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On the Integrability of Representations of Finite Dimensional Real Lie Algebras

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Abstract. We give an integrability criterion for Lie algebra representations in a reflexive Banach space. Applications are given to skewsymmetric Lie algebra representations in Hilbert spaces and to essential skewadjointness of a sum of two skewadjoint operators.

Introduction

Recently it has been shown by Flato, Snellman, Sternheimer and the author [1] that a representation of a real finite dimensional Lie algebra by skewsymmetric operators in a Hilbert space defined on a dense invariant domain of analytic vectors for the representatives of a given basis of the algebra is the differential on its definition domain, of a unitary representation on the Hilbert space of a connected and simply-connected Lie group. This result was then extended to more general representations supposing that the representatives of the basis are the generators of strongly continuous one parameter groups.

The aim of this article is to extend some of these results supposing only the existence of dense domains of analytic vectors (not even necessarily common!) for the adjoint of the representatives of a set of generators of the Lie algebra in case of representations by skewsymmetric operators, and supposing in addition, in more general situations, that these representatives are the generators of strongly continuous one parameter groups. Though some results remain valid in more general spaces we limited our study to the case where the representation space is a reflexive Banach space to avoid a heavier terminology.

1. Notations

In what follows $\mathfrak{g}$ is a real finite dimensional Lie algebra and G is the connected simply connected real Lie group the Lie algebra of which is $\mathfrak{g}$.

A set of generators of $\mathfrak{g}$ is a set of vectors $\{x_1, \ldots, x_n\}$ in $\mathfrak{g}$ such that $\mathfrak{g}$ is generated by linear combinations of the vectors

$$x_1, \ldots, x_n, [x_{i_1}, x_{i_2}], [x_{i_3}, [x_{i_4}, x_{i_5}]], \ldots \quad \text{when} \quad 1 \leqq i_1, i_2, \ldots \leqq n .$$

X is a reflexive Banach space X^* its adjoint space (space of continuous semilinear forms defined on X) and $\langle , \rangle$ denotes the sesquilinear form on $X \times X^*$ defining the duality.

When X is a densely defined operator in X, $D(X)$ denotes its definition set, if $D' \subset D(X)$, $X|_{D'}$ is the restriction of X to D', X^* is the adjoint operator of X and $\bar{X}$ its closure when it is closable.

An analytic vector for X is a vector $\varphi \in \bigcap_{n \in \mathbb{N}} D(X^n)$ such that $\sum_{n=0}^{\infty} \frac{t^n}{n!} \|X^n \varphi\| < +\infty$ for some $t > 0$.

2. Results

Is x, $y \in \mathfrak{g}$ and $t \in \mathbb{R}$, we know that the series $\sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} (\operatorname{ad} x)^\varrho y$ where $(\operatorname{ad} x) y = [x, y]$, is absolutely convergent and we denote by $a(tx, y)$ its sum.

Lemma 1. *If $\{x_1, \ldots, x_n\}$ is a set of generators of $\mathfrak{g}$, the family of vectors*

$$\mathscr{F} = \{x_1, \ldots, x_k, a(tx_{i_1}, x_{i_2}), a(t' x_{i_3}, a(t'' x_{i_4}, x_{i_5})), \ldots\}$$

where $t, t', t'', \ldots \in \mathbb{R}$, $1 \leqq i_1, i_2, \ldots \leqq n$, contains a basis of $\mathfrak{g}$.

If all the commutators $[x_i, x_j]$, $1 \leqq i, j \leqq n$, are linear combinations of $x_1, \ldots, x_n$, this is clear. If one of them $[x_{i_0}, x_{j_0}]$ is not, as $a(tx_{i_0}, x_{j_0}) = x_{j_0} + t[x_{i_0}, x_{j_0}] + O(t^2)$ where $t \to 0$, there exists t_0 in a neighbourhood of zero such that $a(t_0 x_{i_0}, x_{j_0})$ is not a linear combination of $x_1, \ldots, x_n$. Then, by induction, $\mathscr{F}$ contains a family of vectors which generates $\mathfrak{g}$ by linear combinations and therefore contains a basis of $\mathfrak{g}$.

Lemma 2. *If T is a representation of $\mathfrak{g}$, defined on a dense domain $D \subset X$ which is invariant under $T(\mathfrak{g})$, and $\{x_1, \ldots, x_n\}$ is a set of generators of $\mathfrak{g}$ satisfying.*

a) *If $X_i = T(x_i)$, $1 \leqq i \leqq n$, X_i^* has a dense domain $D_i^* \subset X^*$ of analytic vectors such that $D_i^* \subset D(X_j^*)$, $1 \leqq j \leqq n$, $X_j^* D_i^* \subset D_i^*$ and $\overline{X_j^*|_{D_i^*}} = X_j^*$.*

b) *The operators X_i^* are the generators of strongly continuous one parameter groups, $t \to e^{tX_i^*}$ on X^*.*

Then, there exists an extention of T by a representation T' of $\mathfrak{g}$ on an invariant domain $D' \supset D$, and a basis $\{y_1, \ldots, y_r\}$ of $\mathfrak{g}$ such that the operators $Y_i' = T'(y_i)$, $1 \leqq i \leqq r$, have closure which are the generators of one parameter groups leaving D' invariant, and such that:

$$T'(a(ty_i, y_j)) = e^{t\bar{Y}_i'} Y_j' e^{-t\bar{Y}_i'}. \tag{0}$$

Denote by $D' = \bigcap_{p \in \mathbb{N}} D(\bar{X}_{i_1} \dots \bar{X}_{i_p})$, $\tilde{D} = \bigcap_{p \in \mathbb{N}} D(X^*_{i_1} \dots X^*_{i_p})$, $X'_j = \bar{X}_{j|D'}$ and $\tilde{X}_j = -X^*_{j|\tilde{D}}$.

Then $D \subset D'$, $D^*_j \subset \tilde{D}$ and the sets D' and $\tilde{D}$ are invariant respectively under the operators X'_j and $\tilde{X}_j$.

By hypothesis the family

$$\mathscr{F}' = \{x_1, \dots, x_n, [x_{i_1}, x_{i_2}], [x_{i_3}, [x_{i_4}, x_{i_5}]], \dots\} \quad \text{where} \quad 1 \leqq i_1, i_2, \dots \leqq n$$

contains a basis $\{z_1, \dots, z_r\}$ of $\mathfrak{g}$. Moreover denote by $\tilde{Z}_i$ the operator which corresponds to z_i in the operator family

$$\{\tilde{X}_1, \dots, \tilde{X}_n, [\tilde{X}_{i_1}, \tilde{X}_{i_2}], [\tilde{X}_{i_3}, [\tilde{X}_{i_4}, \tilde{X}_{i_5}]], \dots\} \quad \text{where} \quad 1 \leqq i_1, i_2, \dots \leqq n .$$

If $y = \sum_{i=1}^{r} \lambda_i z_i (\lambda_i \in \mathbb{R})$, define $\tilde{Y} \equiv \tilde{T}(y) = \sum_{i=1}^{r} \lambda_i \tilde{Z}_i$. It is straightforward to see, by duality with T, that $\tilde{T}$ is a representation of $\mathfrak{g}$ on the domain $\tilde{D}$ which is invariant under $\tilde{T}(\mathfrak{g})$.

Denote by Z'_i the operator which correspond to z_i in the family

$$\{X'_1, \dots, X'_n, [X'_{i_1}, X'_{i_2}], [X'_{i_3}, [X'_{i_4}, X'_{i_5}]], \dots\}$$

where

$$1 \leqq i_1, i_2, \dots \leqq n .$$

Define $Y' \equiv T'(y) = \sum_{i=1}^{r} \lambda_i Z'_i$. By duality with $\tilde{T}$, we see that T' is a representation of $\mathfrak{g}$ on the domain D' which is invariant under $T'(\mathfrak{g})$.

It is clear that a representation of finite dimensional Lie algebra is strongly continuous on the domain on which it is defined. Therefore, if $x, y \in \mathfrak{g}$, $t \in \mathbb{R}$ and $\varphi \in D'$.

$$\begin{aligned} A'(tX', Y')\varphi \equiv T'(a(tx, y))\varphi &= T'\left(\sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} (\operatorname{ad} x)^\varrho y\right)\varphi \\ &= \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} ((\operatorname{ad} X')^\varrho Y')\varphi \end{aligned}$$

in the same way one has for $\Psi \in \tilde{D}$.

$$\tilde{A}(t\tilde{X}, \tilde{Y})\Psi \equiv \tilde{T}(a(tx, y))\Psi = \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} ((\operatorname{ad} \tilde{X})^\varrho \tilde{Y})\Psi ,$$

where the two series $\sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} ((\operatorname{ad} X')^\varrho Y')\varphi$ and $\sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} ((\operatorname{ad} \tilde{X})^\varrho \tilde{Y})\Psi$ converge absolutely.

Therefore if $\varphi \in D'$ and $\Psi \in D^*_i (1 \leqq i \leqq n)$ the functions

$$a(t) = \langle A'(-tX'_i, X'_j)\varphi, e^{tX^*_i}\Psi\rangle \quad \text{and} \quad b(t) = \langle \varphi, e^{tX^*_i} X^*_j \Psi\rangle$$

are analytic for every $t \in \mathbb{R}$. As

$$\frac{d^\varrho a}{dt^\varrho}(0) = \sum_{q=0}^{\varrho} \binom{\varrho}{q} \langle((\mathrm{ad} - X'_i)^q X'_j)\, \varphi, (X_i^*)^{\varrho-q} \Psi\rangle$$

and

$$\frac{d^\varrho b}{dt^\varrho}(0) = \langle \varphi, (X_i^*)^\varrho X_j^* \Psi\rangle,$$

the formula

$$X'_j X'^{\varrho}_i = \sum_{q=0}^{\varrho} \binom{\varrho}{q} X'^{q}_i (\mathrm{ad} - X'_i)^{\varrho-q} X'_j$$

implies

$$\frac{d^\varrho a}{dt^\varrho}(0) = \frac{d^\varrho b}{dt^\varrho}(0).$$

Therefore $a(t) = b(t)$ for any real t. This means that

$$\langle A'(-tX'_i, X'_j)\, \varphi, e^{tX_i^*} \Psi\rangle = \langle \varphi, e^{tX_i^*} X_j^* \Psi\rangle \tag{1}$$

for $\varphi \in D'$ and $\Psi \in D_i$.

Since X_i^* is the generator of a strongly continuous one parameter group, the Hille-Yosida theorem (cf. Ref. [3], p. 364) for one parameter groups implies that $\bar{X}_i = X_i^{**}$ is the generator of a strongly continuous one parameter group $t \to e^{t\bar{X}_i}$. Now denote by A_i the generator of the group $t \to (e^{tX_i^*})^*$. We see by duality that $A_i \subset \bar{X}_i$. Moreover, for a given $\varphi \in D(A_i)$ the two functions $u(t) = e^{tX_i^*}\varphi$ and $v(t) = e^{tA_i}\varphi$ are solution of differential equation $\frac{d}{dt} f(t) = \bar{X}_i f(t)$ with the same initial value φ. But, for such a solution $\frac{d}{ds}(e^{(t-s)X_i}) f(t) = 0$ and therefore $f(t) = e^{t\bar{X}_i} f(0)$. Thus $u(t) = v(t)$ and the one parameter groups $e^{t\bar{X}_i}$ and $(e^{tX_i^*})^*$ are equal as they coincide on the dense set $D(A_i)$.

Since $\overline{X^*_{j|D_i^*}} = X_j^*$, we have $(X^*_{j|D_i^*})^* = \bar{X}_j$, and from (1) it follows that $e^{t\bar{X}_i}\varphi$ is in the domain of $\bar{X}_j$ for any real number t and every $\varphi \in D'$ by changing t into $-t$ in (1) one gets:

$$T'(a(tx_i\, x_j))\, \varphi = A'(tX'_i\, X'_j)\, \varphi = e^{t\bar{X}_i} \bar{X}_j e^{-t\bar{X}_i} \varphi \tag{2}$$

for $\varphi \in D'$.

Which becomes after multiplication:

$$A'(tX'_i, X'_{j_1}) \dots A'(tX'_i, X'_{j_\varrho})\, \varphi = e^{t\bar{X}_i} \bar{X}_{j_1} \dots \bar{X}_{j_\varrho} e^{-t\bar{X}_i} \varphi$$

and therefore $e^{t\bar{X}_i} D' \subset D'$.

On the other hand, if $\varphi \in D'$ and $\Psi \in \tilde{D}$ we have:

$$\langle \varphi, \tilde{A}(t\tilde{X}_i, \tilde{X}_j)\,\Psi\rangle = \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} \langle \varphi, ((\operatorname{ad}\tilde{X}_i)^\varrho \tilde{X}_j)\,\Psi\rangle$$
$$= \sum_{\varrho=0}^{\infty} \frac{t^\varrho}{\varrho!} \langle -((\operatorname{ad}X'_i)^\varrho X'_j)\,\varphi, \Psi\rangle .$$

Thus

$$\langle \varphi, \tilde{A}(t\tilde{X}_i, \tilde{X}_j)\,\Psi\rangle = \langle -A'(tX'_i, X'_j)\,\varphi, \Psi\rangle, \quad \varphi \in D', \quad \Psi \in \tilde{D} . \tag{3}$$

It then results from (2) that

$$\langle \varphi, \tilde{A}(t\tilde{X}_i, \tilde{X}_j)\,\Psi\rangle = \langle -e^{t\bar{X}_i}\bar{X}_j e^{-t\bar{X}_i}\varphi, \Psi\rangle .$$

Thus $e^{tX_i^*}\Psi \in D(X_j^*)$ and

$$\tilde{A}(t\tilde{X}_i, \tilde{X}_j)\,\Psi = -e^{-tX_i^*}X_j^* e^{tX_i^*}\Psi, \ \Psi \in \tilde{D} \tag{4}$$

and $e^{tX_i^*}\tilde{D} \subset \tilde{D}$.

It results from (3) that

$$A'(tX'_i X'_j)^* \supset -\tilde{A}(t\tilde{X}_i, \tilde{X}_j) .$$

The formula (4) implies then that the set $D_{ij}^*(t) = e^{-tX_i^*}D_j^*$ is a dense set of analytic vectors for the operator $A'(tX'_i, X'_j)^*$, which is invariant under this operator.

From $D_j^* \subset \tilde{D}$ on infers that $D_{ij}^*(t) \subset \tilde{D}$ and thus

$$D_{ij}^*(t) \subset D(X_k^*), \ k = 1, \dots, n .$$

Since D_j^* is stable under the operators X_k^*, the restriction $\tilde{T}_j$ of $\tilde{T}$ to D_j^* is a representation of $\mathfrak{g}$ leaving D_j^* invariant and as

$$\tilde{T}_j(a(-tx_i, x_k))\,\varphi = \tilde{A}(-t\tilde{X}_i, \tilde{X}_k)\,\varphi \quad \text{when} \quad \varphi \in D_j^* .$$

We have $\tilde{A}(-t\tilde{X}_i, \tilde{X}_k)\,D_j^* \subset D_j^*$ and the formula (4) (with t changed into $-t$) implies that $D_{ij}^*(t)$ is invariant under the operators X_k^* for any real number t. Moreover the last inclusion implies that D_k^* is invariant under the operator $A'(tX'_i, X'_j)^*$.

Moreover, the operator $A'(tX'_i, X'_j)^*$ is the generator of the strongly continuous one parameter group

$$t' \to e^{-tX_i^*}e^{t'X_j^*}e^{tX_i^*} .$$

We now consider the set of vectors $\{u_1, \dots, u_{k+1}\}$ defined by $u_i = x_i$, for $1 \leqq i \leqq n$, and $u_{k+1} = a(t_0 x_{i_0}, x_{j_0})$, with t_0, x_{i_0} and x_{j_0} given.

The adjoints of the operators $U'_p = T'(u_p)$ are the generators of strongly continuous one parameter groups, $U_p'^*$ has a dense set of analytic vectors E_p^* on which all the $U_q'^*$ are defined and $U_q'^* E_p^* \subset E_p^*$.

Then, as before, we have

$\langle T'(a(-tu_p, u_q))\varphi, e^{tU_p'^*}\psi\rangle = \langle\varphi, e^{tU_p'^*}U_q'^*\psi\rangle$, for $\varphi\in D'$ and $\psi\in E_p^*$.

D′ being invariant under $e^{t\bar{U}_p'}$ and $(U'^*_{q\,|E_p^*})^* \supset \bar{U}_p'$ we have

$$T'(a(tu_p, u_q)) = e^{t\bar{U}_p'}\,U_q'\,e^{-t\bar{U}_p'}.$$

The conclusion results then by induction from Lemma 1.

Theorem. *Let T be a representation of $\mathfrak{g}$ defined on a dense set $D\subset X$ which is invariant under $T(\mathfrak{g})$. A sufficient condition for T to be the differential of a unique representation $\mathcal{T}$ of G on X is that there exists a set of generators $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$ satisfying the properties a) and b) of Lemma 2.*

If there exists a set of generators of $\mathfrak{g}$ satisfying the properties a) and b), there exists a basis y_1, y_r satisfying the conclusions of Lemma 2. We can then follow the demonstration given in Ref. [1], Theorem 1, the main steps of which we present here.

There exists an open set W in G, diffeomorphic by the exponential to an open neighbourhood U of 0 in $\mathfrak{g}$, satisfying.

1) Any $z\in W$ can be written $z=e^x=e^{t_1y_1}\ldots e^{t_ry_r}$ where $x\in U$ is uniquely defined and the function $z\to(t_1,\ldots,t_r)$ is a coordinate system on W.

2) If $e^x\in W$ and $e^xe^y\in W$ then $e^{tx}e^y\in W$ for $0\leqq t\leqq 1$.

One easily verifies ([1], formulas (1) and (2)) that if e^x, e^y, $e^xe^y\in W$, $e^{tx}=e^{t_1y_1}\ldots e^{t_ry_r}$, $e^{tx}e^y=e^{\alpha_1y_1}\ldots e^{\alpha_ry_r}$ where $t_i=t_i(t)$ and $\alpha_i=\alpha_i(t)$ $(0\leqq t\leqq 1)$ and $e^y=e^{\beta_1y_1}\ldots e^{\beta_ry_r}$, we have the formulas:

$$\left.\begin{aligned} x &= \frac{dt_1}{dt}y_1+\cdots+\frac{dt_r}{dt}a(t_1y_1, a(t_2y_2,\ldots,a(t_{r-1}y_{r-1},y_r)\ldots)\\ x &= \frac{dt_1}{dt}a(-t_ry_r, a(-t_{r-1}y_{r-1},\ldots,a(-t_2y_2,y_1)\ldots)+\cdots+\frac{dt_r}{dt}y_r\\ x &= \frac{d\alpha_1}{dt}y_1+\cdots+\frac{d\alpha_r}{dt}a(\alpha_1y_1,a(\alpha_2y_2,\ldots,a(\alpha_{r-1}y_{r-1},y_r)\cdots)\end{aligned}\right\} \quad (5)$$

We define, for $e^x\in W$, $\mathcal{T}(e^{tx})=e^{t_1\bar{Y}_1}\ldots e^{t_r\bar{Y}_r}$. We easily see that for $\varphi\in D'$ the functions $f(t)=e^{t_1\bar{Y}_1}\ldots e^{t_r\bar{Y}_r}\varphi$ and $h(t)=e^{\alpha_1\bar{Y}_1}\ldots e^{\alpha_r\bar{Y}_r}e^{-\beta_r\bar{Y}_r}\ldots\ldots e^{-\beta_1\bar{Y}_1}\varphi$ have a first order derivative for $0\leqq t\leqq 1$ and have their values in D'. It then results from (0) and (5) that they are solutions of the differential equation $\frac{df}{dt}=X'f(t)$ with the same initial value φ, and $\frac{d}{dt}\mathcal{T}(e^{tx})\varphi=\mathcal{T}(e^{tx})X'\varphi$. Therefore $f(t)=h(t)$ for $0\leqq t\leqq 1$ by the same arguments as in Lemma 2, and in particular $\mathcal{T}(e^xe^y)\varphi$

$= \mathscr{T}(e^x)\,\mathscr{T}(e^y)\,\varphi$ if $\varphi \in D'$ and then for $\varphi \in X$ by continuity. By finite products $\mathscr{T}$ can be extended (from W) to a representation of G. From its definition the differential of $\mathscr{T}$ is equal to T on D, and $\mathscr{T}$ is unique.

Remark. In the theorem, the hypothesis $\overline{X_{i|D_i^*}^*} = X_i^*$ may be suppressed. Indeed, denote by $H_i = \overline{X_{i|D_i^*}^*}$; we have $H_i \subset X_i^*$. Since H_i is closed the serie expansion of $e^{tX_i^*}x$ for t small enough and $x \in D_i^*$ gives $e^{tX_i^*}H_i x = H_i e^{tX_i^*}x$ with $e^{tX_i^*}x \in D(H_i)$. By finite products the same holds for any real numbers t. For the same reason, since for any $y \in D(H_i)$, there exists a sequence $x_n \in D_i^*$ converging to y and such that $X_i^* x_n \to H_i y$, we have then $e^{tX_i^*}y \in D(H_i)$ and $e^{tX_i^*}H_i y = H_i e^{tX_i^*}y$. Therefore for every $y \in D(H_i)$ (cf. [3], Theorem 3.3.2).

$$t^{-1}(e^{tX_i^*}y - y) = t^{-1}\int_0^t H_i e^{sX_i^*}y\,ds = H_i\left(t^{-1}\int_0^t e^{sX_i^*}y\,ds\right). \tag{6}$$

Now, since H_i is closed and $D(H_i)$ is dense, the equality of the two ends of relation (6) extends to all y in X^*. We have $t^{-1}\int_0^t e^{sX_i^*}y\,ds \to y$ when $t \to 0$ for any $y \in X^*$, and therefore, when $y \in D(X_i^*)$, we obtain (when we take $t \to 0$ in (6)) that $X_i^* y = H_i y$. The following corollaries can then immediately be deduced.

Corollary 1. *Let T be a representation of $\mathfrak{g}$ defined on a dense domain $D \subset X$, invariant under $T(\mathfrak{g})$.*

A sufficient condition for T to be the differential (on D) of a unique strongly continuous representation of G on X is that there exists a set of generators $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$ such that

a) *there exists a dense domain $D^* \subset X^*$ of analytic vectors for the adjoints X_i^* of the operators $X_i = T(x_i)$ $(1 \leqq i \leqq n)$ invariant under the operators X_i^*.*

b) *The operators X_i^* are generators of strongly continuous one parameter groups on X^*.*

Corollary 2. *Let T be a representation of $\mathfrak{g}$ defined on a dense domain D in a Hilbert space H, invariant under $T(\mathfrak{g})$, by skewsymmetric operators. Suppose that there exists a set of generators $\{x_1, \ldots, x_n)$ of $\mathfrak{g}$ such that D is a domain of analytic vectors for the operators $X_i = T(x_i)$ $(1 \leqq i \leqq n)$ then T is the differential (on D) of a unique unitary representation of G on H.*

This is an immediate consequence of the above corollary and of Lemma 5.1 of the Ref. [4] which implies that the operators X_i^* are skewadjoint.

Corollary 3. *Let X_1 and X_2 be two skewadjoint operators in a Hilbert space H. Suppose that there exists a common dense domain D of analytic vectors for X_1 and X_2, invariant under X_1 and X_2 and that the operators*

$X_{1|D}$ and $X_{2|D}$ generate a finite dimensional real Lie algebra of operators defined on D, then $X_1 + X_2$ (defined on $D(X_1) \cap D(X_2)$) is an essentially skewadjoint operator.

The demonstration is a trivial consequence of Corollary 2.

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