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On the Geometrical Interpretation of the Harmonic Analysis of the Scattering Amplitude

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Abstract. In this paper we intend to analyze the geometry underlying the various representations of the relativistic scattering amplitudes. More precisely we consider the direct-channel expansion, its euclidean contraction and the crossed-channel representation. In all these representations one can distinguish the factors which express the dynamics from those which reflect the symmetry; starting from the latter, one can try a geometrical interpretation of the harmonic analysis of the scattering amplitude on the Poincaré group.

I. Introduction

In the last decade, many authors [1–5] contributed in making clear the role played in particle physics by the harmonic analysis of the scattering amplitude on the Poincaré group. Consider the elastic scattering of two scalar particles of equal mass m and denote with $T(s, t)$ the scattering amplitude, where s and t are the usual Mandelstam variables; i.e. s is the energy squared and t the momentum-transfer squared. In a paper which appeared in 1962 Joos [5] considered the relativistic phase-shift analysis in the so-called s-channel. In this case one takes $s>0$ fixed and the scattering amplitude is decomposed in functions of $z=\cos\vartheta=1+\frac{2t}{s-4m^2}$; i.e. the cosine of the scattering angle in the center of mass system. These functions are the Legendre polynomials and the little group is the rotation group. However one can also invert the situation and take $t<0$ fixed and decompose the scattering amplitude in functions of s obtaining in this way the so-called t-channel (or crossed-channel) phase-shift analysis. In the latter case, instead of the center of mass system the convenient system is the brick-wall system [4] and the little group is the non-compact group $SO(2, 1)$. One of the great advantages of these decompositions is that the scattering amplitude is thereby separated into its dynamical part, contained in the partialwaves, and its symmetry part which is expressed through the various types of higher transcendental functions which appear in the representations.

Furthermore, quite recently, several mathematical schools [6–7] reconsidered the theory of special functions of mathematical physics and succeeded in giving a group-theoretical foundation to a large part of the enormous amount of properties of these functions. Moreover, Helgason [8] introduced in this framework the powerful methods of differential geometry of Riemannian manifolds. Therefore, many relationships among the algebra of Lie groups, the Riemannian geometry of manifolds and the theory of higher transcendental functions were discovered. Along these lines one can also generalize and give a foundation to a Fourier-analysis on a manifold of constant negative curvature [9], in a sense which we will clarify later on. From this point of view, it seems reasonable to try a geometrical analysis of the various expansions of the scattering amplitude which we have mentioned above. More precisely the main aim of the present note consists in indicating how the factors which express the symmetry of the scattering amplitude are related to the geometry on manifolds of constant, positive and negative, curvature. The mathematical tools which are needed in order to perform this analysis are essentially given by Helgason in Refs. [8] and [9]; therefore, we must borrow some of the methods and theorems which are proved there. Finally, this paper is the continuation and completion of a previous short note [10] where we have analyzed the t-channel representation from the point of view of the non-euclidean Fourier-analysis; this problem is here reconsidered in Section III while Section II is devoted to the s-channel expansion and its euclidean contraction.

II. The Direct-Channel Expansion and Its Euclidean Contraction

In order to introduce the notations, let us write the direct-channel and the crossed-channel expansions. The usual s-channel expansion in its symmetrical from can be written as follows:

$$T(s,t)=\sum_{l=0}^{\infty}(2l+1)f_l(s)\left[P_l(\cos\vartheta)+P_l(-\cos\vartheta)\right] \tag{1}$$

where P_l are the Legendre polynomials, l denotes the angular momentum, $\cos\vartheta=1+\frac{2t}{s-4m^2}$ is the cosine of the scattering angle in the center of mass system (as we mentioned above) and finally $s=(p_1+p_2)^2$, $t=(p_1-p_1')^2$, p_1 and p_2 being the momenta of the incoming particles, p_1' and p_2' the momenta of the outgoing particles.

The crossed-channel or Joos expansion [3] can be written as follows:

$$T(s,t) \tag{2}$$
$$= \frac{1}{i} \int_0^{+\infty} \frac{d\lambda\,\lambda f(\lambda,t)}{\cosh(\pi\lambda)} \left[P_{-\frac{1}{2}+i\lambda}\left(1+\frac{2s}{t-4m^2}\right)+P_{-\frac{1}{2}+i\lambda}\left(-1-\frac{2s}{t-4m^2}\right)\right]$$

where $P_{-\frac{1}{2}+i\lambda}$ are the so-called conical functions (see Ref. [11], Vol. I°, p. 174).

Now we come to the geometry. Let M be a Riemannian manifold of dimension m and $(\varphi, \mathscr{U})$ be a local chart on M; i.e. $\varphi: q \to (x_1(q), \ldots, x_m(q))$ is a coordinate system valid on an open subset $\mathscr{U} \subset M$. We define the functions $g_{ij}, g^{ij}, \bar{g}$ on $\mathscr{U}$ by:

$$\left.\begin{aligned} &g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right); \qquad g^{ij} = (g_{ij})^{-1}; \\ &\bar{g} = |\det(g_{ij})|, \end{aligned}\right\} \tag{3}$$

where g denotes the Riemannian structure on M (see Ref. [8], p. 386). Therefore the Laplace-Beltrami operator can be written in the following form:

$$\Delta: f \to \frac{1}{\sqrt{\bar{g}}} \sum_k \frac{\partial}{\partial x_k}\left(\sum_i g^{ik}\sqrt{\bar{g}}\frac{\partial}{\partial x_i}\right)(f); \qquad f \in C^\infty(\mathscr{U}). \tag{4}$$

More precisely we must work on manifolds of dimension 2 and constant curvature. On these spaces which are the euclidean plane, the 2-sphere S^2 and the hyperbolic plane H^2 we want to introduce a system of geodesic polar coordinates. We consider firstly the 2-sphere S^2, i.e. $M = S^2$. Strictly following Helgason (Ref. [8], p. 403) we denote with o the north-pole on M and with L a semicircle on M joining o to the south-pole. If (φ, r) are ordinary polar coordinates on the tangent space M_0, then we regard (φ, r) as geodesic polar coordinates at $o \in M$. They are valid on $S^2 - L$. In terms of these coordinates the Riemannian structure on S^2 is given by the differential form:

$$dr^2 + (\sin r)^2 d\varphi^2 . \tag{5}$$

We have also: $g_{11} = 1, g_{22} = (\sin r)^2$; $g_{12} = g_{21} = 0$; $\bar{g} = (\sin r)^2$ and therefore from (4) we get for the Laplace-Beltrami operator the following expression:

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial r^2} + \frac{1}{\sqrt{\bar{g}}}\frac{\partial\sqrt{\bar{g}}}{\partial r}\frac{\partial}{\partial r} + g^{22}\frac{\partial^2}{\partial\varphi^2} \\ &= \frac{\partial^2}{\partial r^2} + \frac{\cos r}{\sin r}\frac{\partial}{\partial r} + \frac{1}{(\sin r)^2}\frac{\partial^2}{\partial\varphi^2}. \end{aligned} \tag{6}$$

Now, consider the quotient group G/K where $G=SO(3)$ and $K=SO(2)$. Let Φ be a spherical function on G/K. Then $\Phi(p)=\psi(d(o,p))$ where d denotes distance and $\psi(r)$ is a function which satisfies the differential equation:

$$\frac{d^2\psi}{dr^2}+\frac{\cos r}{\sin r}\frac{d\psi}{dr}=\alpha\psi \qquad (0<r<\pi), \tag{7}$$

α being a complex constant. If $\alpha=-n(n+1)$ where n is an integer $\geqq 0$, a well-known solution of (7) is given by:

$$\psi(r)=P_n(\cos r)=\frac{1}{2\pi}\int_0^{2\pi}(\cos r+i\sin r\cos u)^n du, \tag{8}$$

where the r.h.s. of (8) is the Laplace integral representation of Legendre polynomials. Therefore we can conclude with the following Proposition (Ref. [8], p. 403):

Proposition. *The spherical functions on G/K are precisely the functions*

$$\Phi_n(p)=P_n(\cos(d(o,p))) \tag{9}$$

where P_n are the Legendre polynomials of degree n.

Now, if we identify $\cos r=\cos\vartheta=1+\frac{2t}{s-4m^2}$, where ϑ is the scattering angle in the c.m. system, and if n acquires the meaning of the angular momentum l, then the polynomials (9) give precisely the basis of the direct-channel expansion (1). Recall that the Legendre polynomials form a complete orthogonal system in the space L^2 with respect to the interval $[-1,+1]$. Observe, also, that the operator (6) coincides with the angular part of the operator (3.5) of Vilenkin-Smorodinskij [12][1].

Next we want to consider the contraction of the expansion (1) to its euclidean limit[2]; this implies the transformation of the 2-sphere S^2 into the euclidean plane R^2. We proceed heuristically as follows. Consider the following limit:

$$\lim_{\substack{r\to 0\\ n\to\infty}}(\cos r+i\sin r\cos u)^n=\lim_{\substack{r\to 0\\ n\to\infty}}\left(\cos r+\frac{in\sin r\cos u}{n}\right)^n \tag{10}$$

[1] These authors analyze the Laplace operator on the Lobacevskij space of the relativistic four-velocities; then introduce various coordinate systems and through the "orisphere" method work out expansions of the scattering amplitude which are generalizations of (1) and (2).

[2] More precisely hereafter we will consider the euclidean contraction of the following expansion

$$\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(\cos\vartheta).$$

then for large l and small ϑ we have

$$n \sin r \equiv l \sin \vartheta \to b\sqrt{-t} \tag{11}$$

where b is the impact-parameter of the collision process. Therefore we get:

$$\lim_{\substack{\vartheta \to 0 \\ l \to \infty}} \left(\cos \vartheta + \frac{i l \sin \vartheta \cos u}{l}\right)^l = e^{i b \sqrt{-t} \cos u} \tag{12}$$

and from (8) we obtain:

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i b \sqrt{-t} \cos u} \, du = J_0(b\sqrt{-t}) \approx J_0(l \vartheta) \tag{13}$$

which is a Bessel function of order zero and it must be a solution of the following differential equation:

$$\frac{d^2 \psi}{d r^2} + \frac{1}{r} \frac{d \psi}{d r} = \alpha \psi \qquad (\alpha = -n^2). \tag{14}$$

Now we can observe that the l.h.s. of (14) is the radial part of the following operator:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \tag{15}$$

which is the Laplace-Beltrami operator corresponding to the differential form:

$$d r^2 + r^2 d \varphi^2 \tag{16}$$

which gives the Riemannian structure on $M = R^2$. In this way we have re-obtained the well-known Wigner-Inönü contraction [13].

Now let us recall the standard formulae of Fourier analysis on R^n. For $f \in L^1(R^n)$ put:

$$\tilde{f}(v) = \int_{R^n} f(x) \, e^{-i(x, v)} \, dx \tag{17}$$

(,) denoting the usual inner product on R^n. Then if $f \in C_c^\infty(R^n)$:

$$f(x) = \frac{1}{(2\pi)^n} \int_{R^n} \tilde{f}(v) \, e^{i(x, v)} \, dv. \tag{18}$$

Let us introduce polar coordinates as follows (see Ref. [9], p. 7): $v = \lambda w$, $\lambda \geqq 0$ and where w is a unit vector. Then from (17), (18) we get:

$$\tilde{f}(\lambda w) = \int_{R^n} f(x) \, e^{-i\lambda(x, w)} \, dx \tag{19}$$

and

$$f(x) = \frac{1}{(2\pi)^n} \int_{R^+} \int_{S^{n-1}} \tilde{f}(\lambda w)\, e^{i\lambda(x,w)} \lambda^{(n-1)} d\lambda\, dw. \tag{20}$$

Let us suppose that $\tilde{f}(\lambda w)$ is a function of λ alone and take $e^{i\lambda(x,w)} = e^{ib\sqrt{-t}\cos u}$ this amounts to consider a two-dimensional momentum transfer vector $\boldsymbol{k}$ and a two-dimensional impact parameter $\boldsymbol{b}$ in the plane perpendicular to the incoming beam, such that

$$\begin{aligned} \boldsymbol{k}^2 &= -t\,, \\ \boldsymbol{b}^2 &= b^2\,. \end{aligned} \tag{21}$$

Moreover u denotes the angle between $\boldsymbol{k}$ and $\boldsymbol{b}$. Finally we obtain from formula (20), for $n=2$

$$\begin{aligned} &\frac{1}{(2\pi)^2} \int_0^{+\infty} b\,db\,\tilde{f}(b) \int_0^{2\pi} e^{ib\sqrt{-t}\cos u} du \\ &\qquad = \frac{1}{(2\pi)} \int_0^{+\infty} b\,db\,\tilde{f}(b)\, J_0(b\sqrt{-t}) \end{aligned} \tag{22}$$

which gives the impact-parameter representation for the scattering amplitude, that is widely used in high-energy scattering (see, for instance, Ref. [14]).

Remark. Before going to the next section, where we want to analyze the expansion of the scattering amplitude into the imaginary-mass representations of the Poincaré group (corresponding to $t<0$), it is better to make clear that in the present note we do not discuss what we could call the $t=0$ problem. Any way it is convenient to recall that in the case of equal mass scattering, the little group at $t=0$ is $SO(3,1)$.

III. The Crossed-Channel Expansion

Let M be a manifold of constant negative curvature equal to -1. In geodesic polar coordinates (φ, r) say at the point $i \in M$, the Riemannian structure on M is given by (see Ref. [8], p. 405);

$$dr^2 + (\sinh r)^2\, d\varphi^2\,. \tag{23}$$

More precisely the expression (23) is the elliptic form of the ds^2 for a pseudo-spherical surface (see Ref. [15], Vol. I, p. 224). The corresponding Laplace-Beltrami operator is given by:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\cosh r}{\sinh r}\frac{\partial}{\partial r} + \frac{1}{(\sinh r)^2}\frac{\partial^2}{\partial \varphi^2}\,. \tag{24}$$

This operator coincides with that part of the operator (3.11) of Vilenkin-Smorodinskij [12] which works on the coordinates b and φ following the notation of Ref. [12]. The spherical functions on M have the form $\Phi(p)=\psi(d(i,p))$ where the function $\psi(r)$ satisfies the differential equation:

$$\frac{d^2\psi}{dr^2}+\frac{\cosh r}{\sinh r}\,\frac{d\psi}{dr}=\alpha\psi \qquad (r>0) \tag{25}$$

for same complex α. The general solution of this equation is given by:

$$\psi(r)=P_\varrho(\cosh r)=\frac{1}{2\pi}\int_0^{2\pi}(\cosh r+\sinh r\cos u)^\varrho\,du \tag{26}$$

where ϱ satisfies $\varrho(\varrho+1)=\alpha$ (see Ref. [8], p. 406).

Remark. Up to now we have analyzed the Laplace-Beltrami operator its eigenfunctions and the corresponding integral representations; we have worked essentially with the tecnique of the separation of variables. However there is also a purely algebraic approach to the problem, which makes use only of group invariance properties. With the latter methods, one can derive the integral representations of the Legendre functions, the associated Legendre functions, the Bessel functions and so on; for these methods see Ehrenpreis (Ref. [16], p. 379).

Next we want to introduce the Poincaré model of hyperbolic geometry. As it is well-known, this model can be built in the unit disc D, whose boundary will be called the horizon and denoted with B. Now let us write:

$$\begin{aligned} x&=\tanh\left(\frac{r}{2}\right)\cos\varphi\,,\\ y&=\tanh\left(\frac{r}{2}\right)\sin\varphi\,, \end{aligned} \tag{27}$$

from which we get:

$$|z|^2=x^2+y^2=\left[\tanh\left(\frac{r}{2}\right)\right]^2 \tag{28}$$

where: $z=|z|\,e^{i\varphi}$. With easy computation we have:

$$ds^2=\frac{4[dx^2+dy^2]}{[1-|z|^2]^2}=dr^2+(\sinh r)^2\,d\varphi^2\,, \tag{29}$$

which is exactly the form (23). Moreover it turns out that the shortest distance between the center of the unit disc D and the point z is given by:

$$d(o,z)=\log\frac{1+|z|}{1-|z|} \tag{30}$$

In the hyperbolic geometry the "oricycles" play a fundamental role. Their euclidean images in $|z|<1$ are circles tangent to the horizon $|z|=1$ from within, in a point $w=e^{i\chi}$. The oricycles orthogonally intersect the pencil of parallel "straight-lines" (arcs of circles orthogonal to the unit circle and lying in its interior) passing through the point $w=e^{i\chi}$.

Now let us write the Poisson-Kernel:

$$P(z,w)=\frac{1-|z|^2}{1+|z|^2-2|z|\cos(\varphi-\chi)}, \tag{31}$$

where $z=|z|\,e^{i\varphi}$ and $w=e^{i\chi}$. One of the properties of the Poisson-Kernel $P(z,w)$ is that its level lines[3] are the circles tangent to the unit circle at the point $w=e^{i\chi}$; the euclidean images of oricycles (see Ref. [17], p. 7). Furthermore $P(z,w)$, being the real part of $\frac{w+z}{w-z}$, is a harmonic function of z and $[P(z,w)]^{\mu}$, $\mu\in\mathbb{C}$, is an eigenfunction of the Laplace-Beltrami operator on D [9]. Finally $P(z,w)$ is invariant with respect to any transformation that preserves the unit disc.

Therefore we can say that:

$$e^{\mu\langle z,w\rangle}=\left[\frac{1-|z|^2}{1+|z|^2-2|z|\cos(\varphi-\chi)}\right]^{\mu};\ \mu\in\mathbb{C} \tag{32}$$

is the non-euclidean analog of the plane-wave. In fact $P(z,w)$ is constant on each oricycle of normal w, it is an eigenfunction of the Laplace-Beltrami operator on D and finally $\langle z,w\rangle$ gives the non-euclidean geodetic distance from the center of the unit disc to the oricycle with normal w and passing through z ($\langle z,w\rangle$ is negative if the center of the unit disc o falls inside the oricycle). All these facts allow an extension of the Fourier-Transform to manifolds of negative curvature; this generalization can be performed through the following theorem due to Helgason [9].

Theorem. *(Helgason). For $f\in C_c^{\infty}(D)$ set:*

$$\tilde{f}(\lambda,w)=\int_D e^{(-i\lambda+\frac{1}{2})\langle z,w\rangle}f(z)\,dz;\quad \lambda\in R,\ w\in B, \tag{33}$$

where dz in the volume element on D. Then

$$f(z)=\frac{1}{(2\pi)^2}\int_R\int_B e^{(i\lambda+\frac{1}{2})\langle z,w\rangle}\tilde{f}(\lambda,w)\tanh(\pi\lambda)\,\lambda\,d\lambda\,dw, \tag{34}$$

where dw in the usual angular measure on B[4].

[3] In order to be more precise, it is better to exclude the infinitely distant tangent point which belongs to the horizon.

[4] This theorem is the simplest case of more general results, due to Harish-Chandra [18] and Helgason [9] concerning the Fourier-analysis on symmetric spaces of the non-compact type.

The formulae (33), (34) can be reduced to the classical formulae of Mehler-Transform (see Ref. [11], Vol. I, p. 175, and Ref. [9]). In fact recalling that $|z| = \tanh\left(\frac{r}{2}\right)$ we have:

$$\frac{1-|z|^2}{1+|z|^2-2|z|\cos(\varphi-\chi)} = \frac{1}{\cosh r - \sinh r \cos(\varphi-\chi)}. \tag{35}$$

Furthermore the Jacobian of the transformation (27) is given by: $J = \frac{1}{2}\frac{\tanh\left(\frac{r}{2}\right)}{\left[\cosh\left(\frac{r}{2}\right)\right]^2}$ and therefore the volume element dz in the co-ordinates (φ, r) becomes: $dz = (\sinh r)\, dr\, d\varphi$. Then recalling formulae (26), (32) and (35) we obtain from formula (33):

$$2\pi \int_0^{+\infty} F(r)\, P_{-\frac{1}{2}+i\lambda}(\cosh r) \sinh r\, dr\,, \tag{36}$$

where we have supposed that $f(z) = F(d(o, z))$, F even. In a similar way we obtain from (34):

$$\frac{1}{(2\pi)} \int_0^{+\infty} \lambda\, d\lambda\, \tilde{F}(\lambda)\, P_{-\frac{1}{2}+i\lambda}(\cosh r) \tanh(\pi\lambda)\,, \tag{37}$$

where we have used the eveness of $P_{-\frac{1}{2}+i\lambda}$ with respect to λ and the fact that $\tilde{f}(\lambda, w)$ is supposed to be an even function $\tilde{F}(\lambda)$ of λ alone. Formulae (36) and (37) are the classical formulae of Mehler.

Now concerning the expansion (2) one can start (as Joos [3] has done) from an unsubtracted dispersion relation for the scattering amplitude at fixed, negative, momentum-transfer. The absorptive part of the dispersion integral, let say $f(s, t)$, is expressed through the product of the current operators $j(p_1)\, j(-p_1')$. If p_1 and p_1' lie on the mass shell, then the product $j(p_1)j(-p_1')$ transforms under the Poincaré group according a product representation $(m^2, 0, +) \otimes (m^2, 0, -)$ (recall that we are considering spin-zero particles). Now through the technique of the Clebsh-Gordan-Coefficients, one can decompose this product in "irreducible field operators" corresponding to irreducible tensor operator with respect to the Poincaré group. Then it turns out that the reduced matrix element $f(\lambda, t)$ of the "irreducible field operator" is the Mehler-

Transform of the absorptive part $f(s,t)$; i.e.:

$$f(\lambda,t)=\tfrac{1}{2}\tanh(\pi\lambda)\int_0^{+\infty} d(\cosh r)\,P_{-\frac{1}{2}+i\lambda}(\cosh r)\,f(s,t) \tag{38}$$

with the inversion:

$$f(s,t)=2\int_0^{+\infty}\lambda\,d\lambda\,f(\lambda,t)\,P_{-\frac{1}{2}+i\lambda}(\cosh r), \tag{39}$$

where: $\cosh r=-1-\dfrac{2s}{t-4m^2}$. In other words one obtains for $f(s,t)$ a non-euclidean Fourier-expansion. Finally substituting the expression (39) into the dispersion integral one obtains:

$$T(s,t) \tag{40}$$
$$=\frac{1}{i}\int_0^{+\infty}\frac{\lambda\,d\lambda\,f(\lambda,t)}{\cosh(\pi\lambda)}\left[P_{-\frac{1}{2}+i\lambda}\left(1+\frac{2s}{t-4m^2}\right)+P_{-\frac{1}{2}+i\lambda}\left(-1-\frac{2s}{t-4m^2}\right)\right]$$

which is the Joos expansion (2). Another way of getting the representation (40) consists in applying directly the Mehler-identity to the denominator of the dispersion integral [4]; i.e. to $\dfrac{1}{z'-z+i\varepsilon}$ where $z=-1-\dfrac{2s}{t-4m^2}$.

At this point, it should appear clearly how the expansions of the relativistic scattering amplitude are related to the geometry on manifolds of positive and negative constant curvature. It is better recall that, in the present note, we have considered only the term which corresponds to the principal series of the $SO(2,1)$ group; this term is related to the so-called "background integral" of the complex angular momentum representation (see Refs. [19, 20])[5]. On the other hand, the principal series does not complete the class of unitary representations of the $SO(2,1)$ group; one must consider also the discrete series and the supplementary series. Moreover, if one uses the analogue of the Peter-Weyl theorem, then every square-integrable function over the group manifold can be expressed as a sum of terms which correspond to the principal series and to the discrete series respectively. However if one considers scattering of spin-zero particles, as we have done, then no term of the discrete series will appear [4]. Finally, in order to incorporate more general amplitudes, the expansions must be generalized to non-unitary representations. In the present note, we have avoided these questions in order to make more transparent the analysis of the geometries underlying the expansions which we have considered; we will eventually return to these problems elsewhere.

[5] This relationship has been recently reconsidered by Cronström and Klink [21].

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