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Hydrogen Atom in the Friedman Universe

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Abstract. Within the framework of general relativity the Dirac equation for the hydrogen atom is given in case of a spatially isotropic and homogeneous expanding space-time (Robertson Walker metric). In the special case of the static, closed 3-dimensional spherical space (Einstein Universe) we get a continuous energy spectrum for the H-atom.

Introduction

In Einstein's gravitational theory the existence of so-called standard measures (standard rods, standard clocks) is assumed, which allow to measure explicitly the structure of the 4-dimensional space-time. In general, one supposes that the rods and clocks of the microphysics represents such standard measures.

On the other hand, one should emphasize that also the microphysic is embedded in the space-time and hence is influenced by its structure. Nevertheless, the above assumption seems to be justified in so far as in general a noticeable disturbance of the microphysical standard measures by the metric field is to be expected only if the gradients of the metric inside the system are comparable with the interaction forces. But this will be the case only under extreme conditions.

However, these considerations should be taken carefully. Peres [1], [2], and Callaway [3] have analyzed the influence of the space-curvature on the hydrogen spectrum caused by the mass and charge of the hydrogen nucleus. They got a continuous spectrum for the H-atom with normalized wave functions; until today, the resolution of this paradox has not been found.

Furthermore, an influence of the topology of space-time on the microphysics and therefore also on the microphysical standard measure should be expected. Already 1936 Taub [4], 1938 Schrödinger [5] and 1946 Infeld and Schild [6] have shown that in the case of free electrons embedded in a closed 3-dimensional space of positive curvature there exist differences in the solutions of the Dirac equation for the spherical and the elliptical space. In case of the bounded electrons 1932 McVittie [7] has analyzed the energy spectrum of the hydrogen atom on the basis of the general relativistic Dirac equation using a perturbation method. In case of an expanding, closed or open universe he has found that the energy levels of the hydrogen atom are only slightly influenced by the expansion or contraction of the universe. Recently, Dautcourt [8] has investigated the influence of the topology of space-time on the energy spectrum of the hydrogen atom with the help of a perturbational approach starting from the general relativistic Schrödinger equation. In case of the static Einstein universe he finds a discrete energy spectrum which differs however from that of the hydrogen atom in the flat space. This seems not too much surprisingly because in the last both cases the perturbation methods start from the well-known results for the flat space.

In contrast to this, in this paper the influence of a cosmological metric on the hydrogen atom will be treated exactly. The hydrogen atom is embedded in a spatially isotropic and homogeneous expanding space-time (Robertson Walker metric) and the corresponding general relativistic Dirac equation is discussed neglecting the reaction of the H-atom on the structure of the space-time. Restricting ourselves to the static cosmos we analyze the energy eigenvalue spectrum of the Dirac equation with a method developed by Weyl and Titchmarsh. In case of the *spherical* space (Einstein universe) a continuous spectrum for the H-atom follows in the total range of the energy spectrum in contrast to McVittie's and Dautcourt's results. Analogously to the results of the free electrons mentioned above one should expect a difference in the energy spectrum of the H-atom between the spherical and the elliptical space. But in the latter case our analysis of the eigenvalue problem of the hydrogen atom did not succeed. In case of the hyperbolic space one gets a discrete energy spectrum just in the flat space.

1. Dirac Equation in the Friedman Universe

For the description of the hydrogen atom, we use the Euler-Lagrange equation for the classical Dirac field in a curved space-time, which takes in the 4-spinor-formalism the form:

$$\gamma_A^{\mu B}\left(i\psi_{B||\mu} - \frac{e}{\hbar c} A_\mu \psi_B\right) - \frac{m_0 c}{\hbar}\psi_A = 0 \tag{1.1}$$

($A, B, \ldots = 1, 2, 3, 4$ spinor indices; $\mu, \nu = 1, 2, 3, 4$ tensor indices). A_μ means the electromagnetic vectorpotential according to the general relativistic Maxwell equations; $\gamma_A^{\mu B}$ represent the generalized Dirac matrices defined through the following commutation relation:

$$\left.\begin{aligned} &\gamma_A^{\mu B}\gamma_B^{\nu C}+\gamma_A^{\nu B}\gamma_B^{\mu C}=2g^{\mu\nu}\varepsilon_A^C\,,\\ &\varepsilon_A^C=\begin{pmatrix}1&&&0\\&1&&\\&&1&\\0&&&1\end{pmatrix}.\end{aligned}\right\}\tag{1.2}$$

Eq. (1.2) is satisfied by

$$\gamma_A^{\mu B}=h^{\mu}_{(\varrho)}\gamma_A^{(\varrho)B}\,,\tag{1.3}$$

wherein the $\gamma_A^{(\varrho)B}$ means Dirac's standard matrices

$$\begin{aligned}&\gamma^{(1)}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{pmatrix},\quad \gamma^{(2)}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix},\\ &\gamma^{(3)}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{pmatrix},\quad \gamma^{(4)}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\end{aligned}\tag{1.4}$$

and $h^{\mu}_{(\varrho)}$ describe a orthonormal tetrad ((ϱ) tetrad index) following from:

$$g_{\sigma\alpha}=h_{(\varrho)\sigma}h_{(\mu)\alpha}\varepsilon^{(\varrho\mu)},\qquad \varepsilon^{(\varrho\mu)}=h^{(\varrho)}_{\alpha}h^{(\mu)\alpha}=\begin{pmatrix}-1&&&0\\&-1&&\\&&-1&\\0&&&+1\end{pmatrix}.\tag{1.5}$$

Furthermore, the covariant derivation of the Dirac spinor ψ_B is

$$\psi_{B||\mu}=\psi_{B|\mu}-\Gamma^C_{B\mu}\psi_C\,,\tag{1.6}$$

in which $\Gamma^C_{B\mu}$ means the spinorial connection coefficients given by:

$$\Gamma^C_{B\mu}=\tfrac{1}{4}h^{\alpha}_{(\varrho)||\mu}h^{(\varrho)}_{\sigma}\gamma_B^{\sigma D}\gamma^C_{\alpha D}\,.\tag{1.7}$$

The Robertson-Walker lineelement underlies Friedman's spatially isotropic and homogeneous expanding cosmological models. In angular coordinates this lineelement is

$$ds^2=-R^2(t)\,\{d\chi^2+\xi^2(\chi)\,d\Omega^2\}+dt^2\,,\tag{1.8}$$

in which

$$\xi(\chi)=\begin{cases}\sin\chi & \begin{cases}0\leqq\chi\leqq\pi & \text{spherical 3-dim.space}\\ 0\leqq\chi\leqq\pi/2 & \text{elliptical space}\end{cases}\\ \chi & \quad 0\leqq\chi\leqq\infty \quad \text{flat space-time}\\ \sinh\chi & \quad 0\leqq\chi\leqq\infty \quad \text{hyperbolic 3-dim.space}\end{cases} \tag{1.9}$$

and

$$d\Omega^2=d\vartheta^2+\sin^2\vartheta\,d\varphi^2 \tag{1.10}$$

stands for the element of the spherical angle. The "radius of curvature" $R(t)$ is to be determined by the field equations of gravitation.

Chosing for the tetrad $h^{\mu}_{(\varrho)}$ the tangential vectors on the coordinate lines of the metric (1.8) (which are orthogonal together) the generalized Dirac matrices are according to (1.3):

$$\gamma^1=-\frac{1}{R}\gamma^{(1)},\ \gamma^2=-\frac{1}{R\xi}\gamma^{(2)},\ \gamma^3=-\frac{1}{R\xi\sin\vartheta}\gamma^{(3)},\ \gamma^4=\gamma^{(4)}. \tag{1.11}$$

Correspondingly, the spinorial connection coefficients results according to (1.7) in:

$$\left.\begin{aligned}\Gamma_1&=\frac{1}{2}\frac{dR}{dt}\gamma^{(4)}\gamma^{(1)},\\ \Gamma_2&=\frac{1}{2}\frac{d\xi}{d\chi}\gamma^{(2)}\gamma^{(1)}+\frac{1}{2}\frac{dR}{dt}\xi\gamma^{(4)}\gamma^{(2)},\\ \Gamma_3&=\frac{1}{2}\sin\vartheta\frac{d\xi}{d\chi}\gamma^{(3)}\gamma^{(1)}+\frac{1}{2}\cos\vartheta\gamma^{(3)}\gamma^{(2)}-\frac{1}{2}\xi\frac{dR}{dt}\sin\vartheta\gamma^{(3)}\gamma^{(4)},\\ \Gamma_4&=0.\end{aligned}\right\} \tag{1.12}$$

Furthermore, for the treatment of the H-atom, the knowledge of the vectorpotential A_μ for the Coulombfield of the proton is necessary. Under the assumptions, that the observer being at rest relatively to the material substrat of the Friedman universe perceive no magnetic but only a central symmetric electric vacuum-field (field of a point charge at "rest"), one gets (uniquely to gauge transformations) by integration of the covariant Maxwell equation:

$$\left.\begin{aligned}A_\mu&=(0,0,0,V),\\ V&=\begin{cases}-\dfrac{e}{R}\operatorname{ctg}\chi & \text{for}\quad \varepsilon=1\\ -\dfrac{e}{R\chi} & \text{for}\quad \varepsilon=0\\ -\dfrac{e}{R}(\operatorname{ctgh}\chi-1) & \text{for}\quad \varepsilon=-1.\end{cases}\end{aligned}\right\} \tag{1.13}$$

In case of spherical space ($\varepsilon = 1$), the potential V has at the position $\chi = 0$ a singular *positive* and at the position $\chi = \pi$ a singular *negative* source. Therefore, the electric field E_μ has a behavior just as at the point $\chi = \pi$ a negative point charge would be situated (ghost charge); one gets from (1.13):

$$E_\mu = \left(\frac{e}{R\xi^2}, 0, 0, 0\right). \tag{1.14}$$

2. Separation of the Generalized Dirac Equation

Now, we are able to specify the Dirac equation for the hydrogen atom embedded in the Friedman universe. With respect to (1.13) we obtaine from Eq. (1.1):

$$i\gamma^\mu \psi_{|\mu} - i\gamma^\mu \Gamma_\mu \psi - \frac{e}{\hbar c}\gamma^4 V\psi - \frac{m_0 c}{\hbar}\psi = 0, \tag{2.1}$$

in which with regard to (1.11) and (1.12)

$$i\gamma^\mu \Gamma_\mu = -\frac{3}{2}\frac{1}{R}\frac{dR}{dt}\gamma^{(4)} + \frac{1}{R\xi}\frac{d\xi}{d\chi}\gamma^{(1)} + \frac{1}{2}\frac{\cos\vartheta}{R\xi\sin\vartheta}\gamma^{(2)}. \tag{2.2}$$

It is easy to show, that the second term in (2.1) can be removed by the substitution

$$\psi_A = R^{-\frac{3}{2}}\xi^{-1}\sin^{-\frac{1}{2}}\vartheta\,\Phi_A. \tag{2.3}$$

For Φ_A one gets the simplified differential equation:

$$i\gamma^\mu \Phi_{|\mu} - \frac{e}{\hbar c}\gamma^{(4)} V\Phi - \frac{m_0 c}{\hbar}\Phi = 0. \tag{2.4}$$

With the help of the ansatz

$$\Phi = -\begin{pmatrix} F(\chi, t)\cdot u(\vartheta, \varphi) \\ F(\chi, t)\cdot v(\vartheta, \varphi) \\ iG(\chi, t)\cdot v(\vartheta, \varphi) \\ iG(\chi, t)\cdot u(\vartheta, \varphi) \end{pmatrix}. \tag{2.5}$$

Eq. (2.4) can be separated in the differential equations for the generally time dependent radial parts F and G and the angle parts u and v. One finds:

$$\left.\begin{aligned} &\xi\frac{\partial G}{\partial\chi} + kG + iR\xi\frac{\partial F}{\partial t} - R\xi\left(\frac{m_0 c}{\hbar} + \frac{e}{\hbar c}V\right)F = 0, \\ &\xi\frac{\partial F}{\partial\chi} - kF - iR\xi\frac{\partial G}{\partial t} - R\xi\left(\frac{m_0 c}{\hbar} - \frac{e}{\hbar c}V\right)G = 0 \end{aligned}\right\} \tag{2.6}$$

and

$$\left.\begin{aligned} i\frac{\partial u}{\partial \vartheta} + ku - \frac{1}{\sin\vartheta}\frac{\partial v}{\partial \varphi} &= 0\,, \\ i\frac{\partial v}{\partial \vartheta} - kv - \frac{1}{\sin\vartheta}\frac{\partial u}{\partial \varphi} &= 0\,. \end{aligned}\right\} \tag{2.7}$$

The system (2.7) for the angle parts of the wave function don't differ from that in the flat space-time as a consequence of the symmetry of the metric (1.8). Therefore, the angular momentum eigenfunctions and the eigenvalues belonging to them are the same as in the flat space and shall not be studied furthermore.

3. Stationary Systems

In the following we are interested in stationary solutions of the differential equation (2.6). Therefore, we make the separation ansatz:

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} e^{i\frac{Et}{\hbar c}}, \qquad f = f(\chi), \qquad g = g(\chi)\,. \tag{3.1}$$

The necessary and sufficient condition for realizing the separation (3.1) is

$$R(t) = R_0 = \text{const}\,, \tag{3.2}$$

i.e. the universe is static. Therefore, in case of an expanding universe no stationary solutions in the sense of (3.1) exist for the hydrogen atom. With the restriction (3.2) the system (2.6) passes in view of (3.1) into $\left({}' \mathrel{\hat{=}} \frac{\partial}{\partial \chi}\right)$:

$$\left.\begin{aligned} \xi g' + kg - R_0 \xi \left(\frac{m_0 c}{\hbar} - \frac{E}{\hbar c} + \frac{\alpha V}{e}\right) f &= 0\,, \\ \xi f' - kf - R_0 \xi \left(\frac{m_0 c}{\hbar} + \frac{E}{\hbar c} - \alpha \frac{V}{e}\right) g &= 0 \end{aligned}\right\} \tag{3.3}$$

with the reciprocal compton wavelength $\frac{m_0 c}{\hbar}$, the energy eigenvalues E and the fine structure constant $\alpha = \frac{e^2}{\hbar c}$. In case $\xi = R_0 \chi$ one gets the well-known Dirac equations of the flat space. For $\xi = \sin\chi$ and $\xi = \sinh\chi$ the equations pass into the Dirac equations of the flat space only for $\chi \ll 1$ of course.

In case of the flat space the energy eigenvalues will be caused by the requirement of the quadratic integrability of the wavefunction ψ which demands $\psi(\chi \to \infty) \to 0$ for the behavior in the infinity. From this the well-known energy eigenvalues result:

$$E = \frac{m_0 c^2}{\sqrt{1 + \frac{\alpha^2}{(n + \sqrt{k^2 - \alpha^2})^2}}}, \tag{3.4}$$

in which $n = 0, 1, 2, \ldots$ mean the radial quantum numbers and $k = \pm 1, \pm 2, \pm 3, \ldots$ the angular momentum quantum numbers.

However, in case of the spherical space the circumstances are entirely different from those in the flat space, because the spherical space is finite and closed. In the following we will demonstrate that in this case the Dirac equation possesses no discrete but a continuous energy spectrum. In case of the hyperbolic space one gets a discrete energy spectrum in the same energy range as in the flat space.

4. Theorem of Weyl and Titchmarsh

For the determination of the properties of the energy spectrum of the hydrogen atom in case of the spherical and hyperbolic static space we don't integrate the differential equations (3.3) but use a theorem for the eigenvalues spectrum of systems of differential equations, which H. Weyl and later Titchmarsh [9] for differential equations of the second order have given and which Titchmarsh [10] and Roos and Sangren [11–13] have extended on systems of differential equations of the first order. The explicit integration of the differential equations (3.3) in case of the spherical and hyperbolic space is not yet succeded until today.

The theorems of Weyl, Titchmarsh, Sangren, Roos contain the following *theorem*:

It is given the system of differential equations of first order

$$\left.\begin{aligned} \frac{dx_1(r)}{dr} - \{\lambda a(r) + b(r)\}\, x_2(r) = 0\,, \\ \frac{dx_2(r)}{dr} + \{\lambda c(r) + d(r)\}\, x_1(r) = 0\,. \end{aligned}\right\} \tag{4.1}$$

In this λ is a complex constant and a, b, c, d are real-valued functions of r in the interval $0 \leqq r \leqq \infty$, which are sectionally continuous and from which a and c are positive. Furthermore, the following boundary condition at $r = 0$ may be valid:

$$\cos\beta K^{-1}(0)\, x_1(0) + \sin\beta K(0)\, x_2(0) = 0\,, \tag{4.2}$$

where $K(r) = \{a(r)/c(r)\}^{\frac{1}{4}}$ and β is a real constant.

Now, from the functions a, b, c, d the following auxiliary functions are defined:

$$A(r) = \int_0^r \{[\lambda a(t) + b(t)]\,[\lambda c(t) + d(t)]\}^{\frac{1}{2}}\,dt\,, \tag{4.3}$$

$$F(r, \lambda) = [\lambda a(r) + b(r)]^{\frac{1}{4}}\,[\lambda c(r) + d(r)]^{-\frac{1}{4}}\,, \tag{4.4}$$

$$S(r) = \frac{1}{F(r)} \cdot \frac{d}{dr}\left(\frac{dF}{dr} \Big/ \frac{dA}{dr}\right). \tag{4.5}$$

If $S(r)$ is simply integrable, the following theorem is valid:

The real λ-values associated to the solutions $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of the system of differential equations (4.1) and (4.2) by which any quadratically integrable function $H = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ can be combined by superposition belong to

(a) a continuous λ-spectrum, if $S(r)$, $F(r)$, $A(r)$ are simultaneously real-valued,

(b) a discrete λ-spectrum, if $S(r)$ and $A(r)$ are imaginary and $F(r)$ is complex. In this case the solutions $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ are already quadratically integrable.

Therefore, it depends on the coefficients $a(r)$, $b(r)$, $c(r)$, $d(r)$ of the system of the differential equations (4.1), if the case of a continuous or a discrete spectrum is realized. With the help of the two quotients

$$q_1(r) = \frac{b(r)}{a(r)}, \qquad q_2(r) = \frac{d(r)}{c(r)} \tag{4.6}$$

one can distinguish with regard to the mentioned theorem the following cases because of the asymptotic behavior of $q_1(r)$ and $q_2(r)$ for $r \to \infty$:

$$\text{(A)} \quad \left.\begin{array}{l} 1)\begin{cases} q_1 \to \infty \\ q_2 \to -\infty \end{cases} \\ 2)\begin{cases} q_1 \to -\infty \\ q_2 \to \infty \end{cases} \end{array}\right\} \quad \text{for} \quad -\infty < \lambda < \infty \qquad \text{point spectrum}$$

$$\text{(B)} \quad \left.\begin{array}{l} 1)\begin{cases} q_1 \to \infty \\ q_2 \to \infty \end{cases} \\ 2)\begin{cases} q_1 \to -\infty \\ q_2 \to -\infty \end{cases} \end{array}\right\} \quad \text{for} \quad -\infty < \lambda < \infty \qquad \text{continuous spectrum}$$

(C) $\left.\begin{array}{l} 1)\begin{cases} q_1 \to \infty \\ q_2 \to 0 \end{cases} \\ 2)\begin{cases} q_1 \to 0 \\ q_2 \to \infty \end{cases} \end{array}\right\}$ for $-\infty < \lambda < 0$ point spectrum; $0 < \lambda < \infty$ continuous spectrum

(D) $\left.\begin{array}{l} 1)\begin{cases} q_1 \to -\infty \\ q_2 \to 0 \end{cases} \\ 2)\begin{cases} q_1 \to 0 \\ q_2 \to -\infty \end{cases} \end{array}\right\}$ for $-\infty < \lambda < 0$ continuous spectrum; $0 < \lambda < \infty$ point spectrum

(E) $\left.\begin{cases} q_1 \to Q_1 \\ q_2 \to Q_2 \end{cases}\right\}$ for $-\infty < \lambda < Q_1$ continuous spectrum; $Q_1 < \lambda < Q_2$ point spectrum; $Q_2 < \lambda < \infty$ continuous spectrum

with $-\infty < Q_1 < Q_2 < \infty$.

5. Energy Spectrum of the Hydrogen Atom

Now, we apply the above theorem to the system of differential equation (3.3). For this we set it into the form of the system of differential equation (4.1) carrying out the following transformation of the dependent variables:

$$\left.\begin{aligned} g(\chi) &= x_2(\chi)\exp\left\{-k\int^{\chi}\frac{dy}{\xi(y)}\right\}, \\ f(\chi) &= x_1(\chi)\exp\left\{+k\int^{\chi}\frac{dy}{\xi(y)}\right\}. \end{aligned}\right\} \tag{5.1}$$

Herewith the system of differential equation (3.3) passes into $\left({}' \mathrel{\hat=} \frac{d}{d\chi}\right)$:

$$\left.\begin{aligned} x_1' - R_0\left(\frac{E}{\hbar c} + \frac{m_0 c}{\hbar} - \alpha\frac{V(\chi)}{e}\right) x_2 \exp\left\{-2k\int^{\chi}\frac{dy}{\xi(y)}\right\} &= 0, \\ x_2' + R_0\left(\frac{E}{\hbar c} - \frac{m_0 c}{\hbar} - \alpha\frac{V(\chi)}{e}\right) x_1 \exp\left\{+2k\int^{\chi}\frac{dy}{\xi(y)}\right\} &= 0, \end{aligned}\right\} \tag{5.2}$$

which has already the structure of the Eqs. (4.1). For the immediate applicability of the theorem in the several cases of the spherical, flat and hyperbolic space a suitable transformation of the independent variables is necessary. In case of spherical and flat space ($\varepsilon = 1, 0$) we carry out the following transformation:

$$R_0 \xi(\chi) = \frac{r}{1+\varepsilon \frac{r^2}{4R_0^2}}, \qquad \xi(\chi) = \begin{cases} \sin\chi & \text{for} \quad \varepsilon = 1, \\ \chi & \text{for} \quad \varepsilon = 0. \end{cases} \tag{5.3}$$

Herewith the system of differential equation results in:

$$\left.\begin{aligned} \frac{dx_1}{dr} - \frac{(r/2R_0)^{-2k}}{1+\varepsilon \frac{r^2}{4R_0^2}} \left\{ \frac{E}{\hbar c} + \frac{m_0 c}{\hbar} - \alpha \frac{V(r)}{e} \right\} x_2 = 0, \\ \frac{dx_2}{dr} + \frac{(r/2R_0)^{2k}}{1+\varepsilon \frac{r^2}{4R_0^2}} \left\{ \frac{E}{\hbar c} - \frac{m_0 c}{\hbar} - \alpha \frac{V(r)}{e} \right\} x_1 = 0. \end{aligned}\right. \tag{5.4$$

The comparison from (5.4) with (4.1) yields:

$$\lambda = \frac{E}{\hbar c}, \tag{5.5}$$

$$\left.\begin{aligned} a(r) &= \frac{(r/2R_0)^{-2k}}{1+\varepsilon \frac{r^2}{4R_0^2}}, & b(r) &= \left\{ \mu - \alpha \frac{V(r)}{e} \right\} \frac{(r/2R_0)^{-2k}}{1+\varepsilon \frac{r^2}{4R_0^2}}, \\ c(r) &= \frac{(r/2R_0)^{2k}}{1+\varepsilon \frac{r^2}{4R_0^2}}, & d(r) &= \left\{ -\mu - \alpha \frac{V(r)}{e} \right\} \frac{(r/2R_0)^{2k}}{1+\varepsilon \frac{r^2}{4R_0^2}}, \end{aligned}\right\} \tag{5.6}$$

in which $\mu = \frac{m_0 c}{\hbar}$ is the reciprocal Compton wave length. From this the quotients q_1 and q_2 are calculated accordingly (4.6) to

$$\left.\begin{aligned} q_1(r) &= \mu - \alpha \frac{V(r)}{e}, \\ q_2(r) &= -\mu - \alpha \frac{V(r)}{e}, \end{aligned}\right\} \tag{5.7}$$

in which in view of (1.13) and (5.3)[1]

$$\frac{V(r)}{e} = -\frac{1}{r} \sqrt{\left(1+\varepsilon \frac{r^2}{4R_0^2}\right)^2 - \varepsilon \frac{r^2}{R_0^2}}. \tag{5.8}$$

[1] The application of the theorem of § 4 assumes that the function $S(r)$ (cf. (4.5)) is integrable. This is only the case if the electric field has a behavior as $\xi^{-2\delta}$ with $\delta > 1$ (cf. (1.14)).

For the well-known case of *the flat space* ($\varepsilon = 0$) it follows from (5.8)

$$\frac{V(r \to \infty)}{e} \to -\frac{1}{r} \to 0 \tag{5.9}$$

and therefore according to (5.7):

$$\left.\begin{aligned} q_1(r \to \infty) &\to \mu\,, \\ q_2(r \to \infty) &\to -\mu\,. \end{aligned}\right\} \tag{5.10}$$

With respect to case (E) of the theorem we get (cf. (5.5)) a discrete spectrum in the energy range $-m_0c^2 < E < m_0c^2$, and outside of this interval a continuous spectrum in accordance with the well-known results.

In contrast to this in case of *the spherical space* ($\varepsilon = 1$) one obtains from (5.8)

$$\frac{V(r \to \infty)}{e} \to -\frac{r}{2R_0^2} \to -\infty \tag{5.11}$$

and according to this

$$\left.\begin{aligned} q_1(r \to \infty) &\to \infty\,, \\ q_2(r \to \infty) &\to \infty\,. \end{aligned}\right\} \tag{5.12}$$

Corresponding to case (B) of the theorem (cf. (5.5)) *a continuous spectrum results in the total energy range* $-\infty < E < \infty$.

In case of *the hyperbolic space* ($\varepsilon = -1$) we start immediately with the system of Eqs. (5.2). Calculating in (5.2) the integrals and putting in the potential $V(\chi)$ accordingly (1.14) for $\varepsilon = -1$ one gets with $\lambda = \frac{E}{\hbar c}$ $\left({}' \mathrel{\widehat{=}} \frac{d}{d\chi}\right)$:

$$\left.\begin{aligned} x_1' - R_0 \operatorname{tgh}^{-2k}\frac{\chi}{2}\left\{\lambda + \mu + \frac{\alpha}{R_0}(\operatorname{ctgh}\chi - 1)\right\}x_2 &= 0\,, \\ x_2' + R_0 \operatorname{tgh}^{+2k}\frac{\chi}{2}\left\{\lambda - \mu + \frac{\alpha}{R_0}(\operatorname{ctgh}\chi - 1)\right\}x_1 &= 0\,. \end{aligned}\right\} \tag{5.13}$$

The comparison from (5.13) with (4.1) yields for the functions a, b, c, d after the identification of χ with r:

$$\left.\begin{aligned} a &= R_0 \operatorname{tgh}^{-2k}\frac{r}{2}, & b &= \left\{\mu + \frac{\alpha}{R_0}(\operatorname{ctgh} r - 1)\right\} R_0 \operatorname{tgh}^{-2k}\frac{r}{2}\,, \\ c &= R_0 \operatorname{tgh}^{2k}\frac{r}{2}, & d &= \left\{-\mu + \frac{\alpha}{R_0}(\operatorname{ctgh} r - 1)\right\} R_0 \operatorname{tgh}^{2k}\frac{r}{2}\,. \end{aligned}\right\} \tag{5.14}$$

Herewith the quotients q_1 and q_2 become accordingly (4.6):

$$\left.\begin{aligned} q_1 &= \mu + \frac{\alpha}{R_0}(\operatorname{ctgh} r - 1), \\ q_2 &= -\mu + \frac{\alpha}{R_0}(\operatorname{ctgh} r - 1). \end{aligned}\right\} \tag{5.15}$$

For the asymptotic behaviour one finds:

$$\left.\begin{aligned} q_1(r \to \infty) &\to \mu, \\ q_2(r \to \infty) &\to -\mu, \end{aligned}\right\} \tag{5.16}$$

According to case (E) of the theorem we obtain (cf. (5.5)) a discrete spectrum in the energy interval $-m_0c^2 < E < m_0c^2$, and outside of this a continuous spectrum. Consequently, the range of the discrete energy spectrum is identical with that of the hydrogen atom in the flat space.

Obviously, one obtains a result differing from that in the flat space only in case of the space with positive curvature which is topologically distinct from the flat space. Therefore, in case of a static spherical space (Einstein Universe) the hydrogen atom cannot be used as a standard clock. Thus in general, the hypothesis that the clocks of the microphysics represent always "good" standard clocks cannot be maintained.

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