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The Local b-Completeness of Space-times

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Abstract. It is shown that any point of space-time has a neighbourhood U such that the b-boundary $\dot{U}$ of U coincides with $\bar{U} \setminus U$.

1. Introduction

The b-boundary construction is a device to attach to any space-time a set of boundary points [1]. Such a boundary point can be considered as an equivalence class of inextensible curves in a space-time, whose affine length [2] (i.e. the length measured in a parallely propagated frame) is finite.

It may happen that such a curve is trapped in a compact set and still defines a "boundary point". An example is given by a closed null geodesic, which has moreover the following property. Choose a tangent vector X to the geodesic at a point p and parallelly propagate it along the geodesic. If we return to p with the vector λX, $0 < \lambda < 1$, then the affine length traversed in going round the geodesic n times is $l(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-1})$. Hence the length of the inextensible curve defined by going round again and again is finite. Such a situation occurs in Taub-NUT space [2].

From the above example we learn that the following is possible: If U is an open submanifold of a space-time V^4 with compact closure $\bar{U}$ (relative to V^4), then $\dot{U}$, the b-boundary of the space-time U, possibly contains more points than $\bar{U} \setminus U$, the boundary of U relative to V^4. (From the definition of the b-boundary, $\bar{U} \setminus U \subset \dot{U}$ is obvious.)

Now one can ask the following question, which – as far as the author is aware – was first posed by Hawking: has any point $p \in V^4$ a neighbourhood U such that $\bar{U} \setminus U = \dot{U}$? An affirmative answer to this question is extremely important because otherwise the set of boundary points could be dense in the topological space $V^4 \cup \dot{V}^4$! Clearly the b-boundary would then be useless for a description of singularities.

The main purpose of this paper is to prove that $\bar{U} \setminus U = \dot{U}$ holds, provided there is no null geodesic trapped in $\bar{U}$.

This implies in particular that for normal neighbourhoods N we have always $\bar{N} \setminus N = \dot{N}$, the desired property.

To simplify terminology, we say that a space is locally complete at a point if there is a neighbourhood U of this point with $\bar{U} \setminus U = \dot{U}$.

2. A Counter Example

The b-boundary construction works not only for space-times but for any linear connection. In this section I shall give an example of a linear connection which is at no point locally complete!

The idea is to construct a 2-dimensional connection with the property that there are closed curves in any neighbourhood of a point having a lift of finite length into the frame bundle if we go around the closed curve again and again.

The example of a closed null geodesic suggests that we construct a connection whose holonomy group consists of homothetic linear maps φ, i.e. $\varphi(X) = CX$ for any vector $X \in T_x$.

Consider the following analytic connection on R^2, on which (x^1, x^2) is a global coordinate system, whose only non-vanishing components are

$$\Gamma^1_{11} = x^2, \quad \Gamma^2_{22} = -x^1. \tag{1}$$

The Riemann tensor at the origin is

$$\begin{aligned} R^i{}_{jkl}|_0 &= (\Gamma^i_{j|k|l} - \Gamma^i_{jl|k} + \Gamma^m_{jk}\Gamma^i_{lm} - \Gamma^m_{jl}\Gamma^i_{km})|_0 \\ &= (\Gamma^i_{jk|l} - \Gamma^i_{jl|k})|_0 . \end{aligned} \tag{2}$$

Hence the only non vanishing components are

$$(R^1{}_{112})_0 = 1, \quad (R^2{}_{212})_0 = 1. \tag{3}$$

Obviously all derivatives of $R^i{}_{klm}$ vanish at the origin. Therefore $R^i{}_{k12}$ generates the holonomy group, because the connection is analytic and the manifold simply connected [3]. The linear maps generated by $R^i{}_{k12}$ are, because of (3)

$$X_p \in T_p \to e^\alpha X_p, \quad \alpha \in \mathbb{R}^1 \tag{4}$$

Take an arbitrary curve $x^i(t)$, with $x^i(t+1) = x^i(t)$ for any t, such that for a vector field $X(t)$ which is parallelly propagated along $x^i(t)$, we have $X(1) = q \cdot X(0)$ with $0 < q < 1$. Such curves exist in any neighbourhood of any point because of (4) and because the local holonomy group is isomorphic to the holonomy group for analytic connections [3].

If $X = \xi^i \dfrac{\partial}{\partial x^i}$, $Y = \eta^i \dfrac{\partial}{\partial x^i}$ are two linearly independent vector fields parallel along $x^i(t)$, the components of $\dot{x}^i(t)$ in the frame (X, Y) are

$$\dot{x}^i(t) = \alpha(t)\xi^i + \beta(t)\eta^i \tag{5}$$

and the bundle length of the horizontal lift of $x^i(t)$ defined by $(X(0), Y(0))$ is

$$\int_0^\infty [\alpha^2+\beta^2]^{1/2}\,dt = \sum_{n=0}^\infty q^n \int_0^1 [\alpha^2+\beta^2]^{1/2}\,dt < \infty\,. \tag{6}$$

Therefore this connection is not locally complete at any point.

The invariance groups of all indefinite metrics which are not normal-hyperbolic contain subgroups which are homothetic on sub-spaces of dimension $\geqq 2$. Therefore it is very likely that such spaces are in general not locally complete.

3. Proof of the Theorem

In this Section I shall show that every space-time is locally complete, by proving a slightly more general

Theorem. *If V^4 is a space-time and U an open submanifold with compact closure $\bar{U}$ which contains no trapped null geodesic, then $\dot{U}$, the b-boundary of U, coincides with $\bar{U}\setminus U$, the boundary of U relative to V^4.*

Proof. The definition of the b-boundary [1] obviously implies $\bar{U}\setminus U \subset \dot{U}$. Suppose there is a point in $\dot{U}$ which is not contained in $\bar{U}\setminus U$. Such a point is defined by a Cauchy sequence u_ν in the Lorentz bundle $\mathscr{L}(U)$ endowed with the bundle metric. We can assume that the points u_ν are points on a horizontal curve $u(\lambda)$ of finite length, with $u_\nu := u(\lambda_\nu)$, and having the properties: $x_\nu := \pi(u_\nu) \subset U$, $\{u_\nu; \nu \in N\}$ is not contained in a compact subset of $\pi^{-1}(\bar{U})$. Without loss of generality we can assume that (x_ν) has a limit $x \in U$.

Let us choose a local cross-section around x, i.e. an open neighbourhood N of x and a map $\sigma: N \to \mathscr{L}(V^4)$ satisfying $\pi\circ\sigma = \mathrm{id}_N$. The sequence $\tilde{u}_\nu := \sigma(x_\nu)$ has a limit $\tilde{u}$ because $x_\nu \to x$.

The Lorentz group acts freely on the right on $\mathscr{L}(V^4)$, and simply transitively on the fibres $(u \to R_l u = ul)$.

Therefore there is a uniquely defined sequence of Lorentz transformations l_ν such that

$$\tilde{u}_\nu = u_\nu l_\nu\,. \tag{1}$$

The action of the Lorentz group on the Lorentz frames (e_a) is defined by

$$e_a \to e_b L_a^b \tag{2}$$

where L_a^b is a Lorentz transformation with respect to a fixed orthonormal basis. Hence there is a unique decomposition

$$l_\nu = r_\nu b_\nu \bar{r}_\nu \tag{3}$$

with the properties: r_ν, $\bar{r}_\nu$ are spatial rotations keeping the timelike vectors of the frames fixed; b_ν is a pure Lorentz transformation acting as

$$b_\nu \begin{cases} e_0+e_1 \to e^{\xi_\nu}(e_0+e_1), & \xi_\nu \in \mathbb{R} \\ e_0-e_1 \to e^{-\xi_\nu}(e_0-e_1), \\ e_2, e_3 \quad \to e_2, e_3 . \end{cases} \tag{4}$$

We know that u_ν is not contained in a compact set of $\pi^{-1}(\bar{U})$, therefore $|\xi_\nu| < C$ is impossible. Because of this and the compactness of the rotation group, there is a subsequence such that

$$r_{\nu'} \to r, \ \bar{r}_{\nu'} \to \bar{r}, \ \xi_{\nu'} \to \infty, \ \xi_{\nu'+1} - \xi_{\nu'} > a > 0 \tag{5}$$

holds for arbitrarily chosen a.

Instead of $\tilde{u}_{\nu'}$, we consider now the sequence

$$v_{\nu'} := \tilde{u}_{\nu'}(\bar{r}_{\nu'})^{-1} = u_{\nu'} r_{\nu'} b_{\nu'} \tag{6}$$

which has a limit v because of $\tilde{u}_{\nu'} \to \tilde{u}$ and $\bar{r}_{\nu'} \to \bar{r}$.

The idea is to "slide down" the pieces of the curve $u(\lambda)$, $\lambda_{\nu'} \leqq \lambda \leqq \lambda_{\nu'+1}$ from $u_{\nu'} = u(\lambda_{\nu'})$ to $v_{\nu'}$, and to calculate the change of the length of the pieces. More precisely:

$$C_{\nu'} : u(\lambda), \ \lambda_{\nu'} \leqq \lambda \leqq \lambda_{\nu'+1} \tag{7}$$

is a sequence of curves. If we apply the rotation $r_{\nu'}$ to the curve $C_{\nu'}$ we get another sequence

$$C_{\nu'} r_{\nu'} : \tilde{v}_{\nu'}(\lambda) := u(\lambda) r_{\nu'}, \qquad \lambda_{\nu'} \leqq \lambda \leqq \lambda_{\nu'+1} \tag{8}$$

and finally by applying $b_{\nu'}$

$$\hat{C}_{\nu'} := C_{\nu'} r_{\nu'} b_{\nu'} : v_{\nu'}(\lambda) := \tilde{v}_{\nu'}(\lambda) b_{\nu'}, \ \lambda_{\nu'} \leqq \lambda \leqq \lambda_{\nu'+1} . \tag{9}$$

Because of (8) and (9), the initial points of $\hat{C}_{\nu'}$

$$v_{\nu'}(\lambda_{\nu'}) = u_{\nu'} r_{\nu'} b_{\nu'} = v_{\nu'} \tag{10}$$

have a limit v. The endpoints of $\hat{C}_{\nu'}$ are

$$v_{\nu'}(\lambda_{\nu'+1}) = v_{\nu'+1} b_{\nu'+1}^{-1} r_{\nu'+1}^{-1} r_{\nu'} b_{\nu'} \tag{11}$$

and this, together with (5), shows the existence of a neighbourhood V of v such that all endpoints of $\hat{C}_{\nu'}$ are outside V. We shall need this later.

The bundle length of $u(\lambda)$, $0 \leqq \lambda < \infty$, is finite, hence $L(C_{\nu'})$, the bundle length of $C_{\nu'}$, tends to zero. If we apply a spatial rotation to a horizontal curve, its length does not change, therefore

$$L(C_{\nu'} r_{\nu'}) \to 0 . \tag{12}$$

Using the standard horizontal vector fields [3], $B_u := B_0 + B_1$, $B_v := B_0 - B_1$, B_2, B_3, the tangent vector of $C_{\nu'} r_{\nu'}$ is

$$\dot{\tilde{v}}_{\nu'}(\lambda) = \theta^u_{\nu'} B_u + \theta^v_{\nu'} B_v + \theta^2_{\nu'} B_2 + \theta^3_{\nu'} B_3 \tag{13}$$

and therefore we find for the length of $C_{v'} r_{v'}$

$$L(C_{v'} r_{v'}) = \int_{\lambda_{v'}}^{\lambda_{v'+1}} [(\theta^u_{v'})^2 + (\theta^v_{v'})^2 + (\theta^2_{v'})^2 + (\theta^3_{v'})^2]^{1/2} \, d\lambda \,. \tag{14}$$

The relation $v_{v'}(\lambda) = \tilde{v}_{v'}(\lambda)\, b_{v'}$ implies [3] because of (4),

$$\dot{v}_{v'} = \theta^u_{v'} e^{-\xi_{v'}} B_u + \theta^v_{v'} e^{\xi_{v'}} B_v + \theta^2_{v'} B_2 + \theta^3_{v'} B_3 \,. \tag{15}$$

The length of $\hat{C}_{v'}$ is therefore

$$L(\hat{C}_{v'}) = \int_{\lambda_{v'}}^{\lambda_{v'+1}} [e^{2\xi_{v'}}(\theta^u_{v'})^2 + e^{-2\xi_{v'}}(\theta^v_{v'})^2 + (\theta^2_{v'})^2 + (\theta^3_{v'})^2]^{1/2} \, d\lambda \,. \tag{16}$$

The fact that $L(C_{v'} r_{v'}) \to 0$, together with (14) implies

$$\int_{\lambda_{v'}}^{\lambda_{v'+1}} |\theta^u_{v'}| \, d\lambda \to 0 \,, \quad \int_{\lambda_{v'}}^{\lambda_{v'+1}} |\theta^2_{v'}| \, d\lambda \to 0 \,, \quad \int_{\lambda_{v'}}^{\lambda_{v'+1}} |\theta^3_{v'}| \, d\lambda \to 0 \,. \tag{17}$$

This will now lead to a contradiction.

The initial points $v_{v'}$ of $\hat{C}_{v'}$ tend to the limit v, and we have shown above that there is a compact neighbourhood V of v such that no endpoint and no accumulation point of endpoints is contained in V. The integral curve $g(\lambda)$ of the vector field B_v passing through v projects onto a null geodesic and therefore by our assumptions $g(\lambda)$ leaves $\pi^{-1}(\bar{U})$. This implies the existence of a compact neighbourhood W of v, $W \subset V$, such that all integral curves of B_v in $\pi^{-1}(\bar{U})$ passing through a point in W do not enter $\pi^{-1}(\pi(W))$ again after leaving W, and moreover leave $\pi^{-1}(\bar{U})$.

Choosing W sufficiently small, we can construct a compact tube $T \supset W$ around $g(\lambda)$ which is the union of integral curves of B_v such that: 1. any integral-curve has a part of finite length outside $\pi^{-1}(\bar{U})$ – in both directions –; 2. any hypersurface in T nowhere tangent to B_v is met by an integral curve only once and 3. there exists a hypersurface H in T which is met by any integral curve (comoving coordinates for the vector field B_v).

Then $\pi^{-1}(\pi(W)) \cap T = W$ holds, and therefore the curves $\hat{C}_{v'}$ whose initial points are in T for $v' \geqq N$ must leave T because the endpoints are in $\pi^{-1}(\pi(W))$ but not in W.

If we denote by τ the projection: $\tau: T \to H$ defined by the fibration of T by the integral curves of B_v, the following inequality holds as T is compact

$$0 < A^2 g_b(\tau_* X, \tau_* X) \leqq g_b(X, X) \leqq B^2 g_b(\tau_* X, \tau_* X), \quad X \perp B_v \tag{18}$$

with constants A, B, (g_b being the bundle metric).

Because all curves $\hat{C}_{v'}$ leave T, they have a projection onto H whose length is larger than a certain number, as the initial points converge to v. This, together with (18), contradicts (17); which completes the proof.

Remarks. If one makes no assumption about the null geodesics in $\bar{U}$, the proof given above shows that $\bar{U} \setminus U \neq \dot{U}$ implies the existence of a trapped geodesic. I have not been able to prove that this geodesic is necessarily incomplete. This is easy to show if the null geodesic is closed, or if all its accumulation points not on the geodesic are on a closed geodesic. However, there is the possibility that the accumulation points have a dimension greater than 1.

Corollary. *Every point x of a space-time V^4 has an open neighbourhood U such that $\bar{U} \setminus U = \dot{U}$.*

This is a trivial consequence of the theorem, because every point has a normal neighbourhood in which no geodesics are trapped.

References

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