(1,4)}(i_1, \ldots, i_k) \\ \psi_{(2,3)}(i_1, \ldots, i_k) & \psi_{(2,4)}(i_1, \ldots, i_k) \\ \psi_{(1,7)}(i_1, \ldots, i_k) & \psi_{(1,8)}(i_1, \ldots, i_k) \\ \psi_{(2,7)}(i_1, \ldots, i_k) & \psi_{(2,8)}(i_1, \ldots, i_k) \end{bmatrix} = \phi_1(i_1, \ldots, i_k)\,. \tag{2.6}$$

The Properties 2) and 3) are a consequence of the topology of the lattice. This topology was not present in ϕ_j and $\phi_j(i_1, \ldots, i_k)$.

Because of the spatial invariance of ω, i.e.

$$\phi_1(i_1, \ldots, i_k) = \phi_0(i_1 - 1, \ldots, i_k - 1)\,,$$

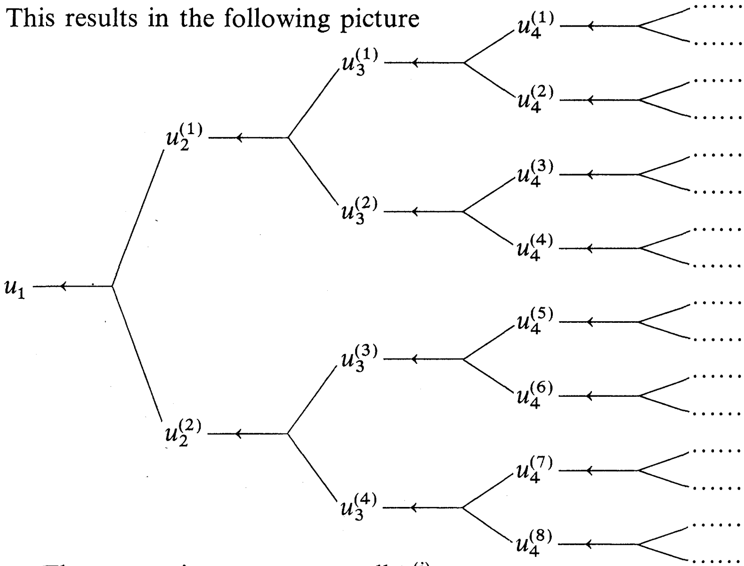
Property 3) implies that $\phi_0(i_1 - 1, \ldots, i_k - 1)$ is contained in $\psi(i_1, \ldots, i_k)$. Consequently $\psi(i_1, \ldots, i_k)$ is contained in $\psi(3, i_1 + 1, \ldots, i_k + 1)$ and $\psi(i_1 + 1, \ldots, i_k + 1)$, independent of $i_1, \ldots, i_k$, which can be shown schematically in the following way

$$\psi(i_1, \ldots, i_k) \longleftarrow \begin{cases} \psi(3, i_1 + 1, \ldots, i_k + 1) \\ \psi(i_1 + 1, \ldots, i_k + 1) \end{cases}$$

Define

$$\begin{array}{llll} u_1 = \psi, & u_2^{(1)} = \psi(3), & u_3^{(1)} = \psi(3, 4), & u_4^{(1)} = \psi(3, 4, 5), \text{ and so on.} \\ & u_2^{(2)} = \psi, & u_3^{(2)} = \psi(4), & u_4^{(2)} = \psi(4, 5), \\ & & u_3^{(3)} = \psi(3), & u_4^{(3)} = \psi(3, 5), \\ & & u_3^{(4)} = \psi, & u_4^{(4)} = \psi(5), \\ & & & u_4^{(5)} = \psi(3, 4), \\ & & & u_4^{(6)} = \psi(4), \\ & & & u_4^{(7)} = \psi(3), \\ & & & u_4^{(8)} = \psi\,. \end{array}$$

This results in the following picture



(2.7)

The properties common to all $u_k^{(i)}$ are

1. Each $u_k^{(i)}$ is contained in two others.

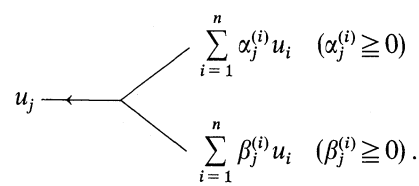
2. Eqs. (2,4), (2.5) and (2.6) imply that only four of the sixteen components of $u_k^{(i)}$ are independent. The independent ones will be taken to be $u_k^{(i)}(1,1)$, $u_k^{(i)}(1,3)$, $u_k^{(i)}(2,1)$ and $u_k^{(i)}(2,3)$ (the notation for the components of $u_k^{(i)}$ is identical to that for $\psi(i_1, \ldots, i_k)$).

3. The components of all $u_k^{(i)}$ should be positive and smaller than one, because they represent expectation values of positive operators with norm 1.

Suppose the solution of scheme (2.7) to be $u_1 = v_1, u_2^{(1)} = v_2, u_2^{(2)} = v_3, \ldots$. Taking only the above mentioned properties into account and noticing the equivalence of all "vertices", other solutions can be constructed with $u_1 = v_2$, $u_1 = v_3$, ... and so on. Because of the linearity of the system (2.7) $\sum_j c_j v_j (c_j \geqq 0)$ is also a solution. This indicates that all solutions for $u_k^{(i)}$ should belong to a class U of the following type

$$U = \left\{ u \,|\, u = \sum_{j=1}^{n} \alpha_j u_j,\ \alpha_j \geqq 0,\ u_j \text{ independent} \right\}.$$

From the above it is obvious that each u_j obeys a relation



(2.8)

There are n independent u_j's, i.e. $4n$ unkowns (Property 2), and $2n^2$ unknowns $\alpha_j^{(i)}$ and $\beta_j^{(i)}$. Eq. (2.8) yields for each j eight equations (in principle sixteen, but due to (2.4) and (2.6) these are dependent). Equating the number of unknowns and the number of equations gives $2n^2+4n = 8n \Rightarrow n=2$. This means that the class U is a two parameter family of solutions. The elements of this class are described by $u(\boldsymbol{p})$, with $\boldsymbol{p}=(p_1, p_2)$, where p_1 and p_2 denote the parameters. In this notation scheme (2.8) reads

$$u(\boldsymbol{p}) \longleftarrow \begin{cases} u(\boldsymbol{p}') \\ u(\boldsymbol{p}'') \end{cases} \quad \forall \boldsymbol{p}, \text{ where } \boldsymbol{p}' \text{ and } \boldsymbol{p}'' \text{ depend on } \boldsymbol{p}. \tag{2.9}$$

Using twice Eq. (2.6) scheme (2.9) is equivalent to

$$\begin{aligned} u_{(1,1)}(\boldsymbol{p}) &= u_{(1,3)}(\boldsymbol{p}') & u_{(2,1)}(\boldsymbol{p}) &= u_{(1,3)}(\boldsymbol{p}'') \\ u_{(1,2)}(\boldsymbol{p}) &= u_{(2,3)}(\boldsymbol{p}') & u_{(2,2)}(\boldsymbol{p}) &= u_{(2,3)}(\boldsymbol{p}'') \\ u_{(1,3)}(\boldsymbol{p}) &= u_{(1,7)}(\boldsymbol{p}') & u_{(2,3)}(\boldsymbol{p}) &= u_{(1,7)}(\boldsymbol{p}'') \\ u_{(1,4)}(\boldsymbol{p}) &= u_{(2,7)}(\boldsymbol{p}') & u_{(2,4)}(\boldsymbol{p}) &= u_{(2,7)}(\boldsymbol{p}''). \end{aligned} \tag{2.10}$$

In order to rewrite Eqs. (2.10) in terms of the real unknowns $u_{(1,1)}(\boldsymbol{p})$, $u_{(1,3)}(\boldsymbol{p})$, $u_{(2,1)}(\boldsymbol{p})$ and $u_{(2,3)}(\boldsymbol{p})$ Eqs. (2.5) and (2.6) have to be used. It easily follows

$$u_{(1,1)}(\boldsymbol{p}) = u_{(1,3)}(\boldsymbol{p}'), \tag{2.11a}$$

$$(HX+1)\,u_{(1,1)}(\boldsymbol{p}) = u_{(2,3)}(\boldsymbol{p}'), \tag{2.11b}$$

$$u_{(1,3)}(\boldsymbol{p}) = (H+1)\,u_{(1,3)}(\boldsymbol{p}') + HY u_{(1,1)}(\boldsymbol{p}'), \tag{2.11c}$$

$$(HX+1)\,u_{(1,3)}(\boldsymbol{p}) = (H+1)\,u_{(2,3)}(\boldsymbol{p}') + HY u_{(2,1)}(\boldsymbol{p}'), \tag{2.11d}$$

$$u_{(2,1)}(\boldsymbol{p}) = u_{(1,3)}(\boldsymbol{p}''), \tag{2.11e}$$

$$(H+1)\,u_{(2,1)}(\boldsymbol{p}) + HY u_{(1,1)}(\boldsymbol{p}) = u_{(2,3)}(\boldsymbol{p}''), \tag{2.11f}$$

$$u_{(2,3)}(\boldsymbol{p}) = (H+1)\,u_{(1,3)}(\boldsymbol{p}'') + HY u_{(1,1)}(\boldsymbol{p}''), \tag{2.11g}$$

$$(H+1)\,u_{(2,3)}(\boldsymbol{p}) + HY u_{(1,3)}(\boldsymbol{p}) = (H+1)\,u_{(2,3)}(\boldsymbol{p}'') + HY u_{(2,1)}(\boldsymbol{p}''). \tag{2.11h}$$

Define

$$u'(\boldsymbol{p}) = \begin{bmatrix} u_{(1,1)}(\boldsymbol{p}) & u_{(2,1)}(\boldsymbol{p}) \\ u_{(1,3)}(\boldsymbol{p}) & u_{(2,3)}(\boldsymbol{p}) \end{bmatrix}.$$

Only the real unknowns appear in $u'(\boldsymbol{p})$. It follows directly from the Eqs. (2.11) that

$$u'(\boldsymbol{p}') = \begin{bmatrix} c_1 & (HX+1)\,c_1 \\ c_2 & (HX+1)\,c_2 \end{bmatrix}, \tag{2.12a}$$

$$u'(\boldsymbol{p}'') = \begin{bmatrix} c_3 & (H+1)\,c_3 + HYc_1 \\ c_4 & (H+1)\,c_4 + HYc_2 \end{bmatrix}, \tag{2.12b}$$

$$u'(\boldsymbol{p}) = \begin{bmatrix} c_2 & c_4 \\ (H+1)\,c_2 + HYc_1 & (H+1)\,c_4 + HYc_3 \end{bmatrix}, \tag{2.12c}$$

where c_1, c_2, c_3 and c_4 are arbitrary. We select as parameter c_2 and c_4, i.e.

$$\begin{aligned} c_1 &= c_1(p_1, p_2) = \alpha_1^{(1)} p_1 + \alpha_1^{(2)} p_2 , \\ c_2 &= p_1 , \\ c_3 &= c_3(p_1, p_2) = \alpha_2^{(1)} p_1 + \alpha_2^{(2)} p_2 , \\ c_4 &= p_2 , \end{aligned}$$

where the linearity of the functions c_1 and c_3 is implied by the form of the class U. The $\alpha_j^{(i)}$ have to be determined by the requirement that $u'(\boldsymbol{p}')$, $u'(\boldsymbol{p}'')$ and $u'(\boldsymbol{p})$ all belong to the same class U, i.e.

$$\forall \boldsymbol{p}', \exists \boldsymbol{p} \text{ such that } u'(\boldsymbol{p}') = u'(\boldsymbol{p})$$

and

$$\forall \boldsymbol{p}'', \exists \boldsymbol{p} \text{ such that } u'(\boldsymbol{p}'') = u'(\boldsymbol{p}) .$$

After some trivial calculations we find

$$u'(\boldsymbol{p}) = \begin{bmatrix} p_1 & p_2 \\ \dfrac{p_1}{\alpha_\pm} & \dfrac{p_2}{\alpha_\pm} \end{bmatrix} \tag{2.13}$$

where

$$\alpha_\pm = \frac{-(H+1) \pm [(H+1)^2 + 4HY]^{\frac{1}{2}}}{2HY} .$$

Substitution of (2.13) into scheme (2.9) enables us to express $\boldsymbol{p}'$ and $\boldsymbol{p}''$ in terms of $\boldsymbol{p}$;

$$\boldsymbol{p}' = L_1 \boldsymbol{p} , \tag{2.14a}$$

$$\boldsymbol{p}'' = L_2 \boldsymbol{p} . \tag{2.14b}$$

The matrices L_1 and L_2 are given by

$$L_1 = \alpha_\pm \begin{pmatrix} 1 & 0 \\ (HX+1) & 0 \end{pmatrix} , \tag{2.15a}$$

$$L_2 = \alpha_\pm \begin{pmatrix} 0 & 1 \\ HY & H+1 \end{pmatrix} . \tag{2.15b}$$

Up till now scheme (2.7) has been solved as far as the common properties of $u_k^{(i)}$ are concerned. To determine the state completely the specific properties of $u_k^{(i)}$ must be taken into account. Once we know u_1 all $u_k^{(i)}$ follow from repeated applications of L_1 and L_2 to the $\boldsymbol{p}$ of u_1. Denoting $u_k^{(i)}$ by $u(\boldsymbol{p}_k^{(i)})$, the vector u_1 is obtained by observing that $u_2^{(2)} = u_1$, i.e. $L_2 \boldsymbol{p}_1 = \boldsymbol{p}_1$, and $u_1(2, 8) = 1$. One finds

$$\boldsymbol{p}_1 = \alpha_\pm^2 \, y_\pm \begin{pmatrix} \alpha_\pm \\ 1 \end{pmatrix} \tag{2.16}$$

where

$$y_{\pm} = \left[2 + \frac{(H-1)}{X}(Y\alpha_{\pm} + 1)\right]^{-1}.$$

The correlation functions of u_1 are given by

$$\omega(n_{j-1}n_j n_{j+1} n_{j+2}) = \alpha_{\pm}^3 y_{\pm}, \quad (2.17\text{a})$$

$$\omega(n_{j-1}n_j n_{j+2}) = (HX+1)\,\alpha_{\pm}^3 y_{\pm}, \quad (2.17\text{b})$$

$$\omega(n_j n_{j+1} n_{j+2}) = \alpha_{\pm}^2 y_{\pm}, \quad (2.17\text{c})$$

$$\omega(n_j n_{j+2}) = (HX+1)\,\alpha_{\pm}^2 y_{\pm}, \quad (2.17\text{d})$$

$$\omega(n_{j-1} n_{j+2}) = ((H+1)^2 X - Y)\,\alpha_{\pm}^3 y_{\pm}, \quad (2.17\text{e})$$

$$\omega(n_j n_{j+1}) = \alpha_{\pm} y_{\pm}, \quad (2.17\text{f})$$

$$\omega(n_j) = y_{\pm}. \quad (2.17\text{g})$$

Both for ferromagnetic ($J>0$) and antiferromagnetic ($J<0$) interactions α_+ has to be taken. This is due to the fact that

1. $J>0$ results in $\alpha_- > 1$. This is not allowed because of e.g. $\omega(n_j n_{j+1}) \leqq \omega(n_j)$.

2. $J<0$ results in $\alpha_- < 0$. This is not allowed because of the positivity of the correlation functions.

This concludes the calculation of the correlation functions.

The knowledge of the correlation functions $\omega(n_j)$ and $\omega(n_j n_{j+1})$ suffices to determine the thermodynamic properties of the system (formula (1.4)). After some calculations the usual results are obtained.

The Ground State ($\beta \to \infty$)

(1) $J>0$; $h>0$; all correlation functions (2.17) are zero ($\alpha_+ = 0$, $y_+ = 0$).

(2) $J>0$; $h<0$; all correlation functions (2.17) are one ($\alpha_+ = 1$, $y_+ = 1$).

(3) $J<0$; $h>0$;
(4) $J<0$; $h<0$; } $\omega(n_j) = \frac{1}{2}$, $\omega(n_j n_{j+2}) = \frac{1}{2}$, all other correlation functions of (2.17) are zero (The state is spatially invariant!).

Taking $h=0$ and afterwards $\beta \to \infty$ yields.

(1) $J>0$, all correlation functions have the value $\frac{1}{2}$.

(2) $J<0$, $\omega(n_j) = \frac{1}{2}$, $\omega(n_j n_{j+2}) = \frac{1}{2}$, all other correlation functions of (2.17) are zero.

The Cluster Property

The question whether the system clusters or not is answered by investigating

$$\lim_{n\to\infty} u_{n+2}^{(2^n)} .$$

Because the system does not exhibit a phase transition the cluster property should hold, i.e.

$$\lim_{n\to\infty} u_{n+2}^{(2^n)} = \omega(n_j)\, u_1 . \tag{2.18}$$

Proof.

$$\lim_{n\to\infty} u_{n+2}^{(2^n)} = \lim_{n\to\infty} L_2^n\, p_2^{(1)} = \lim_{n\to\infty} L_2^n\, L_1\, p_1 .$$

According to Bellman [4]

$$\lim_{n\to\infty} \lambda^{-n} L^n \chi = \frac{(\chi, \psi_0')}{(\psi_0, \psi_0')}\, \psi_0 ,$$

where ψ_0 and ψ_0' are given by

$$L\psi_0 = \lambda\psi_0 ,$$

$$L^T \psi_0' = \lambda\psi_0', \quad \text{where } L^T \text{ is the transposed of } L$$

and λ is the maximal eigenvalue of L. In our case $\lambda = 1$;

$$\psi_0 = \begin{pmatrix} \alpha_+ \\ 1 \end{pmatrix}; \quad \psi_0' = \begin{pmatrix} \alpha_+ H Y \\ 1 \end{pmatrix}; \quad \chi = L_1 p_1 = \alpha_+^4\, y_+ \begin{pmatrix} 1 \\ HX + 1 \end{pmatrix}.$$

Substitution results in

$$\lim_{n\to\infty} u_{n+2}^{(2^n)} = \alpha_+^2\, y_+^2 \begin{pmatrix} \alpha_+ \\ 1 \end{pmatrix} = y_+\, p_1 = \omega(n_j)\, u_1 . \qquad \text{Q.E.D.}$$

Section III

The One-dimensional Ising Model
with Next Nearest Neighbour Interaction

In this case we have

$$v_{i,i+a} = J_1(\delta_{a,1} + \delta_{a,-1}) + J_2(\delta_{a,2} + \delta_{a,-2}) . \tag{3.1}$$

Substituting (3.1) into (1.6) yields

$$\begin{aligned} e^{2\beta(2J_1 + 2J_2 + h)}\, \omega(\exp[-4\beta(J_1 n_{j+1} + J_1 n_{j-1} + J_2 n_{j+2} + J_2 n_{j-2})]\, n_j N) \\ = \omega((1 - n_j)\, N) . \end{aligned} \tag{3.2}$$

Defining $H = e^{2\beta h}$; $X_1 = e^{-4\beta J_1}$; $X_2 = e^{-4\beta J_2}$; $Y_1 = X_1 - 1$; $Y_2 = X_2 - 1$ Eq. (3.2) passes into

$$
\begin{aligned}
\omega(N) = {} & \left(\frac{H}{X_1 X_2} + 1\right)\omega(n_j N) + \frac{H Y_2}{X_1 X_2}\omega(n_{j-2} n_j N) + \frac{H Y_1}{X_1 X_2}\omega(n_{j-1} n_j N) \\
& + \frac{H Y_1}{X_1 X_2}\omega(n_j n_{j+1} N) + \frac{H Y_2}{X_1 X_2}\omega(n_j n_{j+2} N) \\
& + \frac{H Y_1 Y_2}{X_1 X_2}\omega(n_{j-2} n_{j-1} n_j N) + \frac{H Y_1 Y_2}{X_1 X_2}\omega(n_{j-2} n_j n_{j+1} N) \\
& + \frac{H Y_2^2}{X_1 X_2}\omega(n_{j-2} n_j n_{j+2} N) + \frac{H Y_1^2}{X_1 X_2}\omega(n_{j-1} n_j n_{j+1} N) \\
& + \frac{H Y_1 Y_2}{X_1 X_2}\omega(n_{j-1} n_j n_{j+2} N) + \frac{H Y_1 Y_2}{X_1 X_2}\omega(n_j n_{j+1} n_{j+2} N) \\
& + \frac{H Y_1^2 Y_2}{X_1 X_2}\omega(n_{j-2} n_{j-1} n_j n_{j+1} N) + \frac{H Y_1 Y_2^2}{X_1 X_2}\omega(n_{j-2} n_{j-1} n_j n_{j+2} N) \\
& + \frac{H Y_1 Y_2^2}{X_1 X_2}\omega(n_{j-2} n_j n_{j+1} n_{j+2} N) + \frac{H Y_1^2 Y_2}{X_1 X_2}\omega(n_{j-1} n_j n_{j+1} n_{j+2} N) \\
& + \frac{H Y_1^2 Y_2^2}{X_1 X_2}\omega(n_{j-2} n_{j-1} n_j n_{j+1} n_{j+2} N) .
\end{aligned} \tag{3.3}
$$

To generate a scheme analogous to the one previously discussed we take respectively

$$
\begin{aligned}
& N = n_{j-2} n_{j-1} n_{j+1} n_{j+2} N'; \quad N = n_{j-2} n_{j-1} n_{j+1} N'; \quad N = n_{j-2} n_{j-1} n_{j+2} N'; \\
& N = n_{j-2} n_{j+1} n_{j+2} N'; \quad N = n_{j-1} n_{j+1} n_{j+2} N'; \quad N = n_{j-2} n_{j-1} N'; \\
& N = n_{j-2} n_{j+1} N'; \quad N = n_{j-2} n_{j+2} N'; \quad N = n_{j-1} n_{j+1} N'; \\
& N = n_{j-1} n_{j+2} N'; \quad N = n_{j+1} n_{j+2} N'; \quad N = n_{j-2} N'; \\
& N = n_{j-1} N'; \quad N = n_{j+1} N'; \quad N = n_{j+2} N'; \quad N = N',
\end{aligned}
$$

which results into the following sixteen equations

$$
\omega(n_{j-2} n_{j-1} n_{j+1} n_{j+2} N') = (H X_1 X_2 + 1)\,\omega(n_{j-2} n_{j-1} n_j n_{j+1} n_{j+2} N'), \tag{3.4a}
$$

$$
\begin{aligned}
\omega(n_{j-2} n_{j-1} n_{j+1} N') = {} & (H X_1 + 1)\,\omega(n_{j-2} n_{j-1} n_j n_{j+1} N') \\
& + H X_1 Y_2\,\omega(n_{j-2} n_{j-1} n_j n_{j+1} n_{j+2} N'),
\end{aligned} \tag{3.4b}
$$

$$
\begin{aligned}
\omega(n_{j-2} n_{j-1} n_{j+2} N') = {} & (H X_2 + 1)\,\omega(n_{j-2} n_{j-1} n_j n_{j+2} N') \\
& + H X_2 Y_1\,\omega(n_{j-2} n_{j-1} n_j n_{j+1} n_{j+2} N'),
\end{aligned} \tag{3.4c}
$$

$$\omega(n_{j-2}n_{j+1}n_{j+2}N') = (HX_2+1)\,\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') + HX_2Y_1\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'), \tag{3.4d}$$

$$\omega(n_{j-1}n_{j+1}n_{j+2}N') = (HX_1+1)\,\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') + HX_1Y_2\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'), \tag{3.4e}$$

$$\omega(n_{j-2}n_{j-1}N') = (H+1)\,\omega(n_{j-2}n_{j-1}n_jN') + HY_2\,\omega(n_{j-2}n_{j-1}n_jn_{j+2}N') + HY_1\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}N') + HY_1Y_2\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'), \tag{3.4f}$$

$$\omega(n_{j-2}n_{j+1}N') = (H+1)\,\omega(n_{j-2}n_jn_{j+1}N') + HY_2\,\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') + HY_1\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}N') + HY_1Y_2\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'), \tag{3.4g}$$

$$\begin{aligned}\omega(n_{j-2}n_{j+2}N') &= \left(\frac{HX_2}{X_1}+1\right)\omega(n_{j-2}n_jn_{j+2}N') + \frac{HX_2Y_1}{X_1}\,\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') \\ &\quad + \frac{HX_2Y_1}{X_1}\,\omega(n_{j-2}n_{j-1}n_jn_{j+2}N') \\ &\quad + \frac{HX_2Y_1^2}{X_1}\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'),\end{aligned} \tag{3.4h}$$

$$\begin{aligned}\omega(n_{j-1}n_{j+1}N') &= \left(\frac{HX_1}{X_2}+1\right)\omega(n_{j-1}n_jn_{j+1}N') + \frac{HX_1Y_2}{X_2}\,\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') \\ &\quad + \frac{HX_1Y_2}{X_2}\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}N') \\ &\quad + \frac{HX_1Y_2^2}{X_2}\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'),\end{aligned} \tag{3.4i}$$

$$\omega(n_{j-1}n_{j+2}N') = (H+1)\,\omega(n_{j-1}n_jn_{j+2}N') + HY_1\,\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') + HY_2\,\omega(n_{j-2}n_{j-1}n_jn_{j+2}N') + HY_1Y_2\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'), \tag{3.4j}$$

$$\omega(n_{j+1}n_{j+2}N') = (H+1)\,\omega(n_jn_{j+1}n_{j+2}N') + HY_1\,\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') + HY_2\,\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') + HY_1Y_2\,\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'), \tag{3.4k}$$

$$\begin{aligned}\omega(n_{j-2}N') &= \left(\frac{H}{X_1}+1\right)\omega(n_{j-2}n_jN') + \frac{HY_2}{X_1}\omega(n_{j-2}n_jn_{j+2}N') \\ &+ \frac{HY_1}{X_1}\omega(n_{j-2}n_jn_{j+1}N') + \frac{HY_1}{X_1}\omega(n_{j-2}n_{j-1}n_jN') \\ &+ \frac{HY_1Y_2}{X_1}\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') + \frac{HY_1Y_2}{X_1}\omega(n_{j-2}n_{j-1}n_jn_{j+2}N') \\ &+ \frac{HY_1^2}{X_1}\omega(n_{j-2}n_{j-1}n_jn_{j+1}N') + \frac{HY_1^2Y_2}{X_1}\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'),\end{aligned} \tag{3.4l}$$

$$\begin{aligned}\omega(n_{j-1}N') &= \left(\frac{H}{X_2}+1\right)\omega(n_{j-1}n_jN') + \frac{HY_2}{X_2}\omega(n_{j-1}n_jn_{j+2}N') \\ &+ \frac{HY_1}{X_2}\omega(n_{j-1}n_jn_{j+1}N') + \frac{HY_2}{X_2}\omega(n_{j-2}n_{j-1}n_jN') \\ &+ \frac{HY_1Y_2}{X_2}\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') + \frac{HY_2^2}{X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+2}N') \\ &+ \frac{HY_1Y_2}{X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+1}N') + \frac{HY_1Y_2^2}{X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'),\end{aligned} \tag{3.4m}$$

$$\begin{aligned}\omega(n_{j+1}N') &= \left(\frac{H}{X_2}+1\right)\omega(n_jn_{j+1}N') + \frac{HY_2}{X_2}\omega(n_jn_{j+1}n_{j+2}N') \\ &+ \frac{HY_1}{X_2}\omega(n_{j-1}n_jn_{j+1}N') + \frac{HY_2}{X_2}\omega(n_{j-2}n_jn_{j+1}N') \\ &+ \frac{HY_1Y_2}{X_2}\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') + \frac{HY_2^2}{X_2}\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') \\ &+ \frac{HY_1Y_2}{X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+1}N') + \frac{HY_1Y_2^2}{X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'),\end{aligned} \tag{3.4n}$$

$$\begin{aligned}\omega(n_{j+2}N') &= \left(\frac{H}{X_1}+1\right)\omega(n_jn_{j+2}N') + \frac{HY_1}{X_1}\omega(n_jn_{j+1}n_{j+2}N') \\ &+ \frac{HY_1}{X_1}\omega(n_{j-1}n_jn_{j+2}N') + \frac{HY_2}{X_1}\omega(n_{j-2}n_jn_{j+2}N') \\ &+ \frac{HY_1^2}{X_1}\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') + \frac{HY_1Y_2}{X_1}\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') \\ &+ \frac{HY_1Y_2}{X_1}\omega(n_{j-2}n_{j-1}n_jn_{j+2}N') + \frac{HY_1^2Y_2}{X_1}\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N'),\end{aligned} \tag{3.4o}$$

$$\begin{aligned}\omega(N') = {} & \left(\frac{H}{X_1X_2}+1\right)\omega(n_jN') + \frac{HY_2}{X_1X_2}\omega(n_jn_{j+2}N') + \frac{HY_1}{X_1X_2}\omega(n_jn_{j+1}N') \\ & + \frac{HY_1}{X_1X_2}\omega(n_{j-1}n_jN') + \frac{HY_2}{X_1X_2}\omega(n_{j-2}n_jN') + \frac{HY_1Y_2}{X_1X_2}\omega(n_jn_{j+1}n_{j+2}N') \\ & + \frac{HY_1Y_2}{X_1X_2}\omega(n_{j-1}n_jn_{j+2}N') + \frac{HY_1^2}{X_1X_2}\omega(n_{j-1}n_jn_{j+1}N') \\ & + \frac{HY_2^2}{X_1X_2}\omega(n_{j-2}n_jn_{j+2}N') + \frac{HY_1Y_2}{X_1X_2}\omega(n_{j-2}n_jn_{j+1}N') \qquad (3.4\text{p}) \\ & + \frac{HY_1Y_2}{X_1X_2}\omega(n_{j-2}n_{j-1}n_jN') + \frac{HY_1^2Y_2}{X_1X_2}\omega(n_{j-1}n_jn_{j+1}n_{j+2}N') \\ & + \frac{HY_1Y_2^2}{X_1X_2}\omega(n_{j-2}n_jn_{j+1}n_{j+2}N') + \frac{HY_1Y_2^2}{X_1X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+2}N') \\ & + \frac{HY_1^2Y_2}{X_1X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+1}N') + \frac{HY_1^2Y_2^2}{X_1X_2}\omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N').\end{aligned}$$

$$\phi_j(i_1,\ldots,i_k) = \begin{bmatrix} \omega(n_{j-2}n_{j-1}n_jn_{j+1}n_{j+2}N') & \omega(n_{j-2}n_{j-1}n_{j+1}n_{j+2}N') \\ \omega(n_{j-2}n_{j-1}n_jn_{j+1}N') & \omega(n_{j-2}n_{j-1}n_{j+1}N') \\ \omega(n_{j-2}n_{j-1}n_jn_{j+2}N') & \omega(n_{j-2}n_{j-1}n_{j+2}N') \\ \omega(n_{j-2}n_jn_{j+1}n_{j+2}N') & \omega(n_{j-2}n_{j+1}n_{j+2}N') \\ \omega(n_{j-1}n_jn_{j+1}n_{j+2}N') & \omega(n_{j-1}n_{j+1}n_{j+2}N') \\ \omega(n_{j-2}n_{j-1}n_jN') & \omega(n_{j-2}n_{j-1}N') \\ \omega(n_{j-2}n_jn_{j+1}N') & \omega(n_{j-2}n_{j+1}N') \\ \omega(n_{j-2}n_jn_{j+2}N') & \omega(n_{j-2}n_{j+2}N') \\ \omega(n_{j-1}n_jn_{j+1}N') & \omega(n_{j-1}n_{j+1}N') \\ \omega(n_{j-1}n_jn_{j+2}N') & \omega(n_{j-1}n_{j+2}N') \\ \omega(n_jn_{j+1}n_{j+2}N') & \omega(n_{j+1}n_{j+2}N') \\ \omega(n_{j-2}n_jN') & \omega(n_{j-2}N') \\ \omega(n_{j-1}n_jN') & \omega(n_{j-1}N') \\ \omega(n_jn_{j+1}N') & \omega(n_{j+1}N') \\ \omega(n_jn_{j+2}N') & \omega(n_{j+2}N') \\ \omega(n_jN') & \omega(N') \end{bmatrix}$$

with $N' = \prod_{l=1}^{k} n_{i_l}\ (i_l \neq j)$; ϕ_j is defined as the above mentioned expression with $N' = 1$.

Analogous to the nearest neighbour case we define

$$\psi(i_1 \ldots i_k) = (\phi_0(3, i_1 \ldots i_k), \phi_0(i_1 \ldots i_k)), \qquad i_l \neq 0, 1, 2, 3,$$

where the components of $\psi(i_1 \ldots i_k)$ are given by

$$\psi_{(1,n)}(i_1 \ldots i_k) = \phi_0(3, i_1 \ldots i_k)(n)$$
$$\psi_{(2,n)}(i_1 \ldots i_k) = \phi_0(i_1 \ldots i_k)(n)$$

with n denoting the n^{th} component of $\phi_0(\ldots, i_1 \ldots i_k)$, e.g.

$$\psi_{(1,17)}(i_1 \ldots i_k) = \omega(n_{-2} n_{-1} n_1 n_2 n_3 N').$$

Again $\psi = (\phi_0(3), \phi_0)$.

Consider the following subsets of $\psi(i_1 \ldots i_k)$

$$\chi^{(p,q,r,s)}_{(t,u,v,w)}(i_1 \ldots i_k) = \begin{bmatrix} \psi_{(1,p)}(i_1 \ldots i_k) & \psi_{(1,t)}(i_1 \ldots i_k) \\ \psi_{(2,p)}(i_1 \ldots i_k) & \psi_{(2,t)}(i_1 \ldots i_k) \\ \psi_{(1,q)}(i_1 \ldots i_k) & \psi_{(1,u)}(i_1 \ldots i_k) \\ \psi_{(1,p+16)}(i_1 \ldots i_k) & \psi_{(1,t+16)}(i_1 \ldots i_k) \\ \psi_{(1,r)}(i_1 \ldots i_k) & \psi_{(1,v)}(i_1 \ldots i_k) \\ \psi_{(2,q)}(i_1 \ldots i_k) & \psi_{(2,u)}(i_1 \ldots i_k) \\ \psi_{(2,p+16)}(i_1 \ldots i_k) & \psi_{(2,t+16)}(i_1 \ldots i_k) \\ \psi_{(1,q+16)}(i_1 \ldots i_k) & \psi_{(1,u+16)}(i_1 \ldots i_k) \\ \psi_{(2,r)}(i_1 \ldots i_k) & \psi_{(2,v)}(i_1 \ldots i_k) \\ \psi_{(1,s)}(i_1 \ldots i_k) & \psi_{(1,w)}(i_1 \ldots i_k) \\ \psi_{(1,r+16)}(i_1 \ldots i_k) & \psi_{(1,v+16)}(i_1 \ldots i_k) \\ \psi_{(2,q+16)}(i_1 \ldots i_k) & \psi_{(2,u+16)}(i_1 \ldots i_k) \\ \psi_{(2,s)}(i_1 \ldots i_k) & \psi_{(2,w)}(i_1 \ldots i_k) \\ \psi_{(2,r+16)}(i_1 \ldots i_k) & \psi_{(2,v+16)}(i_1 \ldots i_k) \\ \psi_{(1,s+16)}(i_1 \ldots i_k) & \psi_{(1,w+16)}(i_1 \ldots i_k) \\ \psi_{(2,s+16)}(i_1 \ldots i_k) & \psi_{(2,w+16)}(i_1 \ldots i_k) \end{bmatrix}, \qquad (3.5)$$

Translating properties (2.5) and (2.6) of the nearest neighbour model we find

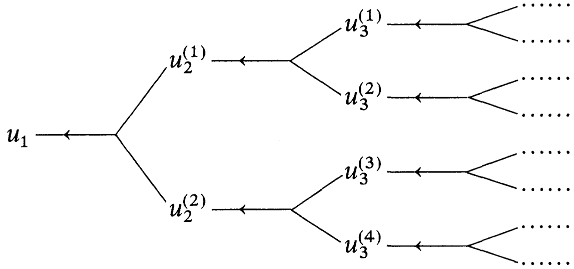
$$\chi^{(1,2,4,7)}_{(3,6,8,12)}(i_1 \ldots i_k) = \phi_1(-2, i_1 \ldots i_k), \qquad (3.6\text{a})$$

$$\chi^{(5,9,11,14)}_{(10,13,15,16)}(i_1 \ldots i_k) = \phi_1(i_1 \ldots i_k). \qquad (3.6\text{b})$$

Again the topology of the lattice is responsible for these properties.
Define

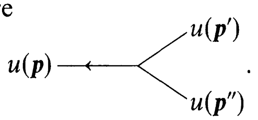
$$
\begin{array}{llll}
u_1=\psi\,; & u_2^{(1)}=\psi(4)\,; & u_3^{(1)}=\psi(4,5)\,; & u_4^{(1)}=\psi(4,5,6)\,; \quad \text{and so on}\\
 & u_2^{(2)}=\psi\,; & u_3^{(2)}=\psi(5)\,; & u_4^{(2)}=\psi(5,6)\,;\\
 & & u_3^{(3)}=\psi(4)\,; & u_4^{(3)}=\psi(4,6)\,;\\
 & & u_3^{(4)}=\psi\,; & u_4^{(4)}=\psi(6)\,;\\
 & & & u_4^{(5)}=\psi(4,5)\,;\\
 & & & u_4^{(6)}=\psi(5)\,;\\
 & & & u_4^{(7)}=\psi(4)\,;\\
 & & & u_4^{(8)}=\psi\,.
\end{array}
$$

If we use Eq. (3.6b) then, because of the assumed spatial invariance of the state ω, i.e. $\phi_1(i_1 \ldots i_k)=\phi_0(i_1-1,\ldots,i_k-1)$, the following scheme results again.



(3.7)

Repeating the arguments used in the nearest neighbour model we find that the solution class U is an eight parameter family denoted by $u(\boldsymbol{p})$. To solve is therefore



. (3.7a)

Because of the Eqs. (3.4) and (3.6) not all components of $u(\boldsymbol{p})$ are independent. The independent ones are chosen to be

$$
\begin{aligned}
&u_{(i,1)}(\boldsymbol{p})\,;\quad u_{(i,2)}(\boldsymbol{p})\,;\quad u_{(i,4)}(\boldsymbol{p})\,;\quad u_{(i,5)}(\boldsymbol{p})\,;\quad u_{(i,7)}(\boldsymbol{p})\,;\\
&u_{(i,9)}(\boldsymbol{p})\,;\quad u_{(i,11)}(\boldsymbol{p})\,;\quad u_{(i,14)}(\boldsymbol{p}) \qquad (i=1,2)\,.
\end{aligned}
$$

In terms of $u_{(i,j)}(\boldsymbol{p})$ Eqs. (3.4) read

$$u_{(i,17)}=(HX_1X_2+1)\,u_{(i,1)}\,, \tag{3.8a}$$

$$u_{(i,18)}=(HX_1+1)\,u_{(i,2)}+HX_1Y_2u_{(i,1)}\,, \tag{3.8b}$$

$$u_{(i,19)}=(HX_2+1)\,u_{(i,3)}+HX_2Y_1u_{(i,1)}\,, \tag{3.8c}$$

$$u_{(i,20)}=(HX_2+1)\,u_{(i,4)}+HX_2Y_1u_{(i,1)}, \tag{3.8d}$$

$$u_{(i,21)}=(HX_1+1)\,u_{(i,5)}+HX_1Y_2u_{(i,1)}, \tag{3.8e}$$

$$u_{(i,22)}=(H+1)\,u_{(i,6)}+HY_2u_{(i,3)}+HY_1u_{(i,2)}+HY_1Y_2u_{(i,1)}, \tag{3.8f}$$

$$u_{(i,23)}=(H+1)\,u_{(i,7)}+HY_2u_{(i,4)}+HY_1u_{(i,2)}+HY_1Y_2u_{(i,1)}, \tag{3.8g}$$

$$\begin{aligned}u_{(i,24)}=&\left(\frac{HX_2}{X_1}+1\right)u_{(i,8)}+\frac{HX_2Y_1}{X_1}u_{(i,4)}+\frac{HX_2Y_1}{X_1}u_{(i,3)}\\&+\frac{HX_2Y_1^2}{X_1}u_{(i,1)},\end{aligned} \tag{3.8h}$$

$$\begin{aligned}u_{(i,25)}=&\left(\frac{HX_1}{X_2}+1\right)u_{(i,9)}+\frac{HX_1Y_2}{X_2}u_{(i,5)}+\frac{HX_1Y_2}{X_2}u_{(i,2)}\\&+\frac{HX_1Y_2^2}{X_2}u_{(i,1)},\end{aligned} \tag{3.8i}$$

$$u_{(i,26)}=(H+1)\,u_{(i,10)}+HY_1u_{(i,5)}+HY_2u_{(i,3)}+HY_1Y_2u_{(i,1)}, \tag{3.8j}$$

$$u_{(i,27)}=(H+1)\,u_{(i,11)}+HY_1u_{(i,5)}+HY_2u_{(i,4)}+HY_1Y_2u_{(i,1)}, \tag{3.8k}$$

$$\begin{aligned}u_{(i,28)}=&\left(\frac{H}{X_1}+1\right)u_{(i,12)}+\frac{HY_2}{X_1}u_{(i,8)}+\frac{HY_1}{X_1}u_{(i,7)}+\frac{HY_1}{X_1}u_{(i,6)}\\&+\frac{HY_1Y_2}{X_1}u_{(i,4)}+\frac{HY_1Y_2}{X_1}u_{(i,3)}+\frac{HY_1^2}{X_1}u_{(i,2)}+\frac{HY_1^2Y_2}{X_1}u_{(i,1)},\end{aligned} \tag{3.8l}$$

$$\begin{aligned}u_{(i,29)}=&\left(\frac{H}{X_2}+1\right)u_{(i,13)}+\frac{HY_2}{X_2}u_{(i,10)}+\frac{HY_1}{X_2}u_{(i,9)}+\frac{HY_2}{X_2}u_{(i,6)}\\&+\frac{HY_1Y_2}{X_2}u_{(i,5)}+\frac{HY_2^2}{X_2}u_{(i,3)}+\frac{HY_1Y_2}{X_2}u_{(i,2)}+\frac{HY_1Y_2^2}{X_2}u_{(i,1)},\end{aligned} \tag{3.8m}$$

$$\begin{aligned}u_{(i,30)}=&\left(\frac{H}{X_2}+1\right)u_{(i,14)}+\frac{HY_2}{X_2}u_{(i,11)}+\frac{HY_1}{X_2}u_{(i,9)}+\frac{HY_2}{X_2}u_{(i,7)}\\&+\frac{HY_1Y_2}{X_2}u_{(i,5)}+\frac{HY_2^2}{X_2}u_{(i,4)}+\frac{HY_1Y_2}{X_2}u_{(i,2)}+\frac{HY_1Y_2^2}{X_2}u_{(i,1)},\end{aligned} \tag{3.8n}$$

$$\begin{aligned}u_{(i,31)}=&\left(\frac{H}{X_1}+1\right)u_{(i,15)}+\frac{HY_1}{X_1}u_{(i,11)}+\frac{HY_1}{X_1}u_{(i,10)}+\frac{HY_2}{X_1}u_{(i,8)}\\&+\frac{HY_1^2}{X_1}u_{(i,5)}+\frac{HY_1Y_2}{X_1}u_{(i,4)}+\frac{HY_1Y_2}{X_1}u_{(i,3)}+\frac{HY_1^2Y_2}{X_1}u_{(i,1)},\end{aligned} \tag{3.8o}$$

$$\begin{aligned} u_{(i,32)} = & \left(\frac{H}{X_1 X_2} + 1\right) u_{(i,16)} + \frac{H Y_2}{X_1 X_2} u_{(i,15)} + \frac{H Y_1}{X_1 X_2} u_{(i,14)} \\ & + \frac{H Y_1}{X_1 X_2} u_{(i,13)} + \frac{H Y_2}{X_1 X_2} u_{(i,12)} + \frac{H Y_1 Y_2}{X_1 X_2} u_{(i,11)} + \frac{H Y_1 Y_2}{X_1 X_2} u_{(i,10)} \\ & + \frac{H Y_1^2}{X_1 X_2} u_{(i,9)} + \frac{H Y_2^2}{X_1 X_2} u_{(i,8)} + \frac{H Y_1 Y_2}{X_1 X_2} u_{(i,7)} + \frac{H Y_1 Y_2}{X_1 X_2} u_{(i,6)} \\ & + \frac{H Y_1^2 Y_2}{X_1 X_2} u_{(i,5)} + \frac{H Y_1 Y_2^2}{X_1 X_2} u_{(i,4)} + \frac{H Y_1 Y_2^2}{X_1 X_2} u_{(i,3)} \\ & + \frac{H Y_1^2 Y_2}{X_1 X_2} u_{(i,2)} + \frac{H Y_1^2 Y_2^2}{X_1 X_2} u_{(i,1)}, \end{aligned} \tag{3.8p}$$

where by $u_{(i,j)}$ is meant $u_{(i,j)}(\boldsymbol{p})$. These relations hold for all $\boldsymbol{p}$ and $i = 1, 2$. The Eqs. (3.6) together with (3.4) yield the remaining ones needed to express all $u_{(i,j)}(\boldsymbol{p})$ in the real unknowns $u_{(k,l)}(\boldsymbol{p})$ $(k = 1, 2; l = 1, 2, 4, 5, 7, 9, 11, 14)$.

These equations are

$$u_{(1,3)} = (H X_1 X_2 + 1)\, u_{(1,1)}, \tag{3.9a}$$

$$u_{(1,6)} = (H X_2 + 1)\, u_{(1,2)} + H X_2 Y_1 u_{(1,1)}, \tag{3.9b}$$

$$u_{(1,8)} = (H X_1 + 1)\, u_{(1,4)} + H X_1 Y_2 u_{(1,1)}, \tag{3.9c}$$

$$u_{(1,10)} = (H X_1 X_2 + 1)\, u_{(1,5)}, \tag{3.9d}$$

$$u_{(1,12)} = (H + 1)\, u_{(1,7)} + H Y_1 u_{(1,4)} + H Y_2 u_{(1,2)} + H Y_1 Y_2 u_{(1,1)}, \tag{3.9e}$$

$$u_{(1,13)} = (H X_2 + 1)\, u_{(1,9)} + H X_2 Y_1 u_{(1,5)}, \tag{3.9f}$$

$$u_{(1,15)} = (H X_1 + 1)\, u_{(1,11)} + H X_1 Y_2 u_{(1,5)}, \tag{3.9g}$$

$$u_{(1,16)} = (H + 1)\, u_{(1,14)} + H Y_1 u_{(1,11)} + H Y_2 u_{(1,9)} + H Y_1 Y_2 u_{(1,5)}, \tag{3.9h}$$

$$u_{(2,3)} = (H X_1 + 1)\, u_{(2,1)} + H X_1 Y_2 u_{(1,1)}, \tag{3.9i}$$

$$u_{(2,6)} = (H + 1)\, u_{(2,2)} + H Y_1 u_{(2,1)} + H Y_2 u_{(1,2)} + H Y_1 Y_2 u_{(1,1)}, \tag{3.9j}$$

$$\begin{aligned} u_{(2,8)} = & \left(\frac{H X_1}{X_2} + 1\right) u_{(2,4)} + \frac{H X_1 Y_2}{X_2} u_{(2,1)} + \frac{H X_1 Y_2}{X_2} u_{(1,4)} \\ & + \frac{H X_1 Y_2^2}{X_2} u_{(1,1)}, \end{aligned} \tag{3.9k}$$

$$u_{(2,10)} = (H X_1 + 1)\, u_{(2,5)} + H X_1 Y_2 u_{(1,5)}, \tag{3.9l}$$

$$u_{(2,12)} = \left(\frac{H}{X_2} + 1\right) u_{(2,7)} + \frac{H Y_1}{X_2} u_{(2,4)} + \frac{H Y_2}{X_2} u_{(2,2)} + \frac{H Y_1 Y_2}{X_2} u_{(2,1)} + \frac{H Y_2}{X_2} u_{(1,7)} + \frac{H Y_1 Y_2}{X_2} u_{(1,4)} + \frac{H Y_2^2}{X_2} u_{(1,2)} + \frac{H Y_1 Y_2^2}{X_2} u_{(1,1)}, \tag{3.9m}$$

$$u_{(2,13)} = (H+1) u_{(2,9)} + H Y_1 u_{(2,5)} + H Y_2 u_{(1,9)} + H Y_1 Y_2 u_{(1,5)}, \tag{3.9n}$$

$$u_{(2,15)} = \left(\frac{H X_1}{X_2} + 1\right) u_{(2,11)} + \frac{H X_1 Y_2}{X_2} u_{(2,5)} + \frac{H X_1 Y_2}{X_2} u_{(1,11)} + \frac{H X_1 Y_2^2}{X_2} u_{(1,5)}, \tag{3.9o}$$

$$u_{(2,16)} = \left(\frac{H}{X_2} + 1\right) u_{(2,14)} + \frac{H Y_1}{X_2} u_{(2,11)} + \frac{H Y_2}{X_2} u_{(2,9)} + \frac{H Y_1 Y_2}{X_2} u_{(2,5)} + \frac{H Y_2}{X_2} u_{(1,14)} + \frac{H Y_1 Y_2}{X_2} u_{(1,11)} + \frac{H Y_2^2}{X_2} u_{(1,9)} + \frac{H Y_1 Y_2^2}{X_2} u_{(1,5)}. \tag{3.9p}$$

Using Eq. (3.6b) scheme (3.7a) is equivalent to

$$\begin{aligned}
&u_{(1,1)}(p) = u_{(1,5)}(p'), && u_{(1,9)}(p) = u_{(2,11)}(p'),\\
&u_{(1,2)}(p) = u_{(2,5)}(p'), && u_{(1,10)}(p) = u_{(1,14)}(p'),\\
&u_{(1,3)}(p) = u_{(1,9)}(p'), && u_{(1,11)}(p) = u_{(1,27)}(p'),\\
&u_{(1,4)}(p) = u_{(1,21)}(p'), && u_{(1,12)}(p) = u_{(2,25)}(p'),\\
&u_{(1,5)}(p) = u_{(1,11)}(p'), && u_{(1,13)}(p) = u_{(2,14)}(p'),\\
&u_{(1,6)}(p) = u_{(2,9)}(p'), && u_{(1,14)}(p) = u_{(2,27)}(p'),\\
&u_{(1,7)}(p) = u_{(2,21)}(p'), && u_{(1,15)}(p) = u_{(1,30)}(p'),\\
&u_{(1,8)}(p) = u_{(1,25)}(p'), && u_{(1,16)}(p) = u_{(2,30)}(p'),
\end{aligned}$$

$$\begin{aligned}
&u_{(2,1)}(p) = u_{(1,5}(p''), && u_{(2,9)}(p) = u_{(2,11)}(p''),\\
&u_{(2,2)}(p) = u_{(2,5)}(p''), && u_{(2,10)}(p) = u_{(1,14)}(p''),\\
&u_{(2,3)}(p) = u_{(1,9)}(p''), && u_{(2,11)}(p) = u_{(1,27)}(p''),\\
&u_{(2,4)}(p) = u_{(1,21)}(p''), && u_{(2,12)}(p) = u_{(2,25)}(p''),\\
&u_{(2,5)}(p) = u_{(1,11)}(p''), && u_{(2,13)}(p) = u_{(2,14)}(p''),\\
&u_{(2,6)}(p) = u_{(2,9)}(p''), && u_{(2,14)}(p) = u_{(2,27)}(p''),\\
&u_{(2,7)}(p) = u_{(2,21)}(p''), && u_{(2,15)}(p) = u_{(1,30)}(p''),\\
&u_{(2,8)}(p) = u_{(1,25)}(p''), && u_{(2,16)}(p) = u_{(2,30)}(p'').
\end{aligned}$$

Substituting (3.8) and (3.9) into these equations we obtain them in terms of the "real" unknowns:

$$u_{(1,1)}(p)=u_{(1,5)}(p'),$$

$$u_{(1,2)}(p)=u_{(2,5)}(p'),$$

$$(HX_1X_2+1)\,u_{(1,1)}(p)=u_{(1,9)}(p'),$$

$$u_{(1,4)}(p)=(HX_1+1)\,u_{(1,5)}(p')+HX_1\,Y_2\,u_{(1,1)}(p'),$$

$$u_{(1,5)}(p)=u_{(1,11)}(p'),$$

$$(HX_2+1)\,u_{(1,2)}(p)+HX_2\,Y_1\,u_{(1,1)}(p)=u_{(2,9)}(p'),$$

$$u_{(1,7)}(p)=(HX_1+1)\,u_{(2,5)}(p')+HX_1\,Y_2\,u_{(2,1)}(p'),$$

$$(HX_1+1)\,u_{(1,4)}(p)+HX_1\,Y_2\,u_{(1,1)}(p)=\left(\frac{HX_1}{X_2}+1\right)u_{(1,9)}(p')$$

$$+\frac{HX_1\,Y_2}{X_2}\,u_{(1,5)}(p')+\frac{HX_1\,Y_2}{X_2}\,u_{(1,2)}(p')+\frac{HX_1\,Y_2^2}{X_2}\,u_{(1,1)}(p'),$$

$$u_{(1,9)}(p)=u_{(2,11)}(p'),$$

$$(HX_1X_2+1)\,u_{(1,5)}(p)=u_{(1,14)}(p'),$$

$$u_{(1,11)}(p)=(H+1)\,u_{(1,11)}(p')+H\,Y_1\,u_{(1,5)}(p')+H\,Y_2\,u_{(1,4)}(p')$$
$$+H\,Y_1\,Y_2\,u_{(1,1)}(p'),$$

$$(H+1)\,u_{(1,7)}(p)+H\,Y_1\,u_{(1,4)}(p)+H\,Y_2\,u_{(1,2)}(p)+H\,Y_1\,Y_2\,u_{(1,1)}(p)$$
$$=\left(\frac{HX_1}{X_1}+1\right)u_{(2,9)}(p')+\frac{HX_1\,Y_2}{X_2}\,u_{(2,5)}(p')+\frac{HX_1\,Y_2}{X_2}\,u_{(2,2)}(p')$$
$$+\frac{HX_1\,Y_2^2}{X_2}\,u_{(2,1)}(p'),$$

$$(HX_2+1)\,u_{(1,9)}(p)+HX_2\,Y_1\,u_{(1,5)}(p)=u_{(2,14)}(p'),$$

$$u_{(1,14)}(p)=(H+1)\,u_{(2,11)}(p')+H\,Y_1\,u_{(2,5)}(p')+H\,Y_2\,u_{(2,4)}(p')$$
$$+H\,Y_1\,Y_2\,u_{(2,1)}(p'),$$

$$(HX_1+1)\,u_{(1,11)}(p)+HX_1\,Y_2\,u_{(1,5)}(p)=\left(\frac{H}{X_2}+1\right)u_{(1,14)}(p')$$

$$+\frac{H\,Y_2}{X_2}\,u_{(1,11)}(p')+\frac{H\,Y_1}{X_2}\,u_{(1,9)}(p')+\frac{H\,Y_2}{X_2}\,u_{(1,7)}(p')+\frac{H\,Y_1\,Y_2}{X_2}\,u_{(1,5)}(p')$$

$$+\frac{H\,Y_2^2}{X_2}\,u_{(1,4)}(p')+\frac{H\,Y_1\,Y_2}{X_2}\,u_{(1,2)}(p')+\frac{H\,Y_1\,Y_2^2}{X_2}\,u_{(1,1)}(p'),$$

$$
\begin{aligned}
&(H+1)u_{(1,14)}(p)+HY_1u_{(1,11)}(p)+HY_2u_{(1,9)}(p)+HY_1Y_2u_{(1,5)}(p)\\
&\quad=\left(\frac{H}{X_2}+1\right)u_{(2,14)}(p')+\frac{HY_2}{X_2}u_{(2,11)}(p')+\frac{HY_1}{X_2}u_{(2,9)}(p')\\
&\quad+\frac{HY_2}{X_2}u_{(2,7)}(p')+\frac{HY_1Y_2}{X_2}u_{(2,5)}(p')+\frac{HY_2^2}{X_2}u_{(2,4)}(p')\\
&\quad+\frac{HY_1Y_2}{X_2}u_{(2,2)}(p')+\frac{HY_1Y_2^2}{X_2}u_{(2,1)}(p')
\end{aligned}
\tag{3.10}
$$

and

$$u_{(2,1)}(p)=u_{(1,5)}(p''),$$

$$u_{(2,2)}(p)=u_{(2,5)}(p''),$$

$$(HX_1+1)u_{(2,1)}(p)+HX_1Y_2u_{(1,1)}(p)=u_{(1,9)}(p''),$$

$$u_{(2,4)}(p)=(HX_1+1)u_{(1,5)}(p'')+HX_1Y_2u_{(1,1)}(p''),$$

$$u_{(2,5)}(p)=u_{(1,11)}(p''),$$

$$(H+1)u_{(2,2)}(p)+HY_1u_{(2,1)}(p)+HY_2u_{(1,2)}(p)+HY_1Y_2u_{(1,1)}(p)=u_{(2,9)}(p''),$$

$$u_{(2,7)}(p)=(HX_1+1)u_{(2,5)}(p'')+HX_1Y_2u_{(2,1)}(p''),$$

$$
\begin{aligned}
&\left(\frac{HX_1}{X_2}+1\right)u_{(2,4)}(p)+\frac{HX_1Y_2}{X_2}u_{(2,1)}(p)+\frac{HX_1Y_2}{X_2}u_{(1,4)}(p)\\
&\quad+\frac{HX_1Y_2^2}{X_2}u_{(1,1)}(p)=\left(\frac{HX_1}{X_2}+1\right)u_{(1,9)}(p'')+\frac{HX_1Y_2}{X_2}u_{(1,5)}(p'')\\
&\quad+\frac{HX_1Y_2}{X_2}u_{(1,2)}(p'')+\frac{HX_1Y_2^2}{X_2}u_{(1,1)}(p''),
\end{aligned}
$$

$$u_{(2,9)}(p)=u_{(2,11)}(p''),$$

$$(HX_1+1)u_{(2,5)}(p)+HX_1Y_2u_{(1,5)}(p)=u_{(1,14)}(p''),$$

$$
\begin{aligned}
u_{(2,11)}(p)=(H+1)u_{(1,11)}(p'')+HY_1u_{(1,5)}(p'')+HY_2u_{(1,4)}(p'')\\
+HY_1Y_2u_{(1,1)}(p''),
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{H}{X_2}+1\right)u_{(2,7)}(p)+\frac{HY_1}{X_2}u_{(2,4)}(p)+\frac{HY_2}{X_2}u_{(2,2)}(p)+\frac{HY_1Y_2}{X_2}u_{(2,1)}(p)\\
&\quad+\frac{HY_2}{X_2}u_{(1,7)}(p)+\frac{HY_1Y_2}{X_2}u_{(1,4)}(p)+\frac{HY_2^2}{X_2}u_{(1,2)}(p)\\
&\quad+\frac{HY_1Y_2^2}{X_2}u_{(1,1)}(p)=\left(\frac{HX_1}{X_2}+1\right)u_{(2,9)}(p'')+\frac{HX_1Y_2}{X_2}u_{(2,5)}(p'')\\
&\quad+\frac{HX_1Y_2}{X_2}u_{(2,2)}(p'')+\frac{HX_1Y_2^2}{X_2}u_{(2,1)}(p''),
\end{aligned}
$$

$$(H+1)u_{(2,9)}(p)+HY_1u_{(2,5)}(p)+HY_2u_{(1,9)}(p)$$
$$+HY_1Y_2u_{(1,5)}(p)=u_{(2,14)}(p''),$$
$$u_{(2,14)}(p)=(H+1)u_{(2,11)}(p'')+HY_1u_{(2,5)}(p'')+HY_2u_{(2,4)}(p'')$$
$$+HY_1Y_2u_{(2,1)}(p''),$$

$$\left(\frac{HX_1}{X_2}+1\right)u_{(2,11)}(p)+\frac{HX_1Y_2}{X_2}u_{(2,5)}(p)+\frac{HX_1Y_2}{X_2}u_{(1,11)}(p)$$
$$+\frac{HX_1Y_2^2}{X_2}u_{(1,5)}(p)=\left(\frac{H}{X_2}+1\right)u_{(1,14)}(p'')+\frac{HY_2}{X_2}u_{(1,11)}(p'')$$
$$+\frac{HY_1}{X_2}u_{(1,9)}(p'')+\frac{HY_2}{X_2}u_{(1,7)}(p'')+\frac{HY_1Y_2}{X_2}u_{(1,5)}(p'')$$
$$+\frac{HY_2^2}{X_2}u_{(1,4)}(p'')+\frac{HY_1Y_2}{X_2}u_{(1,2)}(p'')+\frac{HY_1Y_2^2}{X_2}u_{(1,1)}(p''),$$

$$\left(\frac{H}{X_2}+1\right)u_{(2,14)}(p)+\frac{HY_1}{X_2}u_{(2,11)}(p)+\frac{HY_2}{X_2}u_{(2,9)}(p)+\frac{HY_1Y_2}{X_2}u_{(2,5)}(p)$$
$$+\frac{HY_2}{X_2}u_{(1,14)}(p)+\frac{HY_1Y_2}{X_2}u_{(1,11)}(p)+\frac{HY_2^2}{X_2}u_{(1,9)}(p)$$
$$+\frac{HY_1Y_2^2}{X_2}u_{(1,5)}(p)=\left(\frac{H}{X_2}+1\right)u_{(2,14)}(p'')+\frac{HY_2}{X_2}u_{(2,11)}(p'') \quad (3.11)$$
$$+\frac{HY_1}{X_2}u_{(2,9)}(p'')+\frac{HY_2}{X_2}u_{(2,7)}(p'')+\frac{HY_1Y_2}{X_2}u_{(2,5)}(p'')$$
$$+\frac{HY_2^2}{X_2}u_{(2,4)}(p'')+\frac{HY_1Y_2}{X_2}u_{(2,2)}(p'')+\frac{HY_1Y_2^2}{X_2}u_{(2,1)}(p'').$$

Because we only want to consider the "real" unknowns we define

$$u'(p)=\begin{bmatrix} u_{(1,1)}(p) & u_{(2,1)}(p) \\ u_{(1,2)}(p) & u_{(2,2)}(p) \\ u_{(1,4)}(p) & u_{(2,4)}(p) \\ u_{(1,5)}(p) & u_{(2,5)}(p) \\ u_{(1,7)}(p) & u_{(2,7)}(p) \\ u_{(1,9)}(p) & u_{(2,9)}(p) \\ u_{(1,11)}(p) & u_{(2,11)}(p) \\ u_{(1,14)}(p) & u_{(2,14)}(p) \end{bmatrix}.$$

Eliminating $u'(p)$ from Eqs. (3.10) we get

$$u'(p') = \begin{bmatrix} c_1^{(1)} & c_5^{(1)} \\ (HX_1X_2+1)c_1^{(1)} & (HX_2+1)c_5^{(1)} + HX_2Y_1c_1^{(1)} \\ c_2^{(1)} & c_6^{(1)} \\ c_3^{(1)} & c_7^{(1)} \\ (HX_1X_2+1)c_2^{(1)} & (HX_2+1)c_6^{(1)} + HX_2Y_1c_2^{(1)} \\ (HX_1X_2+1)c_3^{(1)} & (HX_2+1)c_7^{(1)} + HX_2Y_1c_3^{(1)} \\ c_4^{(1)} & c_8^{(1)} \\ (HX_1X_2+1)c_4^{(1)} & (HX_2+1)c_8^{(1)} + HX_2Y_1c_4^{(1)} \end{bmatrix} \tag{3.12}$$

where $c_i^{(1)}$ $(i=1,\ldots,8)$ is still arbitrary. Repeating the procedure for (3.11) in combination with (3.10) gives

$$u'(p'') = \begin{bmatrix} c_1 & c_5 \\ f(c_1, c_1^{(1)}) & g(c_5, c_1, c_5^{(1)}, c_1^{(1)}) \\ c_2 & c_6 \\ c_3 & c_7 \\ f(c_2, c_2^{(1)}) & g(c_6, c_2, c_6^{(1)}, c_2^{(1)}) \\ f(c_3, c_3^{(1)}) & g(c_7, c_3, c_7^{(1)}, c_3^{(1)}) \\ c_4 & c_8 \\ f(c_4, c_4^{(1)}) & g(c_8, c_4, c_8^{(1)}, c_4^{(1)}) \end{bmatrix}, \tag{3.13}$$

where

$$f(z_1, z_2) = (HX_1+1)z_1 + HX_1Y_2z_2$$

$$g(z_1, z_2, z_3, z_4) = (H+1)z_1 + HY_1z_2 + HY_2z_3 + HY_1Y_2z_4$$

and c_i $(i=1,\ldots,8)$ are arbitrary.

Substituting (3.12) and (3.13) into the right hand side of respectively (3.10) and (3.11) the following expression for $u'(p)$ emerges:

$$u'(p) = \begin{bmatrix} c_3^{(1)} & c_3 \\ c_7^{(1)} & c_7 \\ f(c_3^{(1)}, c_1^{(1)}) & f(c_3, c_1) \\ c_4^{(1)} & c_4 \\ f(c_7^{(1)}, c_5^{(1)}) & f(c_7, c_5) \\ c_8^{(1)} & c_8 \\ g(c_4^{(1)}, c_3^{(1)}, c_2^{(1)}, c_1^{(1)}) & g(c_4, c_3, c_2, c_1) \\ g(c_8^{(1)}, c_7^{(1)}, c_6^{(1)}, c_5^{(1)}) & g(c_8, c_7, c_6, c_5) \end{bmatrix} \tag{3.14}$$

Choose the following parameters

$$p_1 = c_3^{(1)};\, p_2 = c_4^{(1)};\, p_3 = c_7^{(1)};\, p_4 = c_8^{(1)};\, p_5 = c_3;\, p_6 = c_4;\, p_7 = c_7;\, p_8 = c_8\,.$$

Evidently $u'(\boldsymbol{p}'')$ spans a bigger class than $u'(\boldsymbol{p}')$; $u'(\boldsymbol{p}')$ and $u'(\boldsymbol{p}'')$ should belong to the same class U. Enclosing $u'(\boldsymbol{p}')$ in $u'(\boldsymbol{p}'')$, i.e. $\forall \boldsymbol{p}', \exists \boldsymbol{p}''$ such that $u'(\boldsymbol{p}') = u'(\boldsymbol{p}'')$ we find

$$c_i(p_1, p_2, p_3, p_4, p_1, p_2, p_3, p_4) = c_i^{(1)}(p_1, p_2, p_3, p_4) \quad (i = 1, 2, 5, 6)\,. \tag{3.15}$$

Eq. (3.15) implies immediately the following form of $u'(\boldsymbol{p})$.

$$u'(\boldsymbol{p}) = \begin{bmatrix} p_1 & p_5 \\ p_3 & p_7 \\ \sum_{l=1}^{4} (A_l + A_{l+4}) p_l & \sum_{l=1}^{8} A_l p_l \\ p_2 & p_6 \\ \sum_{l=1}^{4} (B_l + B_{l+4}) p_l & \sum_{l=1}^{8} B_l p_l \\ p_4 & p_8 \\ \sum_{l=1}^{4} (C_l + C_{l+4}) p_l & \sum_{l=1}^{8} C_l p_l \\ \sum_{l=1}^{4} (D_l + D_{l+4}) p_l & \sum_{l=1}^{8} D_l p_l \end{bmatrix}, \tag{3.16}$$

where the coefficients A_l, B_l, C_l and D_l $(l = 1, \ldots, 8)$ still have to be determined. It is essential to note that we have assumed that the parameter dependence of c_i is linear (Compare the nearest neighbour model).

Contrary to the nearest neighbour case we will not enclose $u'(\boldsymbol{p}'')$ in $u'(\boldsymbol{p})$ but substitute form (3.16) of $u'(\boldsymbol{p})$ into scheme (3.7a), i.e. into Eqs. (3.10) and (3.11), which should hold in the following sense

(1) $\forall \boldsymbol{p}'$, $\exists \boldsymbol{p}$ such that (3.10) are satisfied,

(2) $\forall \boldsymbol{p}''$, $\exists \boldsymbol{p}$ such that (3.11) are satisfied.

This procedure is followed because it shortens the calculations. The coefficients A_l, B_l, C_l and D_l $(l = 1, \ldots, 8)$ are completely determined by this process. Afterwards the same relations yield L_1 and L_2, which give $\boldsymbol{p}'$

and p'' in terms of p by $p' = L_1 p$ and $p'' = L_2 p$. The rather lengthy and tedious calculations result in:

$$A_1 = A_2 = A_3 = A_4 = A_7 = A_8 = 0;\ A_5 = B_7;\ A_6 = B_8\ ;$$
$$B_1 = B_2 = B_3 = B_4 = B_5 = B_6 = 0\ ;$$
$$C_1 = C_2 = C_3 = C_4 = C_7 = C_8 = 0;\ C_5 = D_7;\ C_6 = D_8\ ;$$
$$D_1 = D_2 = D_3 = D_4 = D_5 = D_6 = 0\,,$$

where the coefficients B_7, B_8, D_7 and D_8 have to be determined from the equations

(1) $B_8 D_7 = H X_1 Y_2$,
(2) $D_8 D_7 = (H+1) D_7 + H Y_2 B_7 + H Y_1 Y_2$,
(3) $B_7 + B_8 D_8 = H X_1 + 1$,
(4) $D_7 + D_8^2 = (H+1) D_8 + H Y_2 B_8 + H Y_1$.

The matrices L_1 and L_2 are given by

$$L_1 = \begin{bmatrix} \Lambda & 0 & 0 & 0 \\ (H X_1 X_2 + 1)\Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ H X_2 Y_1 \Lambda & (H X_2 + 1)\Lambda & 0 & 0 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & \Lambda & 0 \\ H X_1 Y_2 \Lambda & 0 & (H X_1 + 1)\Lambda & 0 \\ 0 & 0 & 0 & \Lambda \\ H Y_1 Y_2 \Lambda & H Y_2 \Lambda & H Y_1 \Lambda & (H+1)\Lambda \end{bmatrix},$$

where the 2×2 matrix Λ has the form

$$\Lambda = \begin{bmatrix} -\dfrac{D_8}{D_7} & \dfrac{1}{D_7} \\ 1 & 0 \end{bmatrix}.$$

Take an arbitrary $u_k^{(i)}$ with parameterset p from scheme (3.7). Observe that there exists a sequence p', p'', p''', ... such that

$$p' = L_1 p,\ p'' = L_1 p',\ p''' = L_1 p'', \text{ etc.}$$

Looking only at the first two components this is equivalent to

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = \Lambda \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}; \quad \begin{pmatrix} p_1'' \\ p_2'' \end{pmatrix} = \Lambda \begin{pmatrix} p_1' \\ p_2' \end{pmatrix}; \quad \begin{pmatrix} p_1''' \\ p_2''' \end{pmatrix} = \Lambda \begin{pmatrix} p_1'' \\ p_2'' \end{pmatrix} \text{ etc.},$$

or

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} p_1' \\ p_2' \end{pmatrix}; \quad \begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} p_1'' \\ p_2'' \end{pmatrix}; \quad \begin{pmatrix} p_1'' \\ p_2'' \end{pmatrix} = \Lambda^{-1} \begin{pmatrix} p_1''' \\ p_2''' \end{pmatrix} \text{ etc.},$$

with

$$\Lambda^{-1} = \begin{bmatrix} 0 & 1 \\ D_7 & D_8 \end{bmatrix}.$$

Because all components of all $u_k^{(i)}$ should be positive Λ^{-1} satisfies the conditions of the Perron-Frobenius theorem and therefore by Brascamps lemma [5], $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ is the eigenvector of Λ^{-1} belonging to the largest positive eigenvalue. This gives $p_2 = \lambda p_1$ with $\lambda^2 = D_7 + \lambda D_8$.

Consider again an arbitrary $u_k^{(i)}$ with parameter set $\boldsymbol{p}$; there exists always a parameter set $\boldsymbol{p}'$ such that $\boldsymbol{p}' = L_2 \boldsymbol{p}$ or

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = \Lambda \begin{pmatrix} p_5 \\ p_6 \end{pmatrix}.$$

Because $p_2' = \lambda p_1'$ it directly follows $p_6 = \lambda p_5$. In an analogous way $p_8 = \lambda p_7$, $p_4 = \lambda p_3$. Combining these results with Eq. (3.17) one finds immediately

$$u'(\boldsymbol{p}) = \begin{bmatrix} p_1 & p_5 \\ p_3 & p_7 \\ K(\lambda) p_1 & K(\lambda) p_5 \\ \lambda p_1 & \lambda p_5 \\ K(\lambda) p_3 & K(\lambda) p_7 \\ \lambda p_3 & \lambda p_7 \\ \lambda^2 p_1 & \lambda^2 p_5 \\ \lambda^2 p_3 & \lambda^2 p_7 \end{bmatrix} \tag{3.18}$$

with

$$K(\lambda) = (H X_1 + 1) + H X_1 Y_2 \frac{1}{\lambda}$$

and where λ satisfies the equation

$$\lambda^4 - (H+1)\lambda^3 - H Y_1 \lambda^2 - (H+1) H X_1 Y_2 \lambda - H^2 X_1 Y_2^2 = 0. \tag{3.19}$$

Using the same arguments as in the nearest neighbour case it follows that one must select that root of (3.19) which is bigger than one. This is always possible. To determine u_1 it is sufficient to note that $u_1 = u_2^{(2)}$, i.e. $L_2 \boldsymbol{p}_1 = \boldsymbol{p}_1$. This results in

$$u_1' = \begin{bmatrix} p_1 & \lambda p_1 \\ K(\lambda) p_1 & \lambda^2 p_1 \\ K(\lambda) p_1 & \lambda K(\lambda) p_1 \\ \lambda p_1 & \lambda^2 p_1 \\ K^2(\lambda) p_1 & \lambda^2 K(\lambda) p_1 \\ \lambda K(\lambda) p_1 & \lambda^3 p_1 \\ \lambda^2 p_1 & \lambda^3 p_1 \\ \lambda^2 K(\lambda) p_1 & \lambda^4 p_1 \end{bmatrix} .$$

With the aid of the Eqs. (3.8) and (3.9) u_1 can be calculated. It follows

$$u_1(1,1) = \omega(n_{-2} n_{-1} n_0 n_1 n_2 n_3) = p_1 ,$$

$$u_1(1,2) = \omega(n_{-2} n_{-1} n_0 n_1 n_3) = K(\lambda) p_1 ,$$

$$u_1(1,3) = \omega(n_{-2} n_{-1} n_0 n_2 n_3) = (H X_1 X_2 + 1) p_1 ,$$

$$u_1(1,4) = \omega(n_{-2} n_0 n_1 n_2 n_3) = u_1(1,2) ,$$

$$u_1(1,5) = \omega(n_{-1} n_0 n_1 n_2 n_3) = \lambda p_1 ,$$

$$u_1(1,6) = \omega(n_{-2} n_{-1} n_0 n_3) = [(H X_2 + 1) K(\lambda) + H X_2 Y_1] p_1 ,$$

$$u_1(1,7) = \omega(n_{-2} n_0 n_1 n_3) = K^2(\lambda) p_1 ,$$

$$u_1(1,8) = \omega(n_{-2} n_0 n_2 n_3) = [(H X_1 + 1) K(\lambda) + H X_1 Y_2] p_1 ,$$

$$u_1(1,9) = \omega(n_{-1} n_0 n_1 n_3) = \lambda K(\lambda) p_1 ,$$

$$u_1(1,10) = \omega(n_{-1} n_0 n_2 n_3) = (H X_1 X_2 + 1) \lambda p_1 ,$$

$$u_1(1,11) = \omega(n_0 n_1 n_2 n_3) = \lambda^2 p_1 ,$$

$$u_1(1,12) = \omega(n_{-2} n_0 n_3)$$
$$= [(H+1) K^2(\lambda) + H Y_1 K(\lambda) + H Y_2 K(\lambda) + H Y_1 Y_2] p_1 ,$$

$$u_1(1,13) = \omega(n_{-1} n_0 n_3) = [(H X_2 + 1) K(\lambda) + H X_2 Y_1] \lambda p_1 ,$$

$$u_1(1,14) = \omega(n_0 n_1 n_3) = \lambda^2 K(\lambda) p_1 ,$$

$$u_1(1,15) = \omega(n_0 n_2 n_3) = u_1(1,14) ,$$

$$u_1(1,16) = \omega(n_0 n_3) = [(H+1) \lambda K(\lambda) + H Y_1 \lambda + H Y_2 K(\lambda) + H Y_1 Y_2] \lambda p_1 ,$$

$$u_1(1,17)=\omega(n_{-2}n_{-1}n_1n_2n_3)=u_1(1,3),$$

$$u_1(1,18)=\omega(n_{-2}n_{-1}n_1n_3)=u_1(1,8),$$

$$u_1(1,19)=\omega(n_{-2}n_{-1}n_2n_3)=[(HX_2+1)(HX_1X_2+1)+HX_2Y_1]p_1,$$

$$u_1(1,20)=\omega(n_{-2}n_1n_2n_3)=u_1(1,6),$$

$$u_1(1,21)=\omega(n_{-1}n_1n_2n_3)=u_1(1,9),$$

$$u_1(1,22)=\omega(n_{-2}n_{-1}n_3)=[\{(H+1)(HX_2+1)+HY_1\}K(\lambda) \\ +(H+1)HX_2Y_1+HY_2(HX_1X_2+1)+HY_1Y_2]p_1,$$

$$u_1(1,23)=\omega(n_{-2}n_1n_3)=u_1(1,12),$$

$$u_1(1,24)=\omega(n_{-2}n_2n_3)=u_1(1,22),$$

$$u_1(1,25)=\omega(n_{-1}n_1n_3)=\left[\left(\frac{HX_1}{X_2}+1\right)\lambda K(\lambda)+\frac{HX_1Y_2}{X_2}\lambda \right. \\ \left. +\frac{HX_1Y_2}{X_2}K(\lambda)+\frac{HX_1Y_2^2}{X_2}\right]p_1,$$

$$u_1(1,26)=\omega(n_{-1}n_2n_3)=u_1(1,13),$$

$$u_1(1,27)=\omega(n_1n_2n_3)=\lambda^3p_1,$$

$$u_1(1,28)=\omega(n_{-2}n_3)=\left[\left(\frac{H^2}{X_1}+2H+1\right)K^2(\lambda)\right. \\ +2\left(H^2X_2-\frac{H^2}{X_1}+HY_1+HY_2\right)K(\lambda) \\ \left.+\frac{H^2}{X_1}+H^2X_1X_2^2-2H^2X_2+2HY_1Y_2\right]p_1,$$

$$u_1(1,29)=\omega(n_{-1}n_3)=\left[\left(H^2+HX_2+\frac{HX_1}{X_2}+1\right)\lambda K(\lambda)\right. \\ +\left(H^2X_1X_2-H^2+HX_2Y_1+\frac{HX_1Y_2}{X_2}\right)\lambda \\ \left.+\left(H^2Y_2+\frac{HX_1Y_2}{X_2}\right)K(\lambda)+H^2Y_1Y_2+H^2X_1Y_2^2+\frac{HX_1Y_2^2}{X_2}\right]p_1,$$

$$u_1(1,30)=\omega(n_1n_3)=\left[\left(\frac{H}{X_2}+1\right)\lambda^2K(\lambda)+\frac{HY_2}{X_2}\lambda^2+\frac{HY_2}{X_2}K^2(\lambda)\right. \\ +\frac{HY_1}{X_2}\lambda K(\lambda)+\frac{HY_1Y_2}{X_2}\lambda+\frac{HY_2^2}{X_2}K(\lambda)+\frac{HY_1Y_2}{X_2}K(\lambda) \\ \left.+\frac{HY_1Y_2^2}{X_2}\right]p_1,$$

$$u_1(1,31)=\omega(n_2 n_3)=\lambda^4 p_1\,,$$

$$\begin{aligned}u_1(1,32)=\omega(n_3)=\Bigg[&\left(\frac{H^2}{X_1X_2}+H+\frac{H}{X_1}+\frac{HY_1}{X_1X_2}+1\right)\lambda^2K(\lambda)\\&+\left(\frac{H^2Y_1}{X_1X_2}+HY_1+\frac{HY_1Y_2}{X_1X_2}\right)\lambda^2+\left(\frac{H^2Y_2}{X_1X_2}+\frac{HY_2}{X_2}\right)K^2(\lambda)\\&+\left(\frac{H^2Y_2}{X_1X_2}+\frac{H^2Y_1}{X_1}+HY_2+\frac{HY_1}{X_2}\right)\lambda K(\lambda)\\&+\left(H^2X_2Y_1-\frac{H^2Y_1}{X_1X_2}+HY_1Y_2+\frac{HY_1Y_2}{X_2}\right)\lambda\\&+\left(2H^2Y_2-2\frac{H^2Y_2}{X_1X_2}+\frac{HY_2^2}{X_2}+\frac{HY_1Y_2}{X_1X_2}\right)K(\lambda)\\&+H^2X_2Y_1Y_2+\frac{H^2Y_2^3}{X_2}-\frac{H^2Y_1Y_2}{X_1X_2}+\frac{HY_1Y_2^2}{X_2}\Bigg]p_1\,,\end{aligned}$$

$$u_1(2,1)=\omega(n_{-2}n_{-1}n_0n_1n_2)=u_1(1,5)\,,$$

$$u_1(2,2)=\omega(n_{-2}n_{-1}n_0n_1)=u_1(1,11)\,,$$

$$u_1(2,3)=\omega(n_{-2}n_{-1}n_0n_2)=u_1(1,9)\,,$$

$$u_1(2,4)=\omega(n_{-2}n_0n_1n_2)=u_1(1,9)\,,$$

$$u_1(2,5)=\omega(n_{-1}n_0n_1n_2)=u_1(1,11)\,,$$

$$u_1(2,6)=\omega(n_{-2}n_{-1}n_0)=u_1(1,27)\,,$$

$$u_1(2,7)=\omega(n_{-2}n_0n_1)=u_1(1,14)\,,$$

$$u_1(2,8)=\omega(n_{-2}n_0n_2)=u_1(1,25)\,,$$

$$u_1(2,9)=\omega(n_{-1}n_0n_1)=u_1(1,27)\,,$$

$$u_1(2,10)=\omega(n_{-1}n_0n_2)=u_1(1,14)\,,$$

$$u_1(2,11)=\omega(n_0n_1n_2)=u_1(1,27)\,,$$

$$u_1(2,12)=\omega(n_{-2}n_0)=u_1(1,30)\,,$$

$$u_1(2,13)=\omega(n_{-1}n_0)=u_1(1,31)\,,$$

$$u_1(2,14)=\omega(n_0n_1)=u_1(1,31)\,,$$

$$u_1(2,15)=\omega(n_0n_2)=u_1(1,30)\,,$$

$$u_1(2,16)=\omega(n_0)=u_1(1,32)\,,$$

$$u_1(2,17)=\omega(n_{-2}n_{-1}n_1n_2)=u_1(1,10)\,,$$

$u_1(2,18)=\omega(n_{-2}n_{-1}n_1)=u_1(1,14)$,

$u_1(2,19)=\omega(n_{-2}n_{-1}n_2)=u_1(1,13)$,

$u_1(2,20)=\omega(n_{-2}n_1n_2)=u_1(1,13)$,

$u_1(2,21)=\omega(n_{-1}n_1n_2)=u_1(1,14)$,

$u_1(2,22)=\omega(n_{-2}n_{-1})=u_1(1,31)$,

$u_1(2,23)=\omega(n_{-2}n_1)=u_1(1,16)$,

$u_1(2,24)=\omega(n_{-2}n_2)=u_1(1,29)$,

$u_1(2,25)=\omega(n_{-1}n_1)=u_1(1,30)$,

$u_1(2,26)=\omega(n_{-1}n_2)=u_1(1,16)$,

$u_1(2,27)=\omega(n_1n_2)=u_1(1,31)$,

$u_1(2,28)=\omega(n_{-2})=u_1(1,32)$,

$u_1(2,29)=\omega(n_{-1})=u_1(1,32)$,

$u_1(2,30)=\omega(n_1)=u_1(1,32)$,

$u_1(2,31)=\omega(n_2)=u_1(1.32)$,

$$\begin{aligned}u_1(2,32)=\omega(1)=1=&\left(\frac{H}{X_1X_2}+1\right)u_1(1,32)+\frac{2HY_1}{X_1X_2}u_1(1,31)\\&+\frac{2HY_2}{X_1X_2}u_1(1,30)+\frac{HY_1}{X_1X_2}(Y_1+2Y_2)\,u_1(1,27)\\&+\frac{HY_2^2}{X_1X_2}u_1(1,25)+\frac{2HY_1Y_2}{X_1X_2}u_1(1,14)+\frac{2HY_1^2Y_2}{X_1X_2}u_1(1,11)\\&+\frac{2HY_1Y_2^2}{X_1X_2}u_1(1,9)+\frac{HY_1^2Y_2^2}{X_1X_2}u_1(1,5)\,.\end{aligned}$$

The last equation determines p_1 in terms of λ.

In the absence of a magnetic field, i.e. $H=1$, the roots of Eq. (3.19) are

$$\lambda_{1,2}=e^{-\beta J_1}[\cosh\beta J_1\pm[\sinh^2\beta J_1+e^{-4\beta J_2}]^{\frac{1}{2}}]\,,$$
$$\lambda_{3,4}=e^{-\beta J_1}[\sinh\beta J_1\pm[\cosh^2\beta J_1-e^{-4\beta J_2}]^{\frac{1}{2}}]\,.$$

These roots have a great resemblance with the roots obtained by the transfer matrix method (see e.g. [6]). The thermodynamics of the system can be derived from the internal energy per spin

$$u=-(J_1+J_2+h)+2(2J_1+2J_2+h)\,u_1(1,32)-4J_1u_1(1,31)-4J_2u_1(1,30)\,.$$

The cluster property assumes here exactly the same form as in the nearest neighbour case. For $h=0$ it can easily be checked since $\omega(n_j)=\frac{1}{2}$.

Section IV

Summary and Remarks

The presented method can be summarized as follows: We divide all correlation functions into groups $u_k^{(i)}$, such that the internal relations due to the K.M.S.-condition are the same for each $u_k^{(i)}$, and in such a way that they fit into some kind of "infinitely repeating" scheme ((2.7) and (3.7)). Now we distinguish the properties of $u_k^{(i)}$ into specific properties and those common to them all.

For the moment neglecting the specific properties, we observe that each $u_k^{(i)}$ has the same "pedigree". As shown in the preceding sections this implies that all $u_k^{(i)}$ belong to a finite class U, which can be determined from studying *one* unit of scheme (2.7) or (3.7). The specific properties of u_1 are used to select for each $u_k^{(i)}$ the suitable element from the class U (by means of the matrices L_1 and L_2).

About the number of parameters needed to describe the class U, or equivalently the number of independent vectors in that class, it should be remarked that, in the next-nearest neighbour case, the class U is essentially a four-parameter family as follows from Eq. (3.18); whereas equalizing the numbers of equations and unknowns yields an eight-parameter one. However, as it is impossible to select at the outset the four correct ones, the last approach is preferable.

Besides the number of parameters in the next-nearest neighbour case, the roots of Eq. (3.19) need some comment. It is easily seen that, excepting the case $J_2>0$ $h\geqq 0$, one and only one root of (3.19) exists, which is bigger than one; a necessary requirement for the solution. In the case $J_2>0$ $h\geqq 0$ there exists a "critical" temperature β_c, determined by

$$e^{-2\beta_c h}+e^{-4\beta_c J_2}=1\,,$$

such that we have for

1. $\beta<\beta_c$ one root bigger than one;
2. $\beta=\beta_c$ one root equal to one and one root bigger than one;
3. $\beta>\beta_c$ two unequal roots bigger than one.

Because of the desired continuity of the correlation functions (in particular of the internal energy) the biggest root must be taken. This ambiguity in the choice of the roots for $\beta>\beta_c$ strongly suggest that the method is also applicable to systems which exhibit a phase transition like the two-dimensional model.

To conclude this section we will make some remarks about extending the method to more-dimensional models.

In the case of the one-dimensional models treated here it can be shown that there exist subsets $v_1 \subset u_1$, $v_2 \subset u_2^{(1)}$, $v_3 \subset u_3^{(1)}$, etc., such that scheme (2.7) or (3.7) changes into

$$v_1 \longleftarrow v_2 \longleftarrow v_3 \longleftarrow \cdots .$$

This simplified scheme is analogous to the procedure of the transfer matrix method [6], although they differ in the quantities coupled to each other. These "simplified calculations" are still a lot more complicated than those performed with the transfer matrix method, but they supply again the knowledge of all correlation functions, whereas the transfer matrix method yields the thermodynamical functions only.

It is clear that for more-dimensional models, where the number of next-nearest neighbours is greater than the number of nearest neighbours of a point, the simplified version of the method cannot be used. The method presented in the preceding sections has been constructed to conquer the very difficulties of the more-dimensional lattice.

In the case of the two-dimensional Ising model with nearest neighbour interactions (with or without magnetic field) the topology of the lattice leads to a fanlike scheme analogous to (2.7) or (3.7); the number of coupled equations turns out to be very large (of the order of hundreds). The work on finding explicit solutions of these equations is still in progress.

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