Commun. math. Phys. 27, 103—136 (1972)

The Phase Separation Line in the Two-Dimensional Ising Model*

GIOVANNI GALLAVOTTI
I.H.E.S. Bures sur Yvette, France

Received December 17, 1971

Abstract. We prove that, at low temperature, the line of separation between the two pure phases shows large fluctuations in shape. This implies the translation invariance of the correlation functions associated with some non translation invariant boundary conditions and should be a peculiarity of the dimensionality of the model.

1. The Line of Separation

It has recently been conjectured that the surface of separation between two pure phases is, at low temperature and for short range potential models, rigid in the case of a 3-dimensional model and non rigid in 2-dimensional models [1, 2].

In this paper we prove the truth of the conjecture in the 2-dimensional Ising model.

The precise meaning of what "surface of separation" and "rigid" mean will be given below and has already been discussed in the literature [3].

Let Ω be a $N \times N$ square lattice centered at the origin: let $i = 1, 2, \ldots, N^2$ be a label for the center of each unit square composing Ω. We assume that on each site $i \in \Omega$ is located a spin $\sigma_i = \pm 1$ and that the energy of a spin configuration $\varrho = (\sigma_1, \ldots, \sigma_{N^2})$ is given by:

$$H_N(\sigma) = -\frac{1}{2} \sum_{\langle ij \rangle} \sigma_i \sigma_j - \frac{1}{2} \sum_{i \in \partial^+ \Omega} \sigma_i + \frac{1}{2} \sum_{i \in \partial^- \Omega} \sigma_i \tag{1.1}$$

where $\sum_{\langle ij \rangle}$ means, as usual, sum over the pairs of nearest neighbour couples of points in Ω and $\partial^+ \Omega$ ($\partial^- \Omega$) denote the points adjacent to the upper half (lower half) of the boundary $\partial \Omega$ of Ω.

The physical meaning of (1.1) is that $H_N(\sigma)$ corresponds to the energy of a configuration of spins interacting through a nearest neighbour pair potential and, also, interacting with a set of external fixed spins adjacent

* This work has been partially supported by the Consiglio Nazionale delle Ricerche (Gruppo Nazionale per l'Analisi Funzionale): CNR(GNAFA).

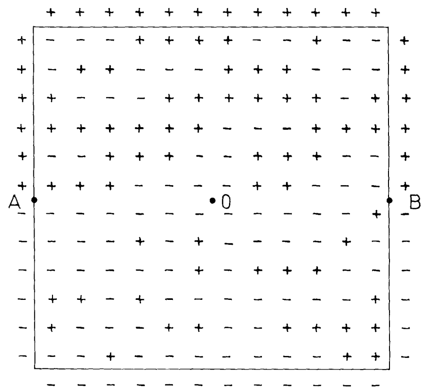

Fig. 1.

to $\partial\Omega$ from the outside: the spins adjacent to $\partial^+\Omega$ being $+1$ and the ones adjacent to $\partial^-\Omega$ being -1.

A picture of this situation is given in Fig. 1 where the boundary condition is illustrated together with a possible spin configuration (in the picture Ω is a 12×12 box).

As usual it will be much more convenient to describe a spin configuration through the lines of separation between regions containing opposite spins. To do so we draw a line on the lattice bonds which separate opposite spins: the set of lines thus obtained splits into several connected components and, at each vertex of the lattice will end 0, 2, 4 lines with the important exception of the vertices A, B (see Fig. 1) where 1 or 3 lines will end.

For an example of such a construction see Fig. 2 where the lines corresponding to the spin configuration of Fig. 1 are drawn.

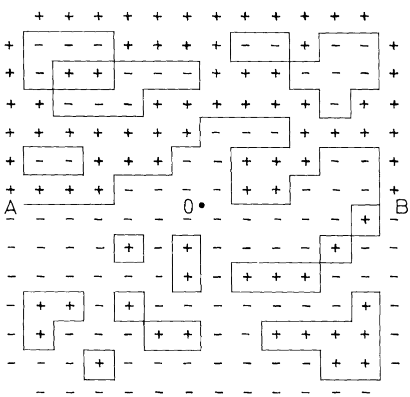

Fig. 2.

We shall consistently call $\gamma_1, \ldots, \gamma_n$ the connected components of the set of lines corresponding to a given spin configuration and which do not contain the points A, B; we shall call λ the component containing A, B (it is easily realized that the component containing A must also contain B).

It is clear that there is a one-to-one correspondence between the "contour configurations" and the spin configurations: a contour configuration being a set $\gamma_1, \ldots, \gamma_n, \lambda$ of $(n+1)$ disjoint sets of lines in Ω such that n of them $(\gamma_1, \ldots, \gamma_n)$ are "closed" (i.e. every vertex $x \in \gamma_i$ belongs to two or four lines of γ) and the other λ is "open" with end points A and B (i.e. every vertex $x \in \lambda$ belongs to two or four lines unless $x = A$ or $x = B$).

It is very important to notice that, if $|\gamma_i|, |\lambda|$ denote the "lengths" of γ_i, λ, then the energy of a spin configuration $\sigma = (\gamma_1, \ldots, \gamma_n, \lambda)$ is given by (see (1.1)):

$$H_N(\sigma) = C_N + |\lambda| + \sum_{i=1}^{n} |\gamma_i| \tag{1.2}$$

where C_N is a suitable constant (i.e. a σ-independent object).

The grand canonical ensemble corresponds to assigning the configuration $\sigma = (\gamma_1, \ldots, \gamma_n, \lambda)$ the probability:

$$p(\lambda, \gamma_1, \ldots, \gamma_n) = \frac{\exp - \beta|\lambda| - \beta \sum_i |\gamma_i|}{Z(\Omega, \beta)} \tag{1.3}$$

where the normalization factor $Z(\Omega, \beta)$ (gran canonical partition function) is:

$$Z(\Omega, \beta) = \sum_{\lambda, \gamma_1, \ldots, \gamma_n} \exp - \beta|\lambda| - \beta \sum_i |\gamma_i| \,. \tag{1.4}$$

At low temperature the system "exhibits long range correlations" corresponding to the fact that there are two possible equilibrium states [4]. The fact that the spins on the upper half of the boundary are fixed to $+1$ will favor the formation of the pure phase with positve magnetization in the upper half of Ω while, for the same reasons, the negatively magnetized phase will be favoured in the lower half of Ω.

These intuitive ideas are put in a precise form by the following theorem: [5, 9]

Theorem (Minlos-Sinai). *If β is large enough then a configuration $\sigma = (\lambda, \gamma_1, \ldots, \gamma_n)$ chosen randomly out of the grand canonical ensemble will have properties i), ii), iii) listed below with a probability approaching* 1 *as $N \to \infty$ (we use the notation $|\Omega| = N^2$):*

i) λ is such that

$$||\lambda| - N| < \frac{C}{\beta} N \quad \textit{for some} \quad C > \log 4 . \tag{1.5}$$

ii) $|\gamma_i| \leqq C_0 \log \Omega$ *for some* $C_0 > 0$. (1.6)

iii) if m^* *denotes the spontaneous magnetization* [6] *and* M_λ^+ (M_λ^-) *denotes the total magnetization (i.e. the sum of the spins) above (below)* λ *then:*

$$|M_\lambda^{\pm} \mp m^* \tfrac{1}{2} |\Omega|| \leqq \varkappa(\beta) \sqrt{|\Omega|} \tag{1.7}$$

where $\varkappa(\beta) \to 0$ *exponentially fast.*

The contour λ will be called the "line of separation" between the two phases: this line is almost straight, because of i), with very large probability.

In the next two sections we introduce a more precise notion of "straight line" and discuss other known results which will be the basis for our investigation of "how straight" λ is.

2. When a Random Line is Straight

Denote Ω_λ^a and Ω_λ^b the regions above and below λ and put:

$$Z_0(\Omega_\lambda^{(i)}, \beta) = \sum_{(\gamma_1, \ldots, \gamma_n) \in \Omega_\lambda^{(i)}} \exp - \beta \sum_i |\gamma_i| \qquad i = a, b \tag{2.1}$$

where the sum runs over the spin configurations above or below λ (if $i = a$ or b) described by contours $\gamma_1, \ldots, \gamma_n$ with no points in common with λ.

In terms of (2.1) one can write the probability $p_N(\lambda)$ that the separation line of a spin configuration coincides with λ as (see (1.3)):

$$p_N(\lambda) = \frac{e^{-\beta|\lambda|} Z_0(\Omega_\lambda^{(a)}, \beta) Z_0(\Omega_\lambda^{(b)}, \beta)}{\sum_{\lambda'} e^{-\beta|\lambda'|} Z_0(\Omega_{\lambda'}^{(a)}, \beta) Z_0(\Omega_{\lambda'}^{(b)}, \beta)} . \tag{2.2}$$

We shall say that the phase separation line λ is "rigid" or "straight" if the probability, after (2.2), that λ passes through a fixed point x does not tend to zero as $N \to \infty$ for some x; in the opposite case we shall say that the line λ is "loose" or "non rigid"

We stress that in the above definition of rigidity, the point x is held fixed as $N \to \infty$ and, therefore, x is at a fixed distance from the center 0 of Ω.

We shall prove that, in the model under discussion, the line λ is loose at low temperature, therefore λ will pass "very far" from any fixed region Q with a probability tending to 1 as $N \to \infty$. In other words we shall prove that there is a function D(N), $D(N)\xrightarrow[N\to\infty]{}\infty$, such that the probability that the distance $d(\lambda, Q)$ exceeds D(N) tends to 1 as $N\to\infty$. The best choice for D(N) will roughly be proportional to $\sqrt{N}$.

It is clear that the techniques used in Ref. [4] combined with such a result will imply that the correlation functions $\langle\sigma_{x_1}, \ldots, \sigma_{x_n}\rangle_\Omega$ will have a limit as $N\to\infty$, at fixed $x_1, \ldots, x_n$, and

$$\lim_{N\to\infty}\langle\sigma_{x_1}, \ldots, \sigma_{x_n}\rangle_\Omega = \tfrac{1}{2}(\langle\sigma_{x_1}, \ldots, \sigma_{x_n}\rangle^+ + \langle\sigma_{x_1}, \ldots, \sigma_{x_n}\rangle^-) \tag{2.3}$$

where $\langle\ \rangle^+, \langle\ \rangle^-$ denote the correlation functions of the two pure phases [4].

Hence the correlation functions associated with the (non translationally invariant) hamiltonian (1.1) will be translationally invariant and, we observe, this phenomenon is quite remarkable since it is not expected to happen in 3-dimensions.

The technique we use can be easily understood from a physical point of view: we picture λ as a sequence of "jumps" each of which is interpreted as a particle of a multicomponent one dimensional lattice gas. Thus a line λ will be regarded as a gas configuration. We show that the gas in question is almost perfect and reduce the problem of the rigidity to the investigation of the fluctuations of an almost perfect gas. We prove, for the needed fluctuations, a "local central limit theorem" following Gnedenko's ideas (with appropriate modifications) [11] and the results, properly reinterpreted, will mean that λ is not rigid.

A more clear and precise idea of the above scheme can be gotten by reading the next two sections.

Our proof of the local central limit theorem can be transformed into a proof of the local central limit theorem for Markov processes which seems to be quite different from Kolmogorov's proof [12]: it is weaker because it seems to need more conditions on the process but seems to apply to more general situations. This remark will be clear to the reader familiar with Gnedenko's and Kolmogorov's theorems and with Spitzer's work on the isomorphism between certain Markov processes and classical lattice gases [13]. We shall pursue this point in a subsequent paper.

3. Technicalities

To proceed we need a more handable form for (2.2).

Let $\mathcal{N}$ be the family of all sets $\Gamma = (\gamma_1, \ldots, \gamma_n)$ of closed (in the sense of § 1) lines lying on the infinite square lattice and such that the set $\{\gamma_1\}\cup\{\gamma_2\}\cup\cdots\cup\{\gamma_n\}$ is connected. Notice that we are not only considering sets Γ of overlapping contours but we even allow identical contours to be part of the same $\Gamma\in\mathcal{N}$.

The knowledge of the statements of the following theorem will be fundamental for understanding this paper:

Theorem 2. *there is a function* $\varphi^T(\Gamma)$ *defined on* $\mathcal{N}$ *such that:*

1) $\varphi^T(\Gamma)$ *is translationally invariant,*

2) *if* $N(\Gamma) =$ *(number of contours in* Γ*) then:*

$$(-1)^{N(\Gamma)-1}\varphi^T(\Gamma) \geqq 0, \tag{3.1}$$

3) *there is a function* $\delta(\beta)\xrightarrow[\beta\to\infty]{}0$ *exponentially fast such that:*

$$\sum_{\Gamma \ni p} |\varphi^T(\Gamma)| \leqq \delta(\beta) \tag{3.2}$$

where the sum is over the $\Gamma \in \mathcal{N}$ *containing the vertex* p.

4) *there exists a function* $\varkappa(\beta)$ *tending to zero exponentially fast and such that the following inequality holds:*

$$\sum_{\substack{\Gamma \ni \gamma \\ N(\Gamma)=n+1}} |\varphi^T(\Gamma)| \leqq \varkappa(\beta)^{n+1} e^{-\frac{1}{2}\beta|\gamma|}, \tag{3.3}$$

5) *if* p *is a vertex of the lattice and* Q *is a set of lattice vertices at a distance* d(p, Q) *from* p *then*

$$\sum_{\substack{\Gamma \ni p \\ \Gamma \cap Q \neq \emptyset}} |\varphi^T(\Gamma)| \leqq \varkappa(\beta)\,(4e^{-\frac{1}{2}\beta})^{\sqrt{d(p,Q)}} \tag{3.4}$$

6) *Using definition* (2.1) *one has:*

$$\frac{Z_0(\Omega_\lambda^{(a)}, \beta) Z_0(\Omega_\lambda^{(b)}, \beta)}{Z_0(\Omega, \beta)} = \exp - \sum_{\substack{\Gamma i \lambda \\ \Gamma \subset \Omega}} \varphi^T(\Gamma) \tag{3.5}$$

where $\Gamma i \lambda$ *means that* $\{\Gamma\} \cap \lambda \neq \emptyset$.

7) *Let* $Y = (y_1, \ldots, y_m)$ *be a set of distinct vertices of the lattice and define*

$$\Phi_0(Y) = \sum_{\Gamma \ni Y} |\varphi^T(\Gamma)| \tag{3.6}$$

where $\Gamma \ni Y$ *means that* $(y_1, \ldots, y_m)$ *are vertices of contours in* Γ, *then one finds:*

$$\begin{cases} \psi(\beta) = \sum\limits_{Y \ni O} \Phi_0(Y) < \infty \\ \sum\limits_{Y \ni O,\, \mathrm{diam}\, Y \geqq d} \Phi_0(Y) \leqq R_0^{-d}\psi(\beta) \quad \text{with} \quad R_0 = 100 \end{cases} \tag{3.7}$$

and $\psi(\beta) \to 0$ *exponentially as* $\beta \to \infty$.

The above theorem is proven with a slightly different notation in Ref. [9] appendix A: the few necessary changes in the notations are discussed in Appendix 1 of this paper.

Clearly (3.5) allows us to write the probability distribution (2.2) in the ensemble $\mathfrak{U}(N)$ of the "lines" from A to B as:

$$p_N(\lambda) = \frac{e^{-\beta|\lambda|} e^{-\sum_{\substack{\Gamma i \lambda \\ \Gamma \supset \Omega}} \varphi^T(\Gamma)}}{(\text{normalization})}. \tag{3.8}$$

We now introduce several auxiliary ensembles of "lines" λ starting in A, ending in some point B′ on the vertical line through B and lying in the vertical strip with base the segment [A, B]. We remember that "open lines starting in A and ending in B′" means a connected set λ of lattice bonds such that every vertex $p \in \lambda$ belongs to two or four bonds of λ except the vertices A, B′ which must belong to one or three bonds of λ.

The auxiliary ensembles are:

I) The ensemble $\mathfrak{U}_0(N)$: it is the set of lines λ from A to B such that

$$||\lambda| - N| < \frac{C}{\beta} N$$

where C is the constant introduced in Theorem 1. The relative weight of $\lambda \in \mathfrak{U}_0(N)$ will be, by definition:

$$\mathscr{W}_0(\lambda) = \exp - \beta|\lambda| - \sum_{\substack{\Gamma i \lambda \\ \Gamma \subset \Omega}} \varphi^T(\Gamma).$$

Notice that, as set of lines, $\mathfrak{U}_0(N)$ is contained in $\mathfrak{U}(N)$ for large β.

II) The ensemble $\tilde{\mathfrak{U}}_i(N)$: it consists of the lines λ starting in A and ending at the point B_i at height i above B ($i = 0, \pm 1, \ldots$). We also require that the elements $\lambda \in \tilde{\mathfrak{U}}_i(N)$ lie in the vertical strip I_N with base the segment [A, B]. By definition the weight of $\lambda \in \tilde{\mathfrak{U}}_i(N)$ will be

$$\tilde{\mathscr{W}}_i(\lambda) = \exp - \beta|\lambda| - \sum_{\substack{\Gamma i \lambda \\ \Gamma \subset I_N}} \varphi^T(\Gamma) \tag{3.9}$$

III) The ensemble $\tilde{\mathfrak{U}}(N)$ is, as a set, given by $\bigcup_{i=-\infty}^{\infty} \tilde{\mathfrak{U}}_i(N)$ and the relative weight of a configuration $\lambda \in \tilde{\mathfrak{U}}(N)$ is, by definition:

$$\tilde{\mathscr{W}}(\lambda) = \exp - \beta|\lambda| - \sum_{\substack{\Gamma i \lambda \\ \Gamma \subset I_N}} \varphi^T(\Gamma) \tag{3.10}$$

(i.e. the same as (3.9)).

It is easy to prove from part i) of the Theorem 1 and (3.3), (3.4), that the line λ is loose in $\mathfrak{U}(N)$ if and only if it is loose in $\tilde{\mathfrak{U}}_0(N)$. The simple proof of this fact can be found in Appendix 2.

From now on we concentrate in proving that λ is losse in $\tilde{\mathfrak{U}}_0(N)$.

4. Geometric Description of $\lambda \in \tilde{\mathfrak{U}}(N)$

This section is devoted to a detailed but purely geometrical description of the elements $\lambda \in \tilde{\mathfrak{U}}(N)$: all the concepts introduced become very clear if one checks, as they are introduced, what they mean for the particular line of Fig. 3.

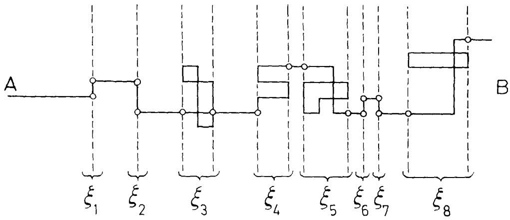

Fig. 3.

Let $\lambda \in \tilde{\mathfrak{U}}(N)$ (see Fig. 3). Consider the $(N+1)$ vertical lattice lines passing through the $(N+1)$ lattice points of the segment $[A, B]$.

A vertical line will be identified with the point $q \in [A, B]$ through which it passes.

A vertical line will have in common with λ at least one point (in fact it will, in general, have in common with λ several isolated points plus several disjoint segments).

Let $q_1, \ldots, q_t$ be the vertical lines which have in common with λ more than one point; group $q_1, \ldots, q_t$ into clusters $\xi_1, \xi_2, \ldots, \xi_s$: a cluster consists of a set of adjacent lines which intersect λ in more than one point and such that any other vertical line drawn between the two extreme lines of a cluster and *not lying* on the lattice will intersect λ in more than one point (notice that such vertical lines intersect λ in an odd number of points). In Fig. 3 we have drawn the the extreme lines of each cluster.

A cluster ξ consists of a set of $(k+1)$ consecutive points $(q_o, q_o+1, \ldots, q_o+k)$ $k \geqq 0$.

We denote $\mathscr{S}_\xi$ the "shape" of the part of λ above a cluster ξ.

Clearly the set of clusters $\xi_1, \ldots, \xi_s$ together with the associated shapes $\mathscr{S}_{\xi_1}, \ldots, \mathscr{S}_{\xi_s}$ completely determine λ. Viceversa we can give arbitrarily a set of disjoint clusters $\xi_1, \ldots, \xi_s$ and of associated shapes $\mathscr{S}_{\xi_1}, \ldots, \mathscr{S}_{\xi_s}$ and uniquely construct a line $\lambda \in \tilde{\mathfrak{U}}(N)$. This representation for λ is due to van Beyeren [14].

It is important, in order to get a clear intuitive picture of λ, to remark that a line λ can be interpreted as a configuration of a multicomponent lattice gas: the particles of the lattice are as many as the shapes above the clusters; there is a hard core which forbids the overlapping of the clusters.

Clearly there is a one-to-one correspondence between the lattice gas configurations compatible with the hard cores and the lines $\lambda \in \tilde{\mathfrak{U}}(N)$.

To each "shape" $\mathscr{S}_\xi$ (or "particle" in the lattice gas language) we associate several "form parameters" and other related notions:

1) The "first" and the "last" point of $\mathscr{S}_\xi$: these points are marked in Fig. 3 and their definition in the general case is easy to infer from this particular case.

2) The "jump", denoted $\delta\mathscr{S}_\xi$, which is the difference in height between the first and the last point of a cluster $\mathscr{S}_\xi$:

$$\delta\mathscr{S}_\xi = 0, \pm 1, \pm 2, \dots .$$

3) The "basis length" $|\xi|$ of ξ is defined as $|\xi| = k$ if

$$\xi = (q_o, q_o + 1, \dots, q_o + k); \qquad k = 0, 1, \dots .$$

4) The "vertical length" at $q \in \xi$ (denoted by $V_q(\mathscr{S}_\xi)$) which is the length of the intersection between λ and a vertical line $q \in \xi$. Remark that if $\xi \equiv q_o$ then $|\delta\mathscr{S}_\xi| = V_{q_o}(\mathscr{S}_\xi)$.

5) The "horizontal length" at $q \in \xi$ (denoted by $h_q(\mathscr{S}_\xi)$) which is one unit less than the number of intersections of $\mathscr{S}_\xi$ with any vertical line between q and $q+1$ if $q = q_o + i$ $i = 0, 1, \dots, k-1$. For $q = q_o + k$ we put $h_q(\mathscr{S}_\xi) = 0$.

6) The "total vertical length" $V(\mathscr{S}_\xi)$ defined as

$$V(\mathscr{S}_\xi) = \sum_{q \in \xi} V_q(\mathscr{S}_\xi) . \tag{4.1}$$

7) The "total horizontal length" $h(\mathscr{S}_\xi)$ defined as

$$h(\mathscr{S}_\xi) = \sum_{q \in \xi} h_q(\mathscr{S}_\xi) . \tag{4.2}$$

8) The "excess length" $|\mathscr{S}_\xi|_q$ at q:

$$|\mathscr{S}_\xi|_q = h_q(\mathscr{S}_\xi) + V_q(\mathscr{S}_\xi) . \tag{4.3}$$

9) The "excess length":

$$|\mathscr{S}_\xi| = h(\mathscr{S}_\xi) + V(\mathscr{S}_\xi) . \tag{4.4}$$

10) The distance of $q \in \xi$ from the extremes of $\xi : d_\xi(q)$.

Clearly if $\lambda = (\xi_1, \dots, \xi_n, \mathscr{S}_{\xi_1}, \dots, \mathscr{S}_{\xi_n})$ we have:

$$|\lambda| = N + \sum_{i=1}^{n} |\mathscr{S}_{\xi_i}| . \tag{4.5}$$

In the next section we push further the analogy with the multicomponent gas and show that the problem of studying the probability distributions of λ in $\tilde{\mathfrak{U}}(N)$ is equivalent to the problem of studying the equilibrium state of the multicomponent lattice gas at certain component activities and under the influence of suitable (many body) potentials.

5. Shape Potentials

Let $X = (\xi_1, \ldots, \xi_n)$ be a set of disjoint clusters and denote $\mathscr{S}_X = (\xi_1, \ldots, \xi_n, \mathscr{S}_{\xi_1}, \ldots, \mathscr{S}_{\xi_n})$ a line $\lambda \in \tilde{\mathfrak{U}}(N)$.

As already mentioned we can interpret $\mathscr{S}_X$ as a configuration of a multicomponent gas. We denote $|\mathscr{S}_X| = \sum_i |\mathscr{S}_{\xi_i}|$.

If $X = T \cup T'$ and $T \cap T' = \emptyset$ we shall also write, with obvious meaning of the symbols, $\mathscr{S}_X = \mathscr{S}_{T \cup T'} = \mathscr{S}_T \cup \mathscr{S}_{T'}$.

Let $\lambda \equiv \mathscr{S}_X$ and interpret the quantity $-\sum_{\Gamma \notin \lambda} \varphi^T(\Gamma)$ in (3.10) as a potential energy of the configuration S_X by writing:

$$\sum_{\Gamma \notin \lambda} \varphi^T(\Gamma) = \sum_{\Gamma \notin [A,B]} \varphi^T(\Gamma) + U(S_X). \tag{5.1}$$

We wish to think $U(\mathscr{S}_X)$ as a sum of many body contributions as:

$$U(\mathscr{S}_X) = \sum_{T \subset X} \Phi(\mathscr{S}_T) \tag{5.2}$$

this is certainly possible and, as a matter of fact, the potentials $\Phi(\mathscr{S}_T)$ are given by the (rather useless) formula:

$$\Phi(\mathscr{S}_T) = \sum_{X \subset T} (-1)^{N(X)} U(\mathscr{S}_X) \tag{5.3}$$

(Möbius inversion formula).

The potentials $\Phi(\mathscr{S}_T)$ will be called, for obvious reasons, shape potentials: they verify the following lemma:

Lemma 1. *If β is large enough the shape potentials verify the following inequality:*

$$|\Phi(\mathscr{S}_X)| \leqq \Phi_0(X)\, |\mathscr{S}_\xi| \tag{5.4}$$

where $\xi \in X$ is either the first or the last cluster in X (i.e. is either to the left of all the other clusters in X or is to the right of all the others).

Furthermore $\Phi_0(X)$ is a translationally invariant function of X and

$$\sup_{\xi_0, T} \sum_{\substack{X \ni \xi_0 \\ X \subset T}} \Phi_0(X) = \psi(\beta) \tag{5.5}$$

where the sup is taken over the allowed (i.e. without overlappings) cluster configurations T which contain ξ_0 and, then, over ξ_0. The function $\psi(\beta)$

in (5.5) *can be taken to be the same as the one in* (3.8) *(hence it tends to zero exponentially as* $\beta \to \infty$*).*
Finally, for $R_o = 100$:

$$\sum_{\substack{X \ni \xi_o \\ X \ni \xi_1}} \Phi_o(X) \leqq 24 (8 R_o^{-1})^{d(\xi_o, \xi_1)} \psi(\beta) \tag{5.6}$$

where $d(\xi_o, \xi_1) =$ *distance of* ξ *from* ξ_o.

The above Lemma 1 is proven in Appendix 3 and is, if course, a basic ingredient to our proof. In fact let us first remark that, because of (3.10), (4.5), (5.1), (5.2), we can interpret the set of lines $\lambda \in \tilde{\mathfrak{U}}(N)$ as a multi-component lattice gas in which a component $\mathscr{S}_\xi$ has an activity $z_{\mathscr{S}_\xi} = \exp - \beta |\mathscr{S}_\xi|$ and in which the interaction between the elements of the gas is described by hard cores and the shape potential: in fact the weight (3.10) is proportional to

$$\tilde{\mathscr{W}}(\lambda) \propto \exp - \beta |\mathscr{S}_X| - \sum_{T \subset X} \Phi(\mathscr{S}_T). \tag{5.7}$$

Lemma 1 tells us that the potentials $\Phi(\mathscr{S}_T)$ are small at low temperature (see (5.5)) and have short range (see (5.6)). Furthermore at low temperature the activities of the components become very small. Hence we can hope that the gas of shapes is almost perfect at low temperature and we shall relate this fact to the non rigidity of λ in $\tilde{\mathfrak{U}}_o(N)$.

We devote the next two sections to make more precise the statement that the gas of shapes is almost perfect: we shall manage to do so with the help of the generalized Kirkwood-Saltzburg equations and the associated cluster expansions (see Ref. [7]).

6. The Gas of Shape Particles is Almost Perfect

We define, as usual (see Ref. [7]), if $X = (\xi_1, \ldots, \xi_n)$ and $\xi_1 < \xi_2 < \cdots < \xi_n$ (in the sense that the clusters are numbered from left to right)

$$U_1(\mathscr{S}_X) = \sum_{\substack{T \subset X \\ T \ni \xi_1}} \Phi(\mathscr{S}_T), \tag{6.1}$$

$$W_1(\mathscr{S}_X, \mathscr{S}_Y) = \sum_{\xi_1 \in T \subset X} \Phi(\mathscr{S}_{T \cup Y}) \qquad \emptyset = Y \cap X,\ Y \neq \emptyset, \tag{6.2}$$

$$I_1(\mathscr{S}_X, \mathscr{S}_Y) = \sum_{R \subset Y} W_1(\mathscr{S}_X, \mathscr{S}_R) \qquad Y \neq \emptyset,\ Y \cap X = \emptyset \tag{6.3}$$

$$K_1(\mathscr{S}_X, \mathscr{S}_Y) = \begin{cases} \displaystyle\sum_{n \geqq 1} \sum_{\substack{(P_1, \ldots, P_n) \\ \cup_i P_i = Y}} \prod_{i=1}^{n} (e^{-W_1(\mathscr{S}_X, \mathscr{S}_{P_i})} - 1) & \text{if } Y \neq \emptyset \\ 1 & Y = \emptyset, \end{cases} \tag{6.4}$$

here $\sum_{\substack{(P_1,\ldots,P_n)\\ \cup_i P_i = Y}}$ runs over all the n^{ples} of different subsets of Y such that $\cup_i P_i = Y$ (notice that we do not require $P_i \cap P_j = \emptyset$ $i \neq j$) and such that $P_i \neq \emptyset$ $i = 1, 2, \ldots, n$.

Define the correlation functions for the gas of shape-particles in [A, B] as

$$\varrho_N(\mathscr{S}_X) = \frac{\sum_{Y \cap X = \emptyset} \sum_{\mathscr{S}'_Y} e^{-\beta|\mathscr{S}_X| - \beta|\mathscr{S}'_Y| - U(\mathscr{S}_X \cup \mathscr{S}'_Y)}}{\text{normalization}} \tag{6.5}$$

where the sum over Y runs over the sets of clusters in [A, B] compatible with $X \subset [A, B]$ and non overlapping.

The correlation functions (6.5) verify (as in [7]) the generalized Kirkwood-Salsburg equations: let $\xi_1 \in X$ then, if $X^{(1)} = (\xi_2, \ldots, \xi_n)$ we have:

$$\begin{aligned} \varrho^N(\mathscr{S}_X) = e^{-\beta|\mathscr{S}_{\xi_1}| - U_1(\mathscr{S}_X)} \sum_{Y \cap X = \emptyset} \sum_{\mathscr{S}'_Y} K_1(\mathscr{S}_X, \mathscr{S}'_Y) \\ \sum_{\substack{\overline{P \cap \xi_1 \neq \emptyset} \\ P \cap (X^{(1)} \cup Y) = \emptyset}} \sum_{\mathscr{S}''_P} (-1)^{N(P)} \varrho(\mathscr{S}_{X^{(1)}} \cup \mathscr{S}'_Y \cup \mathscr{S}''_P) \end{aligned} \tag{6.6}$$

where all the sums run over sets of clusters in [A, B] and where $\overline{P \cap \xi_1 \neq \emptyset}$ means that either *all* the clusters in P have intersection with ξ_1 or $P = \emptyset$. $N(P) =$ number of clusters in P.

We now show that the Neuman series for the inhomogeneous Eq. (6.6) (remember that $\varrho_N(\emptyset) \equiv 1$) converges for large β's.

Let $\mathscr{B}$ be the space of the functions $f(\mathscr{S}_X)$ defined for $X \neq \emptyset$ and without overlapping clusters and such that

$$\|f\| = \sup_{X, \mathscr{S}_X} \frac{|f(\mathscr{S}_X)|}{e^{-(\beta/2)|\mathscr{S}_X|}} < +\infty \tag{6.7}$$

we regard $\mathscr{B}$ as a Banach space with the norm (6.7).

Define the operator $\mathfrak{K} : \mathscr{B} \to \mathscr{B}$

$$\begin{aligned} (\mathfrak{K} f)(\mathscr{S}_X) = e^{-\beta|\mathscr{S}_{\xi_1}| - U_1(\mathscr{S}_X)} \sum_{Y \cap X = \emptyset} \sum_{\mathscr{S}'_Y} K_1(\mathscr{S}_X, \mathscr{S}'_Y) \\ \sum^*_{\substack{P \cap (X^{(1)} \cup Y) = \emptyset \\ \overline{P \cap \xi_1 \neq \emptyset}}} \sum_{\mathscr{S}''_P} (-1)^{N(P)} f(\mathscr{S}_{X^{(1)}} \cup \mathscr{S}''_P \cup \mathscr{S}'_Y) \end{aligned} \tag{6.8}$$

where * means that the term $X^{(1)} = \emptyset$, $P = \emptyset$, $Y = \emptyset$ is omitted.

Let $\chi_N : \mathscr{B} \to \mathscr{B}$ be the operator:

$$(\chi_N f)(\mathscr{S}_X) = \begin{cases} f(\mathscr{S}_X) & \text{if } X \subset [A, B] \\ 0 & \text{if not.} \end{cases} \tag{6.9}$$

Finally let $\alpha \in \mathscr{B}$ be defined as:

$$\begin{aligned} &\alpha(\mathscr{S}_\xi) = \exp - \beta|\mathscr{S}_\xi| - U_1(\mathscr{S}_\xi) \\ &\alpha(\mathscr{S}_X) \equiv 0 \quad \text{if } N(X) \geqq 2\,. \end{aligned} \tag{6.10}$$

Taking into account the definitions (6.8), (6.9), (6.10) and the Eq. (6.6) and the fact that $\varrho_N(\emptyset) \equiv 1$ we realize that (6.6) can be written as an equation on $\mathscr{B}$:

$$\varrho_N = \chi_N \alpha + \chi_N \mathfrak{K} \varrho_N\,. \tag{6.11}$$

It remains to show that $\mathfrak{K}$ is small: in fact we shall prove that $\|\mathfrak{K}\| \leqq k(\beta) < 1$ (the $\|\mathfrak{K}\|$ is the norm of the operator $\mathfrak{K}$ in the space $\mathscr{B}$) and $k(\beta) \to 0$ exponentially fast as $\beta \to \infty$.
In fact, using (6.9)

$$\frac{|(\mathfrak{K}f)(\mathscr{S}_X)|}{e^{-\frac{\beta}{2}|\mathscr{S}_X|}} \leqq \left\{ e^{-\frac{\beta}{2}|\mathscr{S}_{\xi_1}| - U_1(\mathscr{S}_X)} \sum_{Y \cap X = \emptyset} \sum_{\mathscr{S}'_Y} |K_1(\mathscr{S}_X, \mathscr{S}'_Y)| \sum_{\substack{P \cap (X^{(1)} \cup Y) = \emptyset \\ P \cap \xi_1 \neq \emptyset}} \sum_{\mathscr{S}''_P} e^{-\frac{\beta}{2}|\mathscr{S}'_Y|} e^{-\frac{\beta}{2}|\mathscr{S}''_P|} \right\} \|f\|\,. \tag{6.12}$$

A straightforward but very long calculation allows us to estimate the curly bracket in (6.12) and the result is:

$$\frac{|\mathfrak{K}f(\mathscr{S}_X)|}{e^{-\frac{\beta}{2}|\mathscr{S}_X|}} \leqq \|f\|\, k(\beta) \tag{6.13}$$

where $k(\beta) \to 0$ as $\beta \to \infty$ exponentially fast. The details of the computation leading to (6.13) are in Appendix 4.

Formula (6.13) implies

$$\|\mathfrak{K}\| \leqq k(\beta) < 1 \quad \text{for } \beta \text{ large} \tag{6.14}$$

hence the Neuman series for (6.11) converges.

The reader familiar with the Mayer expansion and the proof of its convergence ([10], p. 83) will immediately understand why the result (6.14) can be called a proof of the fact that the gas of shapes is almost perfect.

Unfortunately (6.14) is not quite enough for our purposes and we have to use some more detailed results about the Mayer expansions. The next section provides the additional results we need.

7. Cluster Expansions for the Shape-Particles Gas

The "clusters" in the title of this section do not have any relation with the clusters ξ associated with the shape particles. There should be no confusion between these two concepts.

Consider the space $\mathfrak{F}$ of the symmetric functions defined on the finite ordered sets of configurations of shape-particles obtained by allowing also the configurations not verifying the hard core condition [15].

If $\varphi \in \mathfrak{F}$ then φ associates to every ordered set $\mathscr{S}_X$ of shape particles, where $X = (\xi_1, \ldots, \xi_n)$, a number $\varphi(\mathscr{S}_X)$. We underline the fact that

$$\mathscr{S}_X = (\xi_1, \ldots, \xi_n, \mathscr{S}_{\xi_1}, \ldots, \mathscr{S}_{\xi_n})$$

is a set of shape particles, located in $\xi_1, \ldots, \xi_n$, not necessarily obeying the hard core condition (we also allow the possibility that $\xi_i = \xi_j$ for some $i \neq j$ and $\mathscr{S}_{\xi_i} = \mathscr{S}_{\xi_j}$).

If $\varphi \in \mathfrak{F}$ we call

$$|\varphi|_n = \sup_{\substack{\xi_1, \ldots, \xi_n \\ \mathscr{S}_{\xi_1} \ldots \mathscr{S}_{\xi_n} \\ n \text{ fixed}}} |\varphi(\mathscr{S}_{\xi_1}, \ldots, \mathscr{S}_{\xi_n})| \tag{7.1}$$

and we shall assume that $|\varphi|_n < +\infty$ for all $n \geqq 1$.

It is interesting to introduce some operations on the functions of $\mathfrak{F}$ [10].

If $\varphi_1 \in \mathfrak{F}$, $\varphi_2 \in \mathfrak{F}$ we define the convolution $\varphi_1 \cdot \varphi_2 \in \mathfrak{F}$ as

$$(\varphi_1 \cdot \varphi_2)(\mathscr{S}_X) = \sum_{X_1 \cup X_2 = X}{}' \varphi_1(\mathscr{S}_{X_1}) \varphi_2(\mathscr{S}_{X_2}) \tag{7.2}$$

here X is a general set of clusters (in the sense of Sec. 4) and is determined by the set of different clusters in X and by their multiplicities; the $\sum'_{X_1 \cup X_2 = X}$ is to be regarded as the sum over the ordered couples X_1, X_2 which decompose X into two sets of ordered clusters (the couples $(\emptyset, X)$ and $(X, \emptyset)$ are allowed).

Let us now define the exponential of a function $\varphi \in \mathfrak{F}_o$ where $\mathfrak{F}_o = \{\varphi | \varphi \in \mathfrak{F}, \varphi(\emptyset) = 0\}$:

$$(\operatorname{Exp} \varphi)(\mathscr{S}_X) = \sum_{n \geqq 0} \frac{(\varphi^n)(\mathscr{S}_X)}{n!} \tag{7.3}$$

where φ^n is the n^{th} power of φ in the convolution product (7.2) and we have put $\varphi^o(\mathscr{S}_X) = \mathbf{1}(\mathscr{S}_X)$ with

$$\mathbf{1}(\mathscr{S}_X) = \begin{cases} 0 & \text{if } X \neq \emptyset \\ 1 & \text{if } X = \emptyset . \end{cases} \tag{7.4}$$

It is clear that $\mathbf{1}$ is the identity for the product (7.2). It is also clear that, since $\varphi \in \mathfrak{F}_o$ formula (7.3) makes sense since it involves only a finite sum.

The inverse function to the exponential is defined over the set $\mathfrak{F}_1 = \{\varphi_1 | \varphi_1 \in \mathfrak{F}, \varphi_1(\emptyset) = 1\}$; if $\varphi_1 \in \mathfrak{F}_1$ we can uniquely write $\varphi_1 = \mathbf{1} + \varphi$ with $\varphi \in \mathfrak{F}_o$ and therefore we can define

$$\begin{aligned}\operatorname{Log}\varphi_1 &= \operatorname{Log}(\mathbf{1}+\varphi) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \varphi^n(\mathscr{S}_X) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum'_{X_1 \cup X_2 \cup \dots \cup X_n = X} \varphi(\mathscr{S}_{X_1}) \dots \varphi(\mathscr{S}_{X_n})\end{aligned} \tag{7.5}$$

since, again, the sum runs over a finite set of indices. Clearly $\operatorname{Exp}: \mathfrak{F}_o \to \mathfrak{F}_1$ and $\operatorname{Log}: \mathfrak{F}_1 \to \mathfrak{F}_o$. Furthermore it is easily checked that

$$\operatorname{Exp}\operatorname{Log}\varphi_1 \equiv \varphi_1 \qquad \forall \varphi_1 \in \mathfrak{F}_1 . \tag{7.6}$$

Define, on $\mathfrak{F}$, the operation $D_{\mathscr{S}_X}$:

$$(D_{\mathscr{S}_X} \cdot \varphi)(\mathscr{S}_Y) = \varphi(\mathscr{S}_{X \cup Y}) . \tag{7.7}$$

This operation has the properties:

$$D_{\mathscr{S}_\xi}(\varphi_1 \cdot \varphi_2) = (D_{\mathscr{S}_\xi}\varphi_1) \cdot \varphi_2 + \varphi_1 \cdot (D_{\mathscr{S}_\xi}\varphi_2) , \tag{7.8}$$

$$\begin{aligned}&D_{\mathscr{S}_\xi} \operatorname{Exp}\varphi = (D_{\mathscr{S}_\xi}\varphi) \cdot \operatorname{Exp}\varphi \\ &D_{\mathscr{S}_X} \operatorname{Exp}\varphi = \left(\sum_{n \geqq 1} \frac{1}{n!} \sum_{\substack{\cup_i X_i = X \\ X_i \neq \emptyset}} D_{\mathscr{S}_{X_1}} \varphi \dots D_{\mathscr{S}_{X_n}} \varphi \right) \operatorname{Exp}\varphi .\end{aligned} \tag{7.9}$$

Finally we find that the important formula below holds:

$$\sum_{X, \mathscr{S}_X} (\operatorname{Exp}\varphi)(\mathscr{S}_X)\chi(\mathscr{S}_X) = \exp \sum_{X, \mathscr{S}_X} \varphi(\mathscr{S}_X)\chi(\mathscr{S}_X) \tag{7.10}$$

provided $\sum_{X, \mathscr{S}_X} |\varphi(\mathscr{S}_X)| \, \chi(\mathscr{S}_X)| < +\infty$ and χ is multiplicative i.e. if $X = \xi_1, \dots, \xi_n$:

$$\chi(\mathscr{S}_X) = \prod_{i=1}^{n} \chi(\mathscr{S}_{\xi_i}) .$$

and provided the symbol $\sum_{X, \mathscr{S}_X}$ is consistently given the meaning:

$$\sum_{X, \mathscr{S}_X} \cdot = \sum_{K=o}^{\infty} \frac{1}{K!} \sum_{\xi_1} \cdots \sum_{\xi_n} \sum_{\mathscr{S}_{\xi_1}} \cdots \sum_{\mathscr{S}_{\xi_n}} \cdot \tag{7.11}$$

A special χ will be

$$\chi^{(N)}(\mathscr{S}_X) = \prod_{\xi \in X} \chi^{(N)}(\xi) \tag{7.12}$$

where $\chi^{(N)}(\xi) = 0$ if $\xi \not\subset [A, B]$ but $\chi^{(N)}(\xi) = 1$ if $\xi \subset [A, B]$.

So that

$$\sum_{X\subset[A,B]}\sum_{\mathscr{S}_X}(\operatorname{Exp}\varphi)(\mathscr{S}_X)=\exp\sum_{X\subset[A,B]}\sum_{\mathscr{S}_X}\varphi(\mathscr{S}_X) \tag{7.13}$$

provided $\sum_{X\subset[A,B]}\sum_{\mathscr{S}_X}|\varphi(\mathscr{S}_X)|<+\infty$.

Let β_0 be a large enough number in order that all the series written below in this section converge. Let β be so large that $\beta>\beta_0$ and $\psi(\beta)<\beta_0$ where $\psi(\beta)$ is the constant in (5.5). Consider the element $\bar{\varphi}\in\mathfrak{F}_1$

$$\bar{\varphi}(\mathscr{S}_X)=\begin{cases} e^{-\beta_0|\mathscr{S}_X|}\,e^{-U(\mathscr{S}_X)} & \text{if } (\xi_1,\ldots,\xi_n)=X \text{ and } \xi_i\cap\xi_j=\emptyset\ i\neq j\\ 0 & \text{otherwise}\end{cases}$$

$$\bar{\varphi}(\emptyset)=1 \tag{7.14}$$

Since from (5.5), (5.4), (5.2) it follows

$$|U(\mathscr{S}_X)|\leqq|\mathscr{S}_X|\,\psi|\beta| \tag{7.15}$$

the condition (7.1) is satisfied $\bar{\varphi}$ and therefore $\bar{\varphi}\in\mathfrak{F}_1$. Let $\bar{\varphi}^T=\operatorname{Log}\bar{\varphi}_1$.

We shall also be interested in other elements $\varphi\in\mathfrak{F}_1$ of the type

$$\varphi(\mathscr{S}_X)=\left\{\prod_{\xi\in X}\lambda(\mathscr{S}_\xi)\right\}\bar{\varphi}(\mathscr{S}_X) \tag{7.16}$$

for some suitable real or complex $\lambda(\mathscr{S}_\xi)$ such that $|\lambda(\mathscr{S}_\xi)|\leqq 1$.

Clearly $\operatorname{Log}\varphi$ is such that, see (7.5):

$$\varphi^T(\mathscr{S}_X)=(\operatorname{Log}\varphi)(\mathscr{S}_X)=\left\{\prod_{\xi\in X}\lambda(\mathscr{S}_\xi)\right\}\bar{\varphi}^T(\mathscr{S}_X)\,. \tag{7.17}$$

We now investigate the $\mathscr{S}_X$ dependence of $\bar{\varphi}^T(\mathscr{S}_X)$.

Consider, for this purpose, the function $\bar{\varphi}^{-1}\in\mathfrak{F}$ (i.e. the function $\bar{\varphi}^{-1}$ such that $\bar{\varphi}^{-1}\cdot\bar{\varphi}=\bar{\varphi}\cdot\bar{\varphi}^{-1}=\mathbf{1}$: this function can be inductively defined from (7.2) for all $\bar{\varphi}\in\mathfrak{F}_1$). Define

$$\Delta_{\mathscr{S}_X}(\mathscr{S}_Y)=(\bar{\varphi}^{-1}\cdot D_{\mathscr{S}_X}\bar{\varphi})(\mathscr{S}_Y) \tag{7.18}$$

for $\mathscr{S}_Y$ arbitrary and $\mathscr{S}_X$ such that X contains only non-overlapping clusters.

Then we can write an equation for $\Delta_{\mathscr{S}_X}(\mathscr{S}_Y)$ along the lines of Ref. [7,13].

We have (if Y\T means complement of T in Y):

$$\begin{aligned}\Delta_{\mathscr{S}_{X\cup\xi}}(\mathscr{S}_Y)=e^{-\beta_0|\mathscr{S}_\xi|}\,e^{=U_1(\mathscr{S}_{X\cup\xi})}&\sum_{\substack{T\subset Y\\ T\cap(\xi\cup X)=\emptyset}}K_1(\mathscr{S}_{X\cup\xi},\mathscr{S}_T)\\ &\sum_{\substack{P\cap\xi\neq\emptyset\\ P\subset Y\setminus T}}(-1)^{N(P)}\Delta_{\mathscr{S}_{X\cup T\cup P}}(\mathscr{S}_{Y\setminus P\cup T})\end{aligned} \tag{7.19}$$

where, as in (6.6), $\overline{P \cap \xi \neq \emptyset}$ means that all the elements in P intersect ξ or $P = \emptyset$. Eq. (7.19) is deduced in detail in Appendix 5.

From (7.19) one can deduce very strong results on $\bar{\varphi}^T(\mathscr{S}_X)$. Put

$$I_m = \sup_{\substack{\mathscr{S}_\xi \ldots \mathscr{S}_\xi \\ m \geqq n \geqq 1}} \sum_{\substack{Y \\ N(Y) = m - n}} \sum_{\mathscr{S}'_Y} \frac{|\Delta_{\mathscr{S}_{\xi_1} \ldots \mathscr{S}_{\xi_n}}(\mathscr{S}'_Y)|}{e^{-\frac{\beta_0}{2} \sum_i |\mathscr{S}_{\xi_i}|}} . \tag{7.20}$$

Then, using (7.19), at fixed $X \cup \xi$ and $\mathscr{S}_{\xi \cup X}$:

$$\sum_{\substack{Y \\ N(Y)+N(X)=n}} \sum_{\mathscr{S}'_Y} \frac{|\Delta_{\mathscr{S}_{\xi \cup X}}(\mathscr{S}'_Y)}{e^{-\frac{\beta_0}{2}(|\mathscr{S}_\xi| + |\mathscr{S}_X|)}} \leqq e^{-\frac{\beta_0}{2}|\mathscr{S}_\xi| + U(\mathscr{S}_{X \cup \xi})}$$

$$\sum_{\substack{Y \\ N(X)+N(Y)=n}} \sum_{\mathscr{S}'_Y} \sum_{\substack{T \subset Y \\ T \cap (\xi \cup X) = \emptyset}} |K_1(\mathscr{S}_{X \cup \xi}, \mathscr{S}'_T)| \sum_{\substack{\overline{P \cap \xi \neq \emptyset} \\ P \subset Y \setminus T}} \frac{|\Delta_{\mathscr{S}_{X \cup T \cup P}}(\mathscr{S}'_{Y/P \cup T})|}{e^{-\frac{\beta_0}{2}|\mathscr{S}_{X \cup T \cup P}|}}$$

$$e^{-\frac{\beta_0}{2}|\mathscr{S}_{X \cup T \cup P}|}$$

$$= e^{-\frac{\beta_0}{2}|\mathscr{S}_\xi| + U_1(\mathscr{S}_{X \cup \xi})} \sum_{\substack{T \\ (P \cup T) \cap X = \emptyset \\ P \cap T = \emptyset}} \sum_{\substack{\overline{P \cap \xi \neq \emptyset} \\ P \cap T = \emptyset \\ N(P)+N(T)+N(X) \leqq n}} \sum_{\mathscr{S}'_{P \cup X}} |K_1(\mathscr{S}_{X \cup \xi}, \mathscr{S}'_T)| \tag{7.21}$$

$$\sum_{\substack{Y \supset P \cup T \\ N(Y \setminus P \cup T) + N(P) + N(T) + N(X) = n}} \sum_{\mathscr{S}'_Y \setminus P \cup T} \frac{|\Delta_{\mathscr{S}_{X \cup T \cup P}}(\mathscr{S}'_{Y/P \cup T})|}{e^{-\frac{\beta_0}{2}(\mathscr{S}_{X \cup T \cup P})}} e^{-\frac{\beta_0}{2}|\mathscr{S}_{X \cup T \cup P}|}$$

$$\leqq I_n \left\{ e^{-\frac{\beta_0}{2}|\mathscr{S}_\xi| + U_1(\mathscr{S}_{X \cup \xi})} \sum_{\substack{T \\ T \cap X = \emptyset}} \sum_{\substack{P \cap X = \emptyset \\ P \cap T = \emptyset \\ \overline{P \cap \xi \neq \emptyset}}} \sum_{\mathscr{S}'_{P \cup T}} |K_1(\mathscr{S}_{X \cup \xi}, \mathscr{S}'_T)| \right.$$

$$\left. e^{-\frac{\beta_0}{2}|\mathscr{S}'_{P \cup T}|} \right\} \leqq k(\beta_0) I_n$$

as we easily see after a calculation identical to the one leading to the estimate of the curly bracket in (6.12).

Formula (7.21) says that

$$I_{n+1} \leqq k(\beta_0) I_n, \quad \text{i.e. } I_n \leqq k(\beta_0)^{n-1} I_1 . \tag{7.22}$$

In particular, since $I_1 \leqq e^{-\frac{\beta_0}{2}}$ and

$$\bar{\varphi}^T(\mathscr{S}_{\xi \cup Y}) = \Delta_{\mathscr{S}_\xi}(\mathscr{S}_Y) \tag{7.23}$$

we find (since $\bar{\varphi}^T(\mathscr{S}_\xi) = e^{-\beta_o|\mathscr{S}_\xi|}$):

$$\begin{aligned}\sum_Y \sum_{\mathscr{S}_Y} |\bar{\varphi}^T(\mathscr{S}_{\xi \cup Y})| &= e^{-\beta_o|\mathscr{S}_\xi|} + e^{-\frac{\beta_o}{2}|\mathscr{S}_\xi|} \sum_{m=1}^{\infty} I_m \\ &\leqq e^{-\frac{\beta_o}{2}|\mathscr{S}_\xi|} \frac{\left(e^{-\frac{\beta_o}{2}}\right)}{1-k(\beta_o)} + e^{-\frac{\beta_o}{2}|\mathscr{S}_\xi|} \leqq e^{-\frac{\beta_o}{2}|\mathscr{S}_\xi|} \frac{9e^{-\frac{\beta_o}{2}}}{1-k(\beta_o)}.\end{aligned} \tag{7.24}$$

If β has been chosen large enough.

A simple consequence of (7.24) is that, for β_o large:

$$\sum_{X \subset [A,B]} |\bar{\varphi}^T(\mathscr{S}_X)| < +\infty \tag{7.25}$$

as it follows by combining (7.24) with the observation that the number of shapes $\mathscr{S}_\xi$ with a given $|\mathscr{S}_\xi|$ does not exceed $3^{2|\mathscr{S}_\xi|}$.

We shall be interested in choosing the arbitrary function $\lambda(\mathscr{S}_\xi)$ in (7.16) of the form $\delta\mathscr{S}_\xi$ is defined in Sec. 4):

$$\lambda(\mathscr{S}_\xi) = e^{it\,\delta\mathscr{S}_\xi - (\beta-\beta_o)|\mathscr{S}_\xi| + i\tau\,\delta\mathscr{S}_\xi \tilde{\chi}^{(N)}(\xi)} \tag{7.26}$$

with t, τ real and $\tilde{\chi}^N(\xi) = 1$ if $\xi \subset [A, O]$, $\tilde{\chi}^N(\xi) = 0$ otherwise.

We have now all the instruments to deal with the original problem which is attacked in the next section.

8. Characteristic Functions for the End and Middle Point Displacement

Consider the characteristic function for the random variable $\sum_{\xi \in X} \delta\mathscr{S}_\xi$ defined for every line $\lambda \in \tilde{\mathfrak{U}}(N)$, $\lambda = \mathscr{S}_X$:

$$\left\langle e^{it\sum_\xi \delta\mathscr{S}_\xi} \right\rangle = \frac{\sum_{X \subset [A,B]} \sum_{\mathscr{S}_X} e^{-U(\mathscr{S}_X)} \prod_{\xi \in X} (e^{it\,\delta\mathscr{S}_\xi} e^{-\beta|\mathscr{S}_\xi|})}{\sum_{X \subset [A,B]} \sum_{\mathscr{S}_X} e^{-U(\mathscr{S}_X)} \prod_{\xi \in X} e^{-\beta|\mathscr{S}_\xi|}} \tag{8.1}$$

Here we can apply formula (7.13) to the numerator and denominator of (8.1); we get

$$\left\langle e^{it\sum_\xi \delta\mathscr{S}_\xi} \right\rangle = \exp \sum_{X \subset [A,B]} \sum_{\mathscr{S}_X} \bar{\varphi}^T(\mathscr{S}_X)\, e^{-(\beta-\beta_o)|\mathscr{S}_X|}(e^{it\,\delta\mathscr{S}_X} - 1). \tag{8.2}$$

We now look for estimates of the sums in (8.2). The l.h.s. of (8.2) is periodic with period 2π, hence it is enough to consider it for $-\pi \leqq t \leqq \pi$. We divide the interval $|t| \leqq \pi$ in three regions:

i) $0 \leqq |t| \leqq N^{\frac{1}{5}-\frac{1}{2}}$,

ii) $N^{\frac{1}{3}-\frac{1}{2}} \leqq |t| \leqq \varepsilon N^{-\frac{1}{2}}$,

iii) $\varepsilon N^{-\frac{1}{2}} \leqq |t| \leqq \pi$,

and we estimate the sum in (8.2) in these three regions. The constant ε is a numerical constant that will be chosen later.

Observe that, if $X=\xi$, $|\xi|=0$ and $\delta\mathscr{S}_\xi = j \quad \xi \neq A, B$ we have

$$\bar{\varphi}^T(\mathscr{S}_\xi) = e^{-\beta_0 |j|}\, e^{-\Phi(\mathscr{S}_\xi)} = e^{-\beta_0 |j| - 2e^{-4\beta}|j| + \mathcal{O}(e^{-6\beta})|j|} \tag{8.3}$$

as it follows from the definition of the shape potential $\Phi(\mathscr{S}_\xi)$ and the formula for it in Appendix 3. Hence the contribution to the sum in (8.2) from the one-point clusters of height $j = \pm 1$ is:

$$\exp N e^{-\beta}((e^{it}-1)+(e^{-it}-1)+\mathcal{O}(e^{-\beta})) \tag{8.4}$$

where $\mathcal{O}(e^{-\beta})$ contains the end point contribution ($\xi = A, B$) as well as the corrections coming from the term $\mathcal{O}(e^{-6\beta})$ in (8.3). In formula (8.4), as well as everywhere in this paper, $\mathcal{O}(a)$ means ϑa where ϑ is a function of, a priori, everything possible but such that $|\vartheta| \leqq 1$.

We estimate the contribution of the other clusters to the sum in (8.2) by: (we use (7.24) and the fact that there are at most $3^{|\mathscr{S}_\xi|+|\xi|} \leqq 3^{2|\mathscr{S}_\xi|}$ different shapes over ξ with the same $|\mathscr{S}_\xi|$)

$$\begin{aligned} &2 \sum_{X \subset [A,B]} \sum_{\substack{\mathscr{S}_X \\ |\mathscr{S}_X|>1}} e^{-(\beta-\beta_0)|\mathscr{S}_X|}\, |\bar{\varphi}^T(\mathscr{S}_X)| \\ &\leqq 2 \sum_{\xi \subset [A,B]} \sum_{\mathscr{S}_\xi} e^{-2(\beta-\beta_0)} \sum_{X', \mathscr{S}_{X'}} |\bar{\varphi}^T(\mathscr{S}_\xi \cup \mathscr{S}_{X'})| \\ &\leqq 2\, e^{-2(\beta-\beta_0)} \sum_{\xi \subset [A,B]} \sum_{\mathscr{S}_\xi} e^{-(\beta_0/2)|\mathscr{S}_\xi|} \frac{3 e^{-\beta_0/2}}{1-k(\beta_0)} \\ &\leqq 2 e^{-2(\beta-\beta_0)} N \sum_{p=|\xi|+1}^{\infty} 3^{2p} e^{-(\beta_0/2)p} \frac{3 e^{-\beta_0/2}}{1-k(\beta_0)} \leqq e^{-2\beta} C(\beta_0) N \end{aligned} \tag{8.5}$$

where $C(\beta_0) \to \infty$ if β_0 is large enough.

Hence we have proven that

$$\left\langle e^{it \sum_\xi \delta\mathscr{S}_\xi} \right\rangle = \exp\left(-4N \left(\sin^2 \frac{t}{2} \right) e^{-\beta} + N\mathcal{O}(e^{-2\beta}) \;\; + \mathcal{O}(e^{-\beta}) \right) \tag{8.6}$$

for all $t \in [-\pi, \pi]$.

Clearly the estimate (8.6) is going to be good only in the region iii) and for large β.

To obtain estimates of use in i), ii) we proceed as follows.

Expand $(e^{it\delta\mathscr{S}_X} - 1)$ to third order using the Schömilch formula for the fourth order rest: we get

$$\left\langle e^{it\sum_\xi \delta\mathscr{S}_\xi} \right\rangle = \exp\Big\{ - \sum_{X\subset[A,B]} \sum_{\mathscr{S}_X} \bar{\varphi}^T(\mathscr{S}_X)\, e^{-(\beta-\beta_0)|\mathscr{S}_X|} \frac{(\delta\mathscr{S}_X)^2}{2} t^2 \tag{8.7}$$
$$+ \sum_{X\subset[A,B]} \sum_{\mathscr{S}_X} \bar{\varphi}^T(\mathscr{S}_X)\, e^{-(\beta-\beta_0)|\mathscr{S}_X|} \frac{(\delta\mathscr{S}_X)^4}{24} e^{i\vartheta t\delta\mathscr{S}_X}\, t^4$$

where $|\vartheta| \leqq 1$ and we have used the fact that the first and third order terms vanish for symmetry reasons.

Define $\sigma(\beta)$ as

$$\sigma^2(\beta) = \lim_{N\to\infty} \frac{1}{N} \sum_{X\subset[A,B]} \sum_{\mathscr{S}_X} \bar{\varphi}^T(\mathscr{S}_X)\, e^{-(\beta-\beta_0)|\mathscr{S}_X|} (\delta\mathscr{S}_X)^2 \tag{8.8}$$

the existence of this limit follows from the fact that the functions $\bar{\varphi}^T(\mathscr{S}_X)$ are essentially translationally invariant. We have not dealt with this point and we shall not do so but we leave it as an exercise to the reader. We observe, however, that by taking into account separately the "unit jump" contribution one finds that the sum $\Sigma(N)$ on the r.h.s. of (8.8) is

$$\Sigma(N) = 2Ne^{-\beta} + N\,\mathcal{O}(e^{-2\beta}) + \mathcal{O}(e^{-\beta}) \tag{8.9}$$

and so

$$\sigma^2 = 2e^{-\beta}(1 + \mathcal{O}(e^{-\beta})) \tag{8.10}$$

the real reason we do not insist on the existence of the limit (8.8) is that (8.9) proves that

$$\liminf_{N\to\infty} \frac{\Sigma(N)}{N} = 2e^{-\beta}(1 + \mathcal{O}_1(e^{-\beta})) \text{ and } \limsup_{N\to\infty} \frac{\Sigma(N)}{N} = 2e^{-\beta}(1 + \mathcal{O}_2(e^{-\beta}))$$

and this would be enough for the rest of the proof (modulo minor modifications).

An estimate similar to (8.9) yields

$$\sum_{X\subset[A,B]} \sum_{\mathscr{S}_X} |\bar{\varphi}^T(\mathscr{S}_X)|\, e^{-(\beta-\beta_0)|\mathscr{S}_X|} (\delta\mathscr{S}_X)^4 = N2e^{-\beta}(1 + \mathcal{O}(e^{-\beta})) + \mathcal{O}(e^{-\beta}). \tag{8.11}$$

Hence using (8.7)–(8.9), (8.11) we find that, if $0 \leqq |t| \leqq N^{\frac{1}{5}-\frac{1}{2}}$:

$$\left\langle e^{it\sum_\xi \delta\mathscr{S}_\xi} \right\rangle = \exp\left\{ -\tfrac{1}{2}\Sigma(N)\, t^2 + N^{-}\quad \mathcal{O}(2e^{-\beta}) \right\}$$

and if $N^{\frac{1}{5}-\frac{1}{2}} \leqq |t| \leqq \varepsilon N^{-\frac{1}{2}}$

$$\left\langle e^{it\sum_\xi \delta\mathscr{S}_\xi} \right\rangle \leqq \exp -\tfrac{1}{4}(\Sigma(N)\, t^2 \tag{8.12}$$

provided $\varepsilon < \frac{1}{4}$ (say).

It is now easy to compute (see also [11]) an expression for the probability $P_N(k)$ that, in $\tilde{\mathfrak{U}}(N)$, the line $\lambda \in \tilde{\mathfrak{U}}(N)$ ends in the point B′ at height $k = \sum_i \delta\mathcal{S}_{\xi_i}$:

$$P_N(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\langle e^{it \sum_\xi \delta\mathcal{S}_\xi} \right\rangle e^{-itk} \, dt \tag{8.13}$$

$$= \frac{1}{2\pi\sqrt{N}} \int_{-\pi\sqrt{N}}^{+\pi\sqrt{N}} \left\langle e^{i \frac{t'}{\sqrt{N}} \sum_\xi \delta\mathcal{S}_\xi} \right\rangle e^{-i \frac{t'k}{\sqrt{X}}} \, dt' = \frac{e^{-\frac{k^2}{2N\sigma^2}}}{\sqrt{2\pi\sigma^2 N}} + o\left(\frac{1}{\sqrt{N}}\right).$$

In particular

$$P_N(0) = \frac{1}{\sqrt{2\pi\sigma^2 N}} + o\left(\frac{1}{\sqrt{N}}\right). \tag{8.14}$$

Using a completely analogous procedure we can estimate (see (7.26))

$$\left\langle e^{\sum_{\xi \in X} (it + i\tau \tilde{\chi}^{(N)}(\xi)) \delta\mathcal{S}_\xi} \right\rangle \tag{8.15}$$

and deduce that the probability in $\tilde{\mathfrak{U}}(N)$ that the end point of λ is at height k and the last point of the last cluster in λ contained in [A, O] is at height h:

$$P_N(k, h) = \frac{e^{-\left(\frac{k^2 - 2hk + 2h^2}{N\sigma^2}\right)^2}}{\pi\sigma^2 N} + o\left(\frac{1}{N}\right) \tag{8.16}$$

hence if $|h| \leq h(N)$ and $\frac{h(N)}{\sqrt{N}} \to 0$ as $N \to \infty$

$$\frac{P_N(0, h)}{P_N(0)} \underset{N \to \infty}{\sim} \frac{1}{\sqrt{\pi\sigma^2}} \frac{1}{\sqrt{N}}. \tag{8.17}$$

Since $P_N(0, h)/P_N(0)$ is the probability in $\tilde{\mathfrak{U}}_0(N)$ that the last cluster before O has the last point at height h formula (8.17) tells us that the probability that the height h of this point is such that $|h| \geq h(N)$ tends to 1 as $N \to \infty$ in $\tilde{\mathfrak{U}}_0(N)$.

We now show that the probability, in $\tilde{\mathfrak{U}}(N)$, that λ contains a shape $\mathcal{S}_\xi$ such that $|\mathcal{S}_\xi| > C \log N$ tends to zero *faster* than $\frac{1}{\sqrt{N}}$ if C is suitably chosen. This fact will imply that (see (8.14)) not only in $\tilde{\mathfrak{U}}(N)$ but also in $\tilde{\mathfrak{U}}_0(N)$ the probability that in λ there is a $\mathcal{S}_\xi$ such that $|\mathcal{S}_\xi| > C \log N$ tends to zero as $N \to \infty$. Therefore since the height of the last point of the last cluster such that $\xi \subset [A, O]$ does not exceed h(N) with probability, in $\tilde{\mathfrak{U}}_0(N)$, tending to 1 as $N \to \infty$ we can deduce that the line λ passes above O at an height $h(N) - C \log N$ with a probability tending to 1 as $N \to \infty$.

It is obvious that, once it will have been proven that with probability tending 1 as $N \to \infty$ no shape in λ is such that $|\mathscr{S}_\xi| > C \log N$, a simple modification of the above argument will allow to prove that in $\tilde{\mathfrak{U}}_0(N)$ the probability that the distance of O from λ is larger than $N^{\frac{1}{2}-\varepsilon}$ tends to 1 as $N \to \infty$ at fixed $\varepsilon > 0$.

Therefore we are left with the proof that if

$$\lambda \in \tilde{\mathfrak{U}}(N), \lambda = (\xi_1, \ldots, \xi_n, \mathscr{S}_{\xi_1} \ldots \mathscr{S}_{\xi_n}) \quad \text{then} \quad |\mathscr{S}_{\xi_i}| < C \log N$$

with a probability tending to 1 as $N \to \infty$.

Call P(N, C) this probability: using (6.11), (6.7) and the estimate for the number of shapes above ξ and with given $|\mathscr{S}_\xi|$ used in (8.5):

$$\begin{aligned} \varrho(N,C) &\leqq \sum_{\xi \subset [A,B]} \sum_{|\mathscr{S}_\xi| > C\log N} \varrho_N(\mathscr{S}_\xi) \\ &\leqq N \sum_{|\xi|=0}^{\infty} |\xi| \sum_{|\mathscr{S}_\xi| > C\log N} \frac{3^{2|\mathscr{S}_\xi|} e^{-\frac{\beta}{2}|\mathscr{S}_\xi|}}{1-k(\beta)} \leqq N \sum_{|\xi|=0}^{\infty} |\xi| \sum_{|\mathscr{S}_\xi| > C\log N} \\ &\quad \cdot \left(9e^{-\frac{\beta}{4}}\right)^{|\mathscr{S}_\xi|} \frac{e^{-\frac{\beta}{4}|\xi|}}{1-k(\beta)} \leqq N \left(9e^{-\frac{\beta}{4}}\right)^{C\log N} \frac{1}{1-9e^{-\frac{\beta}{4}}} \frac{1}{\left(1-e^{-\frac{\beta}{4}}\right)^2} \frac{1}{1-k(\beta)} \end{aligned} \tag{8.18}$$

hence if $C = \dfrac{9}{\beta}$ and β is large:

$$P(N,C) \leqq \frac{2}{N^{\frac{5}{4}+\frac{9}{\beta}\log 9}} = o\left(\frac{1}{N}\right). \tag{8.19}$$

Acknowledgements. I wish to express my gratitude to H. Van Beyeren for stimulating my interest in the problem through many discussions and through the description of some partial results which he got using a completely different and much simpler approach [14].

Appendix 1

The basic setting of Ref. [9] is very similar to the one of this paper. Here, for esthetic reasons, we have preferred a different definition of the contours associated with a spin configuration.

What we essentially need are the results of Appendix A of Ref. [9] and to obtain them one has to proceed, word by word, exactly through the same calculations and steps with the understanding that the new definitions of contours (and, therefore, of overlapping contours) are to be taken into account to modify in the obvious way the interpretation of the notation.

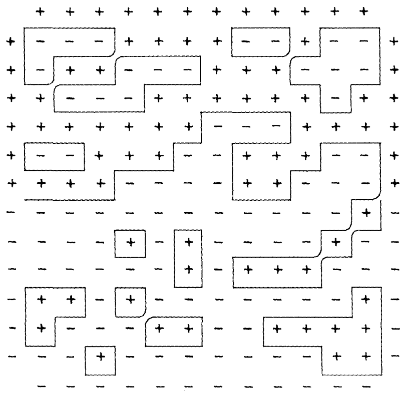

Fig. 4.

We give, as an example, in Fig. 4 the contours associated with the spin configuration of Fig. 1. The reader should compare Fig. 4 with Fig. 2 in order to realize the difference in the definition of contours. We remark that in spite of the difference in definition the number of contours of length $|\gamma|$ is still bounded by $3^{|\gamma|}$.

A mere "translation" changes the results (A.20), (A.19), (A.17), (A.21), (A.24) of Ref. [9] into 1)÷5) of Theorem 2.

Furthermore (A.26) of Ref. [9], duly reinterpreted, can be applied to write (in this paper we call Γ what was X in Ref. [9]):

$$Z_0(\Omega_\lambda^{(a)}, \beta) = \exp \sum_{\Gamma \subset \Omega_\lambda^{(a)}} \varphi^T(\Gamma), \tag{Z.1}$$

$$Z_0(\Omega_\lambda^{(b)}, \beta) = \exp \sum_{\Gamma \subset \Omega_\lambda^{(b)}} \varphi^T(\Gamma), \tag{Z.2}$$

$$Z_0(\Omega_\lambda, \beta) = \exp \sum_{\Gamma \subset \Omega} \varphi^T(\Gamma), \tag{Z.3}$$

hence, by taking the product of (Z.1) and (Z.2) and by comparing the result with (Z.3) we find (3.4).

The last statement of Theorem 2 does not appear in Ref. [9] and we provide here its simple proof.

Observe, from (A.20) of Ref. [9], that

$$\varphi^T(\Gamma) = e^{-\beta|\Gamma|} \tilde{\varphi}^T(\Gamma) \tag{Z.4}$$

where $\tilde{\varphi}^T$ is β-independent.

From (3.3) it follows that

$$\sum_{\substack{\Gamma \ni \gamma \\ N(\Gamma) = n+1}} e^{-\beta_0|\Gamma|} |\tilde{\varphi}^T(\Gamma)| \leqq e^{-\frac{\beta_0}{2}|\gamma|} \varkappa(\beta_0)^{n+1}. \tag{Z.5}$$

Let X be a set of distinct points on the square lattice and let $X=(x_1,\ldots,x_n)$; denoting $\Gamma\ni X$ the fact that the points of X are vertices of contours in Γ we find

$$\begin{aligned}\Phi_0(X)&=\sum_{\Gamma\ni X}|\varphi^T(\Gamma)|=\sum_{\Gamma\ni X}e^{-\beta|\Gamma|}|\tilde\varphi^T(\Gamma)|\\&=\sum_{\Gamma\ni X}e^{-(\beta-\beta_0)|\Gamma|}\,e^{-\beta_0|\Gamma|}|\tilde\varphi^T(\Gamma)|\,.\end{aligned}\tag{Z.6}$$

Let R_0 be an arbitrary large integer, fixed once for all (say $R_0=100$), then if $2R_0e^{-(\beta-\beta_0)}<1$

$$\begin{aligned}\sum_{O\ni X}R^{\operatorname{diam}X}\Phi_0(X)&\leqq\sum_{n=0}^{\infty}\sum_{O\in X}\sum_{\substack{\Gamma\ni X\\N(\Gamma)=n+1}}\\&\quad\cdot R^{\operatorname{diam}X}e^{-(\beta-\beta_0)|\Gamma|}e^{-\beta_0|\Gamma|}|\tilde\varphi^T(\Gamma)|\\&=\sum_{n=0}^{\infty}\sum_{\substack{\Gamma\ni O\\N(\Gamma)=n+1}}\sum_{\substack{X\subset\Gamma\\X\ni O}}R^{\operatorname{diam}X}e^{-(\beta-\beta_0)|\Gamma|}e^{-\beta_0|\Gamma|}|\tilde\varphi^T(\Gamma)|\\&\leqq\sum_{n=0}^{\infty}\sum_{\substack{\Gamma\ni O\\N(\Gamma)=n+1}}(2R_0e^{-(\beta-\beta_0)})^{|\Gamma|}e^{-\beta_0|\Gamma|}|\tilde\varphi^T(\Gamma)|\\&\leqq 2R_0e^{-(\beta-\beta_0)}\sum_{\gamma\ni O}e^{-\frac{\beta_0}{2}|\gamma|}\frac{\varkappa(\beta_0)}{1-\varkappa(\beta_0)}\leqq\frac{\left(3e^{-\frac{\beta_0}{2}}\right)\varkappa(\beta_0)}{1-\varkappa(\beta_0)}2R_0e^{-(\beta-\beta_0)}\end{aligned}\tag{Z.7}$$

and this formula holds for $1\leqq R\leqq R_0$ and β large enough. Taking $R=1$ and $R=R_0$ (Z.7) prove statement 7 in Theorem 2.

Appendix 2

The ensemble $\mathfrak{U}_0(N)$, as a set of configurations in $\mathfrak{U}(N)$, has a probability in $\mathfrak{U}(N)$ approaching 1 as $N\to\infty$ (see i) Theorem 1).

Hence λ is loose in $\mathfrak{U}(N)$ if and only if it is loose in $\mathfrak{U}_0(N)$.

For large β the distance of $\lambda\in\mathfrak{U}_0(N)$ from the upper and lower bases of Ω will be longer than N/2 hence, regarding λ as an element of $\tilde{\mathfrak{U}}_0(N)$, the weights $\mathscr{W}_0(\lambda)$ and $\tilde{\mathscr{W}}_0(\lambda)$ are such that

$$\tilde{\mathscr{W}}_0(\lambda)=\mathscr{W}_0(\lambda)\exp-\sum_{\substack{\Gamma\, i\,\lambda\\\Gamma\not\subset\Omega\\\Gamma\subset I_N}}\varphi^T(\Gamma)\tag{Z.8}$$

and, using (3.4):

$$\sum_{\substack{\Gamma\, i\,\lambda\\\Gamma\subset\Omega}}|\varphi^T(\Gamma)|\leqq\sum_{X\in\lambda}\sum_{\substack{\Gamma\ni X\\\Gamma\cap(\text{bases of }\Omega)}}|\varphi^T(\Gamma)|\leqq N\left(1+\frac{c}{\beta}\right)\varkappa(\beta)\left(4e^{-\frac{\beta}{2}}\right)^{\sqrt{\frac{N}{2}}}\tag{Z.9}$$

therefore, for large β:

$$\tilde{\mathscr{W}}_0(\lambda)=\mathscr{W}_0(\lambda)\exp\mathcal{O}\left(e^{-\frac{\beta}{3}\sqrt{\frac{N}{2}}}\right) \tag{Z.10}$$

where $\mathcal{O}\left(e^{-\frac{\beta}{3}\sqrt{\frac{N}{2}}}\right)=\vartheta(\lambda,\beta,\ldots)\,e^{-\frac{\beta}{3}\sqrt{\frac{N}{2}}}$ and $|\vartheta|\leqq 1$.

Furthermore the set $\mathfrak{U}_0(N)$, as a subset of $\tilde{\mathfrak{U}}_0(N)$, has in $\tilde{\mathfrak{U}}_0(N)$ a probability that tends to 1 as $N\to\infty$. In fact, since, using (3.2),

$$\sum_{\Gamma\,\cap\,\lambda\neq\emptyset}|\varphi^T(\Gamma)|\leqq|\lambda|\sum_{\Gamma\ni O}|\varphi^T(\Gamma)|\leqq|\lambda|\,\delta(\beta) \tag{Z.11}$$

we have

$$P_{\tilde{U}_0(N)}\left(\big||\lambda|-N\big|>\frac{C}{\beta}\right)\leqq\frac{\sum\limits_{|\lambda|>\left(1+\frac{C}{\beta}\right)N}e^{-\beta|\lambda|}e^{|\lambda|\,\delta(\beta)}}{e^{-\beta N-N\delta(\beta)}} \tag{Z.12}$$

where the denominator is a lower bound to the contribution to the normalization constant to (3.9) coming from the single line $\lambda=[A,B]$.

The r.h.s. of (Z.12) is bounded above as:

$$\begin{aligned}(Z.11)&\leqq\frac{\sum\limits_{k\geqq\left(1+\frac{C}{\beta}\right)N}(3\,e^{-\beta+\delta(\beta)})^k}{e^{-\beta N}e^{-N\delta(\beta)}}\\&=\frac{1}{1-3\,e^{-\beta+\delta(\beta)}}\left(3^{1+\frac{C}{\beta}}e^{2\delta(\beta)+\frac{C\delta(\beta)}{\beta}}e^{-C}\right)^N\leqq(4e^{-C})^N\end{aligned} \tag{Z.13}$$

if $C>\log 4$ and if β is large enough.

Hence $\mathfrak{U}_0(N)$, as a subset of $\tilde{\mathfrak{U}}_0(N)$, has in $\tilde{\mathfrak{U}}_0(N)$ probability tending to 1 as $N\to\infty$. Clearly this fact combined with (Z.10) implies that λ is loose in $\mathfrak{U}_0(N)$ (hence in $\mathfrak{U}(N)$) if and only if it is loose in $\tilde{\mathfrak{U}}_0(N)$.

Appendix 3. Theory of the Shape Potentials

Consider a configuration $\mathscr{S}_X=(\xi_1,\ldots,\xi_s,\mathscr{S}_{\xi_1},\ldots,\mathscr{S}_{\xi_s})$ (see Fig. 3 for reference).

We divide the sets of contours $\Gamma\in\mathfrak{N}$ and lying in I_N and intersecting λ in three classes: to the first class belong the Γ's which have in common with λ some points and which have a projection on the segment [A, B] which does not have points in common with any of the vertical lines through $\xi_1,\ldots,\xi_s$. To the second class belong the Γ's which have intersection with some "shape" $\mathscr{S}_{\xi_i}$ of λ. To the third class belong the Γ's which do not intersect any of the $\mathscr{S}_\xi$'s but have a projection over [A, B] which overlaps with some of the clusters $\xi_1,\ldots,\xi_s$.

Call $C_1(\lambda), C_2(\lambda), C_3(\lambda)$ the three classes and call $C_1'(\xi_1,\ldots,\xi_s)$ the set of Γ's which intersect the segment [A, B] and have a projection on

[A, B] overlapping with some of the clusters $\xi_1, \ldots, \xi_s$. We easily find:

$$\begin{aligned}\sum_{\substack{\Gamma \imath \lambda \\ \Gamma \subset I_N}} \varphi^T(\Gamma) &= \sum^*_{\Gamma \in C_1(\lambda)} \varphi^T(\Gamma) + \sum^*_{\Gamma \in C_2(\lambda)} \varphi^T(\Gamma) + \sum^*_{\Gamma \in C_3(\lambda)} \varphi^T(\Gamma) \\ &= \sum^*_{\Gamma \imath [A,B]} \varphi^T(\Gamma) - \sum^*_{\Gamma \in C_1(\xi_1' \ldots \xi_s)} \varphi^T(\Gamma) + \sum^*_{\Gamma \in C_2(\lambda)} \varphi^T(\Gamma) + \sum^*_{\Gamma \in C_3(\lambda)} \varphi^T(\Gamma)\end{aligned} \tag{Z.14}$$

where the * reminds us that $\Gamma \subset I_N$.

We recognize $U(\mathscr{S}_X)$ in the sum of the last three addends of (Z.14).

We now show that each of the last three sums in (Z.14) is of the form $\sum_{T \subset X} \Phi^{(i)}(\mathscr{S}_T)$, where $i = 1, 2, 3$ is an index that labels the last three sums in (Z.14). Furthermore each of the three functions $\Phi^{(i)}(\mathscr{S}_T)$ verifies Lemma 1.

Consider first $\sum^*_{\Gamma \in C_1'(\xi_1 \ldots \xi_s)} \varphi^T(\Gamma)$: We have

$$\begin{aligned}\sum^*_{\Gamma \in C_1'(\xi_1 \ldots \xi_s)} \varphi^T(\Gamma) &= \sum_{j=1}^{s} \left(\sum^*_{\Gamma \imath \xi_j} \varphi^T(\Gamma) \right) \\ &- \sum_{i<j}^{1,s} \left(\sum^*_{\substack{\Gamma \imath \xi_i \\ \Gamma \imath \xi_j}} \varphi^T(\Gamma) \right) + \sum_{i<j<k}^{1,s} \left(\sum^*_{\substack{\Gamma \imath \xi_i \\ \Gamma \imath \xi_j \\ \Gamma \imath \xi_k}} \varphi^T(\Gamma) \right) - \cdots .\end{aligned} \tag{Z.15}$$

Hence, if we define

$$\Phi^{(1)}(\mathscr{S}_{\xi_1} \cup \mathscr{S}_{\xi_2} \cup \cdots \cup \mathscr{S}_{\xi_N}) = (-1)^{n+1} \sum^*_{\substack{\Gamma \imath \xi_j \\ j = 1 \ldots n}} \varphi^T(\Gamma) . \tag{Z.16}$$

We have for some $C_2 > 0$, using (3.7) and $|\mathscr{S}_\xi| \geqq (|\xi| + 1)$:

$$|\Phi^{(1)}(\mathscr{S}_X)| \leqq |\mathscr{S}_{\xi_i}| \, \psi(\beta) \, R_o^{-\delta(X)} |\mathscr{S}_{\xi_1}| \, \Phi_o^{(1)}(X) \tag{Z.17}$$

where $\delta(X) =$ distance between the first and the last cluster in X.

Hence if T is a set of non overlapping clusters:

$$\sum_{\substack{X \ni \xi \\ X \subset T}} \Phi_o^{(1)}(X) \leqq \psi(\beta) \left(\sum_{p=1}^{\infty} p 2^p R_o^{-p} \right) \leqq \psi(\beta) . \tag{Z.18}$$

Consider next $\sum_{\Gamma \in C_2(\lambda)} \varphi^T(\Gamma)$. Suppose $\lambda = \mathscr{S}_X$ (in the following we sometimes write $\lambda = (X, \mathscr{S}_X)$) fixed; introduce the following symbols:

a) $(\mathscr{S}_P)$: this symbol means sum of $\varphi^T(\Gamma)$ over the Γ's which intersect all the shapes of the line $\lambda' = (P, \mathscr{S}_P)$.

The next symbols will be defined when the clusters of Q are between the extreme clusters of P.

b) $(\mathscr{S}_P | Q)$: this symbol means sum of the $\varphi^T(\Gamma)$'s over the Γ's which do intersect all the shapes with base P of the line $\lambda'' = (P \cup Q, \mathscr{S}_{P \cup Q})$ but do not intersect all the shapes $\mathscr{S}_P$ in the case we considered only the line $\lambda' = (P, \mathscr{S}_P)$.

c) $(\mathscr{S}_P \| Q)$: this symbol means sum of the $\varphi^T(\Gamma')$, over the Γ's which *would* intersect all the shapes of $(P, \mathscr{S}_P)$ but *do not do so* if the shapes $\mathscr{S}_P$ are considered part of a line $\lambda'' = (\mathscr{S}_{P\cup Q}, P\cup Q)$.

d) $(\mathscr{S}_P)_Q$: this symbol means sum of the $\varphi^T(\Gamma)$'s over the Γ's which intersect all the shapes of $\mathscr{S}_P$ in the line

$$\lambda'' = (P\cup Q, \mathscr{S}_{P\cup Q}).$$

Notice that

$$(\mathscr{S}_P)_Q = (\mathscr{S}_P) + (\mathscr{S}_P | Q) - (\mathscr{S}_P \| Q). \tag{Z.19}$$

Then we can say that if $X = \xi_1$

$$\sum_{\Gamma\in C_2(\lambda)} \varphi^T(\Gamma) = (\mathscr{S}_{\xi_1}) \tag{Z.20}$$

if $X = (\xi_1, \xi_2)$ and $\xi_1 < \xi_2$ (i.e. ξ_1 is to the left of ξ_2):

$$\sum_{\Gamma\in C_2(\lambda)} \varphi^T(\Gamma) = (\mathscr{S}_{\xi_1}) + (\mathscr{S}_{\xi_2}) - (\mathscr{S}_{\xi_1\xi_2}). \tag{Z.21}$$

If $X = (\xi_1, \xi_2, \xi_3)$ and $\xi_1 < \xi_2 < \xi_3$

$$\begin{aligned}\sum_{\Gamma\in C_2(\lambda)} \varphi^T(\Gamma) = {} & (\mathscr{S}_{\xi_1}) + (\mathscr{S}_{\xi_2}) + (\mathscr{S}_{\xi_3}) - (\mathscr{S}_{\xi_1\xi_2}) - (\mathscr{S}_{\xi_2\xi_3}) \\ & - (\mathscr{S}_{\xi_1\xi_3})_{\xi_2} + (\mathscr{S}_{\xi_1\xi_2\xi_3}).\end{aligned} \tag{Z.22}$$

Therefore

$$\begin{aligned}\sum_{\Gamma\in C_2(\lambda)} \varphi^T(\Gamma) = {} & (\mathscr{S}_{\xi_1}) + (\mathscr{S}_{\xi_2}) + (\mathscr{S}_{\xi_3}) - (\mathscr{S}_{\xi_1\xi_2}) - (\mathscr{S}_{\xi_2\xi_3}) \\ & - (\mathscr{S}_{\xi_1\xi_3}) + \{(\mathscr{S}_{\xi_1\xi_2\xi_3}) - (\mathscr{S}_{\xi_1\xi_3} | \xi_2) + (\mathscr{S}_{\xi_1\xi_3} \| \xi_2)\}.\end{aligned} \tag{Z.23}$$

If $X = (\xi_1, \xi_2, \xi_3, \xi_4)$, $\xi_1 < \xi_2 < \xi_3 < \xi_4$ we find

$$\begin{aligned}\sum_{\Gamma\in C_2(\lambda)} \varphi^T(\Gamma) = {} & (\mathscr{S}_{\xi_1}) + (\mathscr{S}_{\xi_2}) + (\mathscr{S}_{\xi_3}) + (\mathscr{S}_{\xi_4}) - (\mathscr{S}_{\xi_1\xi_2}) - (\mathscr{S}_{\xi_2\xi_3}) \\ & - (\mathscr{S}_{\xi_3\xi_4}) - (\mathscr{S}_{\xi_1\xi_3})_{\xi_2} - (\mathscr{S}_{\xi_1\xi_4})_{\xi_2\xi_3} - (\mathscr{S}_{\xi_2\xi_4})_{\xi_3} \\ & + (\mathscr{S}_{\xi_1\xi_2\xi_3}) + (\mathscr{S}_{\xi_2\xi_3\xi_4}) + (\mathscr{S}_{\xi_1\xi_3\xi_4})_{\xi_2} + (\mathscr{S}_{\xi_1\xi_2\xi_4})_{\xi_3} - (\mathscr{S}_{\xi_1\xi_2\xi_3\xi_4})\end{aligned} \tag{Z.24}$$

and using (Z.19) we deduce:

$$\begin{aligned}\sum_{\Gamma\in C_2(\lambda)} \varphi^T(\Gamma) = {} & (\mathscr{S}_{\xi_1}) + (\mathscr{S}_{\xi_2}) + (\mathscr{S}_{\xi_3}) + (\mathscr{S}_{\xi_4}) - \sum_{i<j}^{1,4} (\mathscr{S}_{\xi_i\xi_j}) \\ & + \{(\mathscr{S}_{\xi_1\xi_2\xi_3}) - (\mathscr{S}_{\xi_1\xi_3} | \xi_2) + (\mathscr{S}_{\xi_1\xi_3} \| \xi_2)\} \\ & + \{(\mathscr{S}_{\xi_2\xi_3\xi_4}) - (\mathscr{S}_{\xi_2\xi_4} | \xi_3) + (\mathscr{S}_{\xi_2\xi_4} \| \xi_3)\} \\ & + \{(\mathscr{S}_{\xi_1\xi_2\xi_4}) - (\mathscr{S}_{\xi_1\xi_4} | \xi_1) + (\mathscr{S}_{\xi_1\xi_4} \| \xi_2)\} \\ & + \{(\mathscr{S}_{\xi_1\xi_3\xi_4}) - (\mathscr{S}_{\xi_1\xi_4} | \xi_3) + (\mathscr{S}_{\xi_1\xi_4} \| \xi_3)\} \\ & + \{-(\mathscr{S}_{\xi_1\xi_2\xi_3\xi_4}) + (\mathscr{S}_{\xi_1\xi_4} | \xi_2) - (\mathscr{S}_{\xi_1\xi_4} \| \xi_2) + (\mathscr{S}_{\xi_1\xi_4} | \xi_3) \\ & - (\mathscr{S}_{\xi_1\xi_4} \| \xi_3) - (\mathscr{S}_{\xi_1\xi_4} | \xi_2\xi_3) + (\mathscr{S}_{\xi_1\xi_4} \| \xi_2\xi_3) + (\mathscr{S}_{\xi_1\xi_2\xi_4} | \xi_3) \\ & - (\mathscr{S}_{\xi_1\xi_2\xi_4} \| \xi_3) + (\mathscr{S}_{\xi_1\xi_3\xi_4} | \xi_2) - (\mathscr{S}_{\xi_1\xi_3\xi_4} \| \xi_2)\}.\end{aligned} \tag{Z.25}$$

In general, as it can be seen by induction, if $\xi < X < \xi'$ $X = (\xi_2, \ldots, \xi_{n-1})$ we find:

$$\Phi(\mathscr{S}_{\xi \cup X \cup \xi'}) = (-1)^{N(X)+1} \{(\mathscr{S}_{\xi \cup X \cup \xi'}) + \sum_{\substack{P \subset X \\ Q \subset X \\ P \cap Q = \emptyset \\ Q \neq \emptyset}} (-1)^{N(Q)} [(\mathscr{S}_{\xi \cup P \cup \xi'} | Q) - (\mathscr{S}_{\xi \cup P \cup \xi'} \| Q)]\} \tag{Z.26}$$

it is clear from (3.7) that ($\delta(\xi, \xi')$ is defined in (Z.17) or (5.6)):

$$|(\mathscr{S}_{\xi \cup X \cup \xi'})| \leqq |\mathscr{S}_{\xi'}| \, \psi(\beta) \, R_o^{-\delta(\xi, \xi')}, \tag{Z.27}$$

$$|(\mathscr{S}_{\xi \cup P \cup \xi'} | Q)| \leqq \psi(\beta) \, R_o^{-\delta(\xi, \xi')} |\mathscr{S}_{\xi}|, \tag{Z.28}$$

$$|(\mathscr{S}_{\xi \cup P \cup \xi'} \| Q)| \leqq \psi(\beta) \, R_o^{-\delta(\xi, \xi')} |\mathscr{S}_{\xi'}|, \tag{Z.29}$$

clearly the role of ξ and ξ' is symmetric and no similar inequalities hold with ξ interchanged with ξ'. Hence

$$|\Phi(\mathscr{S}_{\xi \cup X \cup \xi'})| \leqq 4^{\delta(\xi, \xi')} \psi(\beta) \, R_o^{-\delta(\xi, \xi')} |\mathscr{S}_{\xi}| \tag{Z.30}$$

because $\sum_{\substack{P \subset X \\ Q \subset X}}$ contains at most $4^{\delta(\xi, \xi')}$ terms.

Hence calling $\Phi^{(2)}(\xi \cup X \cup \xi')$ the coefficient of $|\mathscr{S}_{\xi}|$ in (Z.30) we find (remember that $R_o = 100$).

$$\sum_{\substack{X \ni \xi_0 \\ X \subset T}} \Phi^{(2)}(X) \leqq \psi(\beta) \sum_{p=1}^{\infty} p 4^p R_o^{-p} \leqq \psi(\beta). \tag{Z.31}$$

It remains to deal with $\sum_{\Gamma \in C_3(\lambda)} \varphi^T(\Gamma)$: A method very similar to the case just treated works and, actually, one could find for this term much better estimates (see sketch in Appendix 6).

Formula (5.6) follows from (Z.30), (Z.17) and the analogous result for the contribution from $C_3(\lambda)$.

Appendix 4. Estimate of $\|\mathfrak{R}\|$

The starting point is the term in the curly bracket (6.12).

We remember that ξ_1 is the first cluster of $X = (\xi_1, \xi_2, \ldots, \xi_m)$ $(\xi_1 < \xi_2 < \xi_3 < \cdots < \xi_m)$ looking from the left to right. Therefore, using (5.4), (5.5), (6.1), we have

$$|U_1(\mathscr{S}_X)| \leqq |\mathscr{S}_{\xi_1}| \, \psi(\beta), \tag{Z.32}$$

$$\sum_i |W_1(\mathscr{S}_X, \mathscr{S}_{P_i})| \leqq (|\mathscr{S}_{\xi_1}| + |\mathscr{S}_{\cup_i P_i}|) \, \psi(\beta) \tag{Z.33}$$

if all the sets $P_1, P_2, \ldots, P_n$ are different and $P_i \cap X = \emptyset$. Therefore the curly bracket in (6.12) is such that (using $e^a - 1 \leqq a e^a$ for $a \geqq 0$):

$$\begin{aligned}
\{\cdot\} &\leqq e^{-\frac{\beta}{2}|\mathscr{S}_{\xi_1}| + \psi(\beta)|\mathscr{S}_{\xi_1}|} \sum_{n=0}^{\infty} \sum_{\substack{(P_1\ldots P_n)\\ P_i\cap X=\emptyset}}^{*} \sum_{\mathscr{S}_{\cup_i P_i}} e^{-\frac{\beta}{2}|\mathscr{S}_{\cup_i P_i}|} \\
&\quad \cdot e^{\sum_i |W_1(\mathscr{S}_X|\mathscr{S}_{P_i})|} \prod_i |W_1(\mathscr{S}_X|\mathscr{S}_{P_i})| \sum_{P''\cap\xi_1\neq\emptyset}^{*} \sum_{\mathscr{S}''_P} e^{-\frac{\beta}{2}|\mathscr{S}''_P|} \\
&\leqq e^{\left(-\frac{\beta}{2}+2\psi(\beta)\right)|\mathscr{S}_{\xi_1}|} \sum_{n=0}^{\infty} \sum_{\substack{(P_1\ldots P_n)\\ P_i\cap X=\emptyset}}^{*} \sum_{\mathscr{S}_{\cup_i P_i}} \left(e^{-\frac{\beta}{2}+\psi(\beta)}\right)^{|\mathscr{S}_{\cup_i P_i}|} \\
&\quad \cdot \prod_i |W_1(\mathscr{S}_X|\mathscr{S}_{P_i})| \sum_{P\cap\xi_1\neq\emptyset}^{*} \sum_{\mathscr{S}''_P} e^{-\frac{\beta}{2}|\mathscr{S}''_P|}
\end{aligned} \tag{Z.34}$$

where the * reminds us that $P_1, \ldots, P_n$, P are made with clusters taken out of a configuration of non overlapping clusters (see (6.4), (6.6)).

The last two sums in (Z.34) can be written as, if $P = (\pi_1, \ldots, \pi_t)$:

$$\begin{aligned}
&\sum_{t=0}^{\infty} \sum_{\substack{(\pi_1\ldots\pi_t)\\ \pi_i\cap\xi_1\neq\emptyset}} \sum_{\mathscr{S}''_{\pi_1}\ldots\mathscr{S}''_{\pi_t}} \prod_{i=1}^{t} e^{-\frac{\beta}{2}|\mathscr{S}''_{\pi_i}|} \leqq \sum_{t=0}^{\infty} \sum_{\substack{\pi_1\ldots\pi_t\\ \pi_i\cap\xi_1\neq\emptyset}} \prod_{t=1}^{t} \left(\sum_{q=|\prod_i|+1}^{\infty} 3^{2q} e^{-\frac{\beta}{2}q} \right) \\
&\leqq \exp\left\{ (|\xi_1|+1) \sum_{r=0}^{\infty} \sum_{q=r+1}^{\infty} \left(3^2 e^{-\frac{\beta}{2}}\right)^q \right\} \leqq \exp\{|\mathscr{S}_{\xi_1}|\,\psi_1(\beta)\}
\end{aligned} \tag{Z.35}$$

where $\psi_1(\beta) \to 0$ exponentially fast as $\beta \to \infty$.

Hence

$$\begin{aligned}
\{\cdot\} &\leqq \left(e^{-\frac{\beta}{2}+2\psi(\beta)+\psi_1(\beta)}\right)^{|\mathscr{S}_{\xi_1}|} \sum_{n=0}^{\infty} \sum_{\substack{P_1\ldots P_n\\ P_i\cap X=\emptyset}}^{*} \sum_{\mathscr{S}'_{\cup_i P_i}} \left(e^{-\frac{\beta}{2}+\psi(\beta)}\right)^{|\mathscr{S}'_{\cup_i P_i}|} \\
&\quad \cdot \prod_i W_1(\mathscr{S}_X, \mathscr{S}'_{P_i}) \\
&\leqq \left(e^{-\frac{\beta}{2}+\psi_1(\beta)+2\psi(\beta)}\right)^{|\mathscr{S}_{\xi_1}|} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\substack{P_1\ldots P_n\\ P_i\cap X=\emptyset\\ P_i \ell \xi_1}} \sum_{\substack{Q_1\ldots Q_m\\ Q_i\cap X=\emptyset\\ Q_i \imath \xi_1}} \sum_{\mathscr{S}'_{\cup_i P_i \cup \cup_j Q_j}} \\
&\quad \cdot \left(e^{-\frac{\beta}{2}+\psi(\beta)}\right)^{|\mathscr{S}'_{\cup_2 P_i \cup \cup_j Q_j}|} \prod_{i=1}^{n} |W(\mathscr{S}_X|\mathscr{S}'_{P_i})| \prod_{j=1}^{m} |W(\mathscr{S}_X|\mathscr{S}'_{Q_j})|
\end{aligned} \tag{Z.36}$$

where $P_i \ell \xi_1$ means that the first cluster in P_i is to the left of ξ_1 while $Q_j \imath \xi_i$ means that the first cluster in Q_j is to the right of ξ_1. Hence using

$$|W_1(\mathscr{S}_X|Q_j)| \leqq |\mathscr{S}_{\xi_1}| \sum_{\xi_1\in T\subset X} |\Phi_0(T\cup Q_j)| \tag{Z.37}$$

and the property (5.5) and summing over the shapes associated with clusters appearing *in* $\cup_j Q_j$ *and not* in $\cup_i P_i$ we find from (Z.36) for β large enough:

$$\{\cdot\} \leqq \left(e^{\left(-\frac{\beta}{2}+\psi_1(\beta)+3\psi(\beta)\right)|\mathscr{S}_{\xi_1}|}\right) \sum_{n=0}^{\infty} \sum^*_{\substack{P_1\dots P_n\\ P_i\cap X=\emptyset\\ P_i \not\ni \xi_1}} \sum_{\mathscr{S}'_{\cup_i P_i}} \left(e^{\left(-\frac{\beta}{2}+\psi(\beta)\right)|\mathscr{S}'_{\cup_i P_i}|}\right) \cdot \prod_{i=1}^{n} |W_1(\mathscr{S}_X|\mathscr{S}_{P_i})| \tag{Z.38}$$

since the sum over the shapes considered is $\leqq 1$ for β large.

Let $P_1, \dots, P_n$ be an element of the sum in (Z.38) and call $\delta_1, \dots, \delta_k$ the different first clusters of $P_1, \dots, P_n$. Since $P_1, P_2, \dots, P_n$ are subclusters of a set of non overlapping clusters, we have that $\delta_i \cap \delta_j = \emptyset$ $i \neq j$. Suppose $\delta_1 < \delta_2 < \dots < \delta_k < \xi_1$. Then the r.h.s. of (Z.38) can be rewritten as:

$$\begin{aligned} \text{r.h.s. (Z.38)} \leqq & \left(e^{\left(-\frac{\beta}{2}+3\psi(\beta)+\psi_1(\beta)\right)|\mathscr{S}_{\xi_1}|}\right) \\ & \cdot \Bigg\{1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{\delta_1<\dots<\delta_k<\xi_1} \sum_{n_1+n_2+\dots+n_k=n} \sum{}' \sum_{\mathscr{S}'_{\cup_i P_i}} \\ & \cdot \left(e^{\left(-\frac{\beta}{2}+\psi(\beta)\right)|\mathscr{S}'_{\cup_i P_i}|}\right) (|\mathscr{S}'_{\delta_1}|)^{n_1} \dots (|\mathscr{S}'_{\delta_k}|)^{n_k} \\ & \cdot \prod_{i=1}^{n} \left(\sum_{\xi_1 \in T \subset X} \Phi_0(T \cup P_i) \right) \Bigg\} \end{aligned} \tag{Z.39}$$

where Σ' runs over the sets of clusters $P_1, P_2, \dots, P_n$ such that $P_1, P_2, \dots, P_{n_1}$ contain δ_1 as first cluster, $P_{n_1+1} \dots P_{n_2}$ contain δ_2 as a first cluster, … …, $P_{n_1+\dots+n_{k-1}}, \dots, P_{n_1+\dots+n_k}$ contain δ_k as a first cluster. We have again used (5.5). Let us perform the sum over the shapes in $\cup_i P_i$ other than $\mathscr{S}_{\delta_1}, \dots, \mathscr{S}_{\delta_k}$ we get (since the sum over these shapes is $\leqq 1$ for large β):

$$\begin{aligned} \text{(r.h.s.) (Z.39)} \leqq & \left(e^{\left(-\frac{\beta}{2}+3\psi(\beta)+\psi_1(\beta)\right)|\mathscr{S}_{\xi_1}|}\right) \\ & \cdot \Bigg(1 + \sum_{k=1}^{\infty} \sum_{\delta_1<\delta_2<\dots<\delta_k<\xi_1} \sum_{\mathscr{S}'_{\delta_1}\dots\mathscr{S}'_{\delta_k}} \sum_{n_1=1}^{\infty} \dots \sum_{n_k=1}^{\infty} \prod_{i=1}^{k} (|\mathscr{S}'_{\delta}|) \left(e^{\left(-\frac{\beta}{2}+\psi(\beta)\right)|\mathscr{S}'_{\delta_i}|}\right) \\ & \cdot \sum{}' \prod_i \left(\sum_{X \supset T \supset \xi_1} \Phi_0(T \cup P_i) \right) \\ \leqq & \left(e^{\left(-\frac{\beta}{2}+3\psi(\beta)+\psi_1(\beta)\right)|\mathscr{S}_{\xi_1}|}\right) \Bigg(1 + \sum_{k=1}^{\infty} \sum_{\delta_1<\dots<\delta_k<\xi_1} \sum_{n_1\dots n_k}^{1,\infty} \sum_{\mathscr{S}'_{\delta_1}\dots\mathscr{S}'_{\delta_k}} \\ & \cdot \prod_{i=1}^{k} \frac{(|\mathscr{S}'_{\delta_i}|)^{n_i}}{n_i!} \left(\sum_{\substack{T\ni\xi_1\\ T\ni\delta_i}} \Phi_0(T) \right)^{n_i} \left(e^{\left(-\frac{\beta}{2}+\psi(\beta)\right)|\mathscr{S}'_{\delta_i}|}\right) \end{aligned} \tag{Z.40}$$

If d_1 is the distance between δ_1 and δ_2, d_2 the distance between δ_2 and $\delta_3, \ldots$, (see (5.6)):

$$\sum_{\substack{T \ni \xi_1 \\ T \ni \delta_1}} \Phi_o(T) \leqq 24(8 R_o^{-1})^{d_i} \psi(\beta) \tag{Z.41}$$

hence, using (Z.41) $\leqq 1$ for large β and $n_i \geqq 1$:

$$\begin{aligned}
&(\text{r.h.s.})\,(\text{Z.40}) \leqq \left(e^{-\frac{\beta}{2}+3\psi(\beta)+\psi_1(\beta)}\right)^{|\mathscr{S}_{\xi_1}|} \Bigg(1+\sum_k \sum_{|\delta_1|\ldots|\delta_k|}^{0,\infty} \sum_{d_1,\ldots,d_k}^{1,\infty} \\
&\quad \cdot \sum_{n_1\ldots n_k}^{1,\infty} \sum_{|\mathscr{S}'_{\delta_1}|\ldots|\mathscr{S}'_{\delta_k}|}^{|\delta|+1,\infty} \prod_{i=1}^{k} \left\{\left(3^2 e^{-\frac{\beta}{2}}\right)^{|\mathscr{S}'_{\delta_i}|}\right\} \prod_{i=1}^{k} \left\{\frac{(|\mathscr{S}'_{\delta}|)^{n_i}}{n_i!} \cdot 24(8R_o^{-1})^{d_i}\psi(\beta)\right\} \\
&\leqq \left(e^{-\frac{\beta}{2}+3\psi(\beta)+\psi_1(\beta)}\right)^{|\mathscr{S}_{\xi_1}|} \Bigg(1+\sum_{k=\emptyset}^{\infty} \sum_{|\delta_1|\ldots|\delta'_k|}^{0,\infty} \sum_{d_1\ldots d_k}^{1,\infty} \\
&\quad \cdot \prod_{i=1}^{k} \frac{\left(3^2 e^{-\frac{\beta}{2}}\right)^{|\delta_i|+1} e^{(|\delta_i|+1)}}{1-3^2 e^{-\frac{\beta}{2}+1}} 24\psi(\beta)(8R_o^{-1})^{d_i}\Bigg) \\
&\leqq \left(e^{-\frac{\beta}{2}+3\psi(\beta)+\psi_1(\beta)}\right)^{|\mathscr{S}_{\xi_1}|} \\
&\quad \cdot \left(1+\sum_{k=\emptyset}^{\infty} \left(\frac{24\psi(\beta)}{1-3^2 e^{-\frac{\beta}{2}+1}}\right)^k \left(\frac{3^2 e^{-\frac{\beta}{2}}}{1-3^2 e^{-\frac{\beta}{2}+1}}\right)^k \left(\frac{8R_o^{-1}}{1-8R_o^{-1}}\right)^k\right. \\
&\leqq e^{-\frac{\beta}{2}+3\psi(\beta)+\psi_1(\beta)} \frac{1}{1-3456.\,\dfrac{R_o^{-1} e^{-\frac{\beta}{2}+1}\,\psi(\beta)}{1-3^2 e^{-\frac{\beta}{2}+1}}} = k(\beta)
\end{aligned} \tag{Z.42}$$

and $k(\beta) \to 0$ exponentially fast as $\beta \to \infty$.

Appendix 5. Derivation of Equations (7.19)

$$\begin{aligned}
\Delta_{\mathscr{S}_{\xi\cup X}}(\mathscr{S}_Y) &= (\bar{\varphi}^{-1} D_{\mathscr{S}_{\xi\cup X}} \bar{\varphi})(\mathscr{S}_Y) = \sum_{Y_1\cup Y_2=Y}{}' \varphi^{-1}(\mathscr{S}_{Y_1})\, \varphi(\mathscr{S}_{\xi\cup X\cup Y_2}) \\
&\equiv \sum_{\substack{Y_1\cup Y_2=Y \\ Y_2\cap(\xi\cup X)=\emptyset}}^{*} \varphi^{-1}(\mathscr{S}_{Y_1})\, e^{-U(\mathscr{S}_{\xi\cup X\cup Y_2})}\, e^{-\beta_0|\mathscr{S}_\xi|}\, e^{-\beta_0|\mathscr{S}_X|}\, e^{-\beta_0|\mathscr{S}_{Y_2}|}
\end{aligned} \tag{Z.43}$$

where the * remembers us that the clusters of Y_2 can be assumed to be non overlapping (see (7.14)).

The chain of equations continues as:

$$\Delta_{\mathscr{S}_\xi \cup X}(\mathscr{S}_Y) = e^{-\beta_0|\mathscr{S}_\xi|} e^{-U_1(\mathscr{S}_{X\cup\xi})} \sum_{\substack{Y_1\cup Y_2=Y\\ Y_2\cap(\xi\cup X)=\emptyset}}'^{*} \varphi^{-1}(\mathscr{S}_{Y_1})\,\varphi(\mathscr{S}_{X\cup Y_2})$$
$$\cdot \prod_{R\subset Y_2} e^{-W_1(\mathscr{S}_X|\mathscr{S}_R)}$$
$$= e^{-\beta_0|\mathscr{S}_\xi|-U_1(\mathscr{S}_{X\cup\xi})} \sum_{\substack{Y_1\cup Y_2=Y\\ Y_2\cap(\xi\cup X)=\emptyset}}'^{*} \varphi^{-1}(\mathscr{S}_{Y_1})\,\varphi(\mathscr{S}_{X\cup Y_2}) \sum_{T\subset Y_2} K_1(\mathscr{S}_{X\cup\xi},\mathscr{S}_T)$$
$$= e^{-\beta_0|\mathscr{S}_\xi|-U_1(\mathscr{S}_{X\cup\xi})} \sum_{\substack{T\subset Y\\ T\cap(\xi\cup X)=\emptyset}}' K_1(\mathscr{S}_{X\cup\xi},\mathscr{S}_T) \sum_{\substack{Y_1\cup Y_2=Y\backslash T\\ Y_2\cap(\xi\cup X\cup T)=\emptyset}}' \varphi^{-1}(\mathscr{S}_{Y_1})\,\cdot$$
$$\cdot\,\varphi(\mathscr{S}_{X\cup T\cup Y_2}) = e^{-\beta_0|\mathscr{S}_\xi|-U_1(\mathscr{S}_{X\cup\xi})} \sum_{\substack{T\subset Y\\ T\cap(\xi\cup X)=\emptyset}} K_1(\mathscr{S}_{X\cup\xi},\mathscr{S}_T) \tag{Z.44}$$
$$\cdot \sum_{\substack{P\cap\xi_1\neq\emptyset\\ P\subset Y\backslash T}} (-1)^{N(P)}\,\Delta_{X\cup T\cup P}(\mathscr{S}_{Y\backslash P\cup T})\,.$$

here the sum over T or P is over the subset of Y or Y\T regarded as consisting of different elements.

Appendix 6

Let $\lambda\in\tilde{\mathfrak{U}}(N)$, $\lambda=\mathscr{S}_X$ and $X=(\xi_1,\ldots,\xi_n)$.

Denote

1) $(i,i+1,\ldots,i+k)_\lambda$ = Sum of $\varphi^T(\Gamma)$ over the Γ's in $C_3(\lambda)$ having a projection on $[A,B]$ which intersects only $\xi_i,\ldots,\xi_{i+k}$.

2) $(j,j+1,\ldots,j+k/j+k+1)$ = Sum of $\varphi^T(\Gamma)$ over the Γ's having a preojection over $[A,B]$ crossing the clusters $\xi_j,\ldots,\xi_{j+k}$ and going, to the right, beyond the first vertical line of ξ_{j+k+1} and, finally, which would be in in $C_3(\lambda')$, where λ' is obtained by continuing λ horizontally to the right of ξ_{j+k}.

3) $(j,\ldots,j+k/j-1)$ = Sum of $\varphi^T(\Gamma)$ over the Γ's having a projection over $[A,B]$ crossing the clusters $\xi_j,\ldots,\xi_{j+k}$ and going, to the left, beyond the last vertical line of ξ_{j-1} and, finally, which are in $C_3(\lambda'')$ where λ'' is obtained by continuing λ horizontally to the left of ξ_j.

4) $(j,\ldots,j+k/j-1,j+k+1)$ = Sum of the $\varphi^T(\Gamma)$ over the Γ's having a projection over $[A,B]$ intersecting the clusters $\xi_j,\ldots,\xi_{j+k}$ and going both to the right and to the left of the last vertical line of ξ_{j-1} or the first vertical line of ξ_{j+k+1}, respectively. Finally the Γ's are required to be in $C_3(\lambda''')$ where λ''' is obtained by continuing λ horizontally to the right and to the left of ξ_{j+k} and ξ_j respectively.

5) $(j,j+1,\ldots,j+k)$ = Sum of the $\varphi^T(\Gamma)$'s which belong to $C_3(\lambda'''')$ where λ'''' is obtained by continuing λ horizontally before ξ_j and after ξ_{j+k}.

One then finds:

$$\sum_{\Gamma\in C_3(\lambda)} \varphi^{T}(\Gamma) = \sum_{i=1}^{n} (i)_\lambda + \sum_{i=1}^{n-1} (i, i+1)_\lambda + \sum_{i=1}^{n-2} (i, i+1, i+2)_\lambda + \cdots + (1, 2, \ldots, n)_\lambda . \tag{Z.45}$$

And using

$$\begin{aligned}(j, j+1, \ldots, j+k)_\lambda &= (j, j+1, \ldots, j+k) - (j, j+1, \ldots, j+k/j-1) \\ &\quad - (j, \ldots, j+k/j+k+1) + (j, \ldots, j+k/j-1, j+k+1)\end{aligned} \tag{Z.46}$$

one gets from (Z.45)

$$\sum_{\Gamma\in C_3(\lambda)} \varphi^{T}(\Gamma) = \sum_{i=1}^{n} \psi_i + \sum_{i=1}^{n-1} \psi_{i,i+1} + \sum_{i=1}^{n-2} \psi_{i,i+1,i+2} + \cdots + \psi_{1,2\ldots n} \tag{Z.47}$$

where (the meaningless symbols are to be set equal to zero):

$$\begin{aligned}\psi_{j,j+1,\ldots,j+k} &= (j, \ldots, j+k) - (j, \ldots, j+k-1/j+k) \\ &\quad + (j+1, \ldots, j+k/j) + (j+1, \ldots, j+k-1/j+k, j) .\end{aligned} \tag{Z.48}$$

Let d = distance between ξ_j and $\xi_{j+k} = d(\xi_j, \xi_{j+k})$ and use the fact that a line appearing in $(j, \ldots, j+k)$ or $(j, \ldots, j+k-1/j+k)$ etc. must cross a horizontal part of the line obtained by continuing horizontally to the left and to the right the part of λ containing $\mathscr{S}_{\xi_{j+1}}, \ldots, \mathscr{S}_{\xi_{j+k+1}}$. Then one gets from (Z.48) and (3.7)

$$|\psi_{j\ldots j+k}| \leqq 4\left(d^2 + \frac{d}{1-2R_o^{-1}}\right)(R_o^{-1})^d\, \psi(\beta) .$$

Hence if $X = \xi \cup R \cup \xi'$ and $\xi < R < \xi'$ from the Möbius inversion formula one gets: (remark that $\psi_{j,\ldots,j+k}$ depends only on $\mathscr{S}_{\xi_j}, \ldots, \mathscr{S}_{\xi_{j+k}}$, see (Z.48))

$$\Phi^{(3)}(\mathscr{S}_X) = \sum_{P\subset X} (-1)^{N(P)}\, U(\mathscr{S}_P) = \sum_{T\subset R} (-1)^{N(T)}\, \psi_{\xi\cup R\cup\xi'} \tag{Z.49}$$

hence

$$\begin{aligned}|\Phi^{(3)}(\mathscr{S}_X)| &\leqq \psi(\beta)\,(2R_o^{-1})^d\, 4\left(d^2 + \frac{d}{1-2R_o^{-1}}\right) \\ &\leqq \psi(\beta)\,(8R_o^{-1})^d \max_{1\leqq d\leqq\infty}\left\{4\left(\frac{d^2 + \frac{d}{1-2R_o^{-1}}}{4^d}\right)\right\}.\end{aligned} \tag{Z.50}$$

Notice that no $|\mathscr{S}_\xi|$ appears in (Z.50) (as a priori forseeable) and this is why we say in Appendix 3, that $\sum_{\Gamma\in C_3(\lambda)}$ gives much stronger results. One could have avoided the $|\mathscr{S}_\xi|$ also in $|\Phi^{(1)}(\mathscr{S}_X)|$ with little extra effort but, of course, this factor cannot be eliminated from $|\Phi^{(2)}(\mathscr{S}_X)|$.

References

1. Dobrushin, R. L.: Functional Anal. Appl. **8**, 302 (1968) (English edition, see p. 309).
2. Minlos, R.: Russian Math. Surveys **23**, 137 (1968).
3. Gallavotti, G.: talk at the 1971 S.I.F. meeting. Internal report of the physics Dept. of the Univ. of Roma n° 347. To appear in Riv. Nuovo Cimento.
4. — Miracle-Sole, S.: Equilibrium states of the Ising model in the two phase region. Phys. Rev. **5** B, 2555 (1972).
5. Minlos, R., Sinai, Ya.: Math. Sbornik **73**, 115 (1967).
6. See [4], see also R. B. Griffiths: Phys. Rev. **152**, 240 (1966) and A. Martin-Lof: preprint.
7. Gallavotti, G., Miracle-Sole, S.: Commun. math. Phys. **7**, 274 (1968).
8. See [3].
9. — Martin-Löf, A.: Surface tension in the two dimensional Ising model. Commun. math. Phys. **25**, 87—126 (1972).
10. Ruelle, D.: Statistical mechanics. New York: Benjamin 1969; p. 83 and p. 86; see also [7] p. 285—288
11. Fisz, M.: Probability theory and mathematical statistics III. edition, p. 211. New York: J. Wiley.
12. Kolmogorov, A. N.: Selected translations in Math. Statistics and Probability. IMS and AMS 1962 (translation of the AMS) p. 109.
13. Spitzer, F.: Am. Math. Monthly **78**, 142 (1971).
14. Van Beyeren, H.: private communication. To be published.
15. The functions φ and φ^T of this section are obviously not the same as the ones of Theorem 2: there should be no confusion between them since they are defined on completely different spaces.

Giovanni Gallavotti
Istituto di Matematica
Università di Roma
Piazzale delle Scienze
I-00187, Roma, Italy