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The Information on the Pion Electromagnetic Form Factor Inside Its Analyticity Region Provided by Bounds on Its Modulus on the Cut ($t \geqq 4m^2$)*

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Abstract. We concentrate on the mathematical aspects connected with the derivation of the model independent information one can get on the pion electromagnetic form factor $F(t)$ inside the analyticity region (the cut t-plane) from the knowledge of upper and lower bounds of its modulus on the cut $t \geqq 4m^2$ using analyticity, reality, and the normalization $F(0)=1$. It turns out that (in a certain sense) this information depends only on the upper bound, whereas the lower one is irrelevant.

1. Introduction

We consider in this paper the mathematical steps involved in the derivation of the model independent information on the pion electromagnetic form factor $F(t)$, contained in the $e^+e^- \to \pi^+\pi^-$ scattering data.

The cross section of this process is given, up to a kinematic factor, by the square of the modulus of $F(t)$:

$$\sigma(t) = \mathrm{K}(t)\,|F(t)|^2, \qquad t \geqq 4m^2, \tag{1.1}$$

$$K(t) = \left(\frac{e^2}{4\pi}\right)^2 \frac{\pi}{3}(t-4m^2)^{\frac{3}{2}}\, t^{-\frac{5}{2}}, \qquad \frac{e^2}{4\pi} = \frac{1}{137} \tag{1.2}$$

where t has the meaning of the total energy squared in the c.m. system and m that of the pion mass.

We admit that $F(t)$ is a real analytic function in the complex t-plane cut along $t \geqq 4m^2$, normalized at $t=0$ to $F(0)=1$. The cross section (1.1) then provides us with data on the boundary values of the modulus of this function on the cut. These data take, due to the experimental errors, the form of upper and lower bounds,

$$|I(t)| \leqq |F(t)| \leqq |S(t)|, \qquad t \geqq 4m^2. \tag{1.3}$$

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So far experiment covers only part of the cut and we therefore supplement (1.3) for the rest with a *hypothesis* of the same kind. The (experimental or theoretical) origin of our knowledge of the functions $|S(t)|$ and $|I(t)|$ is, however, ultimately important only for the physical aspect of the problem, which has been discussed in Ref. [1, 2]. There also references to earlier work on the subject are given.

The questions we want to answer are:

a) if there is (at least) one function real analytic in the cut t-plane and normalized to $F(0)=1$, such that its boundary values satisfy (1.3),

b) if there are more such functions than one, which are, for a given t inside the analyticity region, their possible values,

c) how do the results of (a) and) b) depend on the information given by (1.3).

The answers will be given by reducing the problem to an interpolation problem of the Pick-Nevanlinna type, its solution, and the discussion of some properties of the solution.

2. The Interpolation Problem

We perform a conformal mapping of the cut t-plane onto the unit disc $|z|<1$,

$$z=\frac{1-\left(1-\dfrac{t}{4m^2}\right)^{\frac{1}{2}}}{1+\left(1-\dfrac{t}{4m^2}\right)^{\frac{1}{2}}}, \tag{2.1}$$

by which we get the real analytic function $f(z)\equiv F(t)$ normalized to $f(0)=1$ and with the boundary values satisfying

$$|i(\tau)|\leqq|f(\tau)|\leqq|s(\tau)|, \qquad \tau=\exp(i\theta), \qquad -\pi\leqq\theta\leqq\pi\,. \tag{2.2}$$

Temporarily we will ignore the information contained in the l.h.s. of (2.2) (i.e. we put $|i(\tau)|\equiv 0$) and proceed with the condition

$$|f(\tau)|\leqq|s(\tau)| \tag{2.3}$$

instead of (2.2). Later we will come back and investigate, as part of the answer to c), the implications of $|i(\tau)|\not\equiv 0$.

To tackle the so modified problem we first make use of a *theorem of Szegö* [3]:

For any function $|s(\tau)|$ (nonnegative and) of period 2π, which is such that $\ln|s(\tau)|$ and $|s(\tau)|^{\delta}$, $\delta>0$, are summable, there exists a function $g(z)$, unique up to a constant factor of modulus 1, belonging to the class H_{δ} and maximal for $|s(\tau)|$, i.e. such that $|g(\tau)|=|s(\tau)|$ almost everywhere and

$|g(z)| > |f(z)|$ for $|z| < 1$, if $f(z)$ is a function of class H_δ which satisfies $|f(\tau)| \leqq |s(\tau)|$ almost everywhere.

The function is given by

$$g(z) = \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln|s(\tau)| \frac{\tau + z}{\tau - z} d\theta\right), \tag{2.4}$$

if we make an appropriate choice for the phase. It has evidently no zeros in $|z| < 1$ and therefore also $g^{-1}(z)$ is analytic.

We suppose that the pion form factor *is* of class H_δ and that the upper bound (2.3) obeys the requirements of the theorem. Then from it we get the result that

$$\varphi(z) = f(z)\, g^{-1}(z) \tag{2.5}$$

is analytic and bounded, i.e.

$$|\varphi(z)| \leqq 1, \qquad |z| < 1 \tag{2.6}$$

and that the equality sign is valid only for $f(z) = e^{i\lambda} g(z)$ with $g(z)$ given by (2.4) and $\lambda =$ a real constant. From the reality of $f(z)$ and from $|s(\tau)| = |s(\tau^*)|$, which implies the reality of $g(z)$, it follows that $\varphi(z)$ is also real.

To $\varphi(z)$ we apply the Pick-Nevanlinna interpolation technique in its simplest form: To find all functions $\varphi(z)$ (real) analytic in $|z| < 1$ (if they exist), satisfying (2.6) and taking the value

$$\varphi(0) = g^{-1}(0) \tag{2.7}$$

which is a consequence of the normalization of the form factor.

The existence of at least one function $\varphi(z)$ with the required properties depends on the value of $g(0)$. If $g(0) < 1$, then (2.6) and (2.7) contradict each other and no function $\varphi(z)$ exists. The value $g(0) = 1$ determines $\varphi(z)$ uniquely: $\varphi(z) \equiv 1$. For values $g(0) > 1$ $\varphi(z)$ is not uniquely determined. To construct for this case all allowed functions $\varphi(z)$ we apply the Schwarz lemma in the form given by Pick [4] (essentially the basis of the Pick-Nevanlinna interpolation procedure) which asserts that if $\varphi(z)$ takes for $z = z_0$ ($|z_0| < 1$) the value $\varphi(z_0)$ ($|\varphi(z_0)| < 1$), then

$$\frac{\varphi(z) - \varphi(z_0)}{1 - \varphi^*(z_0)\varphi(z)} = \chi(z) \frac{z - z_0}{1 - z_0^* z}, \tag{2.8}$$

where $\chi(z)$ is analytic and bounded in the unit disc,

$$|\chi(z)| \leqq 1, \qquad |z| < 1 \tag{2.9}$$

but is otherwise undetermined. Those $\chi(z)$ which satisfy the sign of equality in one point necessarily have the form $\chi(z) = e^{i\lambda}$ and give rise to functions $\varphi(z)$ performing a conformal mapping of the unit disc onto itself:

$$\frac{\varphi(z)-\varphi(z_0)}{1-\varphi^*(z_0)\,\varphi(z)} = e^{i\lambda}\frac{z-z_0}{1-z_0^* z}\,. \tag{2.10}$$

The solutions of the problem take thus in this case, with $z_0=0$ and $\varphi(0)=g^{-1}(0)$, the form

$$\varphi(z) = \frac{g^{-1}(0)+z\chi(z)}{1+g^{-1}(0)\,z\chi(z)}\,. \tag{2.11}$$

The condition that an upper bound $|s(\tau)|$ does not contradict the normalization of the form factor is thus given by

$$g(0) = \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln|s(\tau)|\,d\theta\right) \geqq 1\,. \tag{2.12}$$

Admissible bounds, i.e. those which satisfy (2.12), give possible form factors

$$f(z) = g(z)\frac{g^{-1}(0)+z\chi(z)}{1+g^{-1}(0)\,z\chi(z)}\,, \tag{2.13}$$

whose values, for a given z, lie all, if $\chi(z)$ covers the whole unit disc, in a disc $\mathscr{D}(g,z)$, as it follows from the linearity of the (conformal) transformation from χ to f.

In our case $\chi(z)$ has, however, to be real and this does not allow it to take, for a given z ($|z|<1$) all values in the unit disc. For instance, of all the functions $\chi(z)=e^{i\lambda}$ taking their values on the boundary, only $\chi(z)=\pm 1$ are real. So the domain of values of $f(z)$, $\mathscr{L}(g,z)$, lies inside $\mathscr{D}(g,z)$ and only the images of the two points $\chi(z)=\pm 1$ touch its boundary. The region $\mathscr{L}(g,z)$ will be determined in the next Section.

3. Consequences of Reality

We investigate first the implications of reality on the values allowed for $w=\chi(z)$ satisfying (2.9) and thus start by requiring that for $-1<z=a<1$ it takes the values $-1\leqq \chi(a)=b\leqq 1$ (the signs of equality allowed only for $\chi(z)\equiv \pm 1$). Then, following Carathéodory [5], we apply again the lemma of Pick, (2.8), in a form convenient for this purpose. Namely if

$$\left|\frac{z-a}{1-az}\right| = \varrho\,, \tag{3.1}$$

then

$$\left|\frac{w-b}{1-bw}\right| \leqq \varrho\,, \tag{3.2}$$

i.e. if a value of z lies on a circle of (noneuclidean) radius ϱ and centre a, then the values of $\chi(z)$ (with $\chi(a)=b$) lie inside or on a circle of the same (noneuclidean) radius with centre b.

We consider now the family of all circles (3.1) of constant (noneuclidean) radius ϱ with centres on $(-1, 1)$. They cover the area between their envelopes, the two lines of constant (noneuclidean) distance ϱ from the diameter $(-1, 1)$, which are arcs of circles going through ± 1. If b goes through the values $-1 \leqq b \leqq 1$ the family (3.2) covers the same area. It follows then that for a given z ($|z|<1$) the values of the real function $\chi(z)$ are allowed to lie inside or on the boundary of the lens $\mathscr{L}(z)$ determined by the arcs of circle passing through the points $-1, 1, z$ and $-1, 1, -z$, respectively. For further convenience we note that the first of these circles passes through the point z^{-1} and the second through $-z^{-1}$, as can be easily verified with the anharmonic ratio. The extremal functions, i.e. those, whose values lie on the boundary, are given according to the lemma of Pick by

$$\frac{w-b}{1-bw} = \pm \frac{z-a}{1-az}. \tag{3.3}$$

Their geometric meaning is particularly simple:

$$\frac{w-b}{1-bw} = \frac{z-a}{1-az} \tag{3.4}$$

are hyperbolic transformations with fix points ± 1 and thus leave all circles passing through these points, i.e. all distance lines of the diameter $(-1, 1)$, invariant; the other functions (3.3) are hyperbolic transformations of the same kind combined with reflections,

$$\frac{w-b}{1-bw} = \frac{v+a}{1+av}, \qquad v=-z. \tag{3.5}$$

Coming now back to (2.11) we notice that, with $\chi(z)$ taking values in $\mathscr{L}(z)$, $z\chi(z)$ will take values in a lens determined by the two circles passing through the points z, $-z$, z^2 (and 1) and z, $-z$, $-z^2$ (and -1), respectively. The transformation (2.11) from $z\chi(z)$ to $\varphi(z)$ being again linear and parabolic, it follows that the values of $\varphi(z)$ also lie in a lens, bounded by the two circles passing through the points $\dfrac{g^{-1}(0) \pm z}{1 \pm g^{-1}(0)z}$, 1 and $\dfrac{g^{-1}(0) \pm z}{1 \pm g^{-1}(0)z}$, -1, respectively, and that these points lie on the boundary of the lens $\mathscr{L}(z)$. The value of $f(z)$, finally, lie in the lens $\mathscr{L}(g, z)$, the image of the region of values of $\varphi(z)$ by the transformation

$$f(z) = g(z)\,\varphi(z). \tag{3.6}$$

4. Influence of the Lower Bound

Now we come back to the information

$$|i(\tau)| \leqq |f(\tau)| \tag{4.1}$$

which we have so far ignored in deriving the above given results. It restricts the set of functions $\chi(z)$ entering (2.13) by imposing on their boundary values $\chi(\tau)$ ($|\chi(\tau)| \leqq 1$) the condition

$$\left|\frac{g^{-1}(0) + \tau\chi(\tau)}{1 + g^{-1}(0)\,\tau\chi(\tau)}\right| \geqq \frac{|i(\tau)|}{|s(\tau)|} \quad \text{(almost everywhere)} \tag{4.2}$$

which forces $\chi(\tau)$ to lie on or outside the circle of noneuclidean radius $\left|\frac{i(\tau)}{s(\tau)}\right|$ and centre $-g^{-1}(0)\tau^{-1}$. In investigating its consequences we first observe that the boundary of $\mathscr{L}(g, z)$ is given by functions $f(z)$, (2.13), which are the images of the boundary points

$$\chi(z) = \frac{b \pm z}{1 \pm bz} \tag{4.3}$$

of $\mathscr{L}(z)$ (which follow from (3.3) by putting, without loss of generality, $a = 0$). Since (4.3) satisfies $|\chi(\tau)| = 1$, *these* functions $f(z)$ have boundary values $|f(\tau)| = |s(\tau)|$ almost everywhere. Further we show that also *for any inner point* of $\mathscr{L}(z)$ there exist functions $\chi(z)$ with the property $|\chi(\tau)| = 1$, which implies that the whole region $\mathscr{L}(g, z)$ is covered already by functions with $|f(\tau)| = |s(\tau)|$ almost everywhere and proves that the information (4.1) is, in fact, useless.

Such functions $\chi(z)$ we construct in the following way: From z on the boundary of $\mathscr{L}(z)$ we get, by multiplication with $\frac{\alpha + z}{1 + \alpha z}$ $(-1 < \alpha < 1)$, the point

$$u = z\frac{\alpha + z}{1 + \alpha z} \quad (-1 < \alpha < 1) \tag{4.4}$$

inside $\mathscr{L}(z)$, on the arc of circle defined by the points z, $-z$ and 1. The (noneuclidean) distance of this point to the diameter $(-1, 1)$ depends on the value of α. To it we apply the parabolic transformation

$$\chi(z) = \frac{\beta + u}{1 + \beta u} \quad (-1 < \beta < 1) \tag{4.5}$$

and move it *on* a distance line of the diameter $(-1, 1)$. For a given z we can thus reach with (4.5), by a suitable choice of the parameters α and β, all inner points of $\mathscr{L}(z)$. Because of $|u(\tau)| = 1$ we also have for the functions (4.5) $|\chi(\tau)| = 1$.

5. Region without Zeros

The representation (2.13) with $\chi(z) \in \mathscr{L}(z)$ enables us to deduce immediately the region $\mathscr{Z}(g)$, where zeros of $f(z)$ are not allowed to exist according to the information $|f(\tau)| \leqq |s(\tau)|$. This consists of the set of all points z for which $-g^{-1}(0) \notin z\mathscr{L}(z)$. The boundary of the region is given by the solutions of the equation

$$g^{-1}(0) + z\chi(z) = 0\,, \tag{5.1}$$

where $\chi(z)$ is a boundary point of $\mathscr{L}(z)$. But of these points only $\chi(z) = \dfrac{a+z}{1+az}$, for which $z\chi(z) = z\dfrac{a+z}{1+az}$ $(-1<a<1)$ intersects the real diameter at negative values, can satisfy (5.1). The corresponding points z are the solutions z_1, z_2 of the equation

$$z^2 + a(1+g^{-1}(0))z + g^{-1}(0) = 0\,. \tag{5.2}$$

For $a^2 < a_c^2$, where

$$a_c = \frac{2g^{-\frac{1}{2}}(0)}{1+g^{-1}(0)} \qquad (0 < a_c \leqq 1)\,, \tag{5.3}$$

they are complex and situated, since $z_1 z_2 = g^{-1}(0)$, on the circle of radius $g^{-\frac{1}{2}}(0)$ and centre $z=0$. If $a = \pm a_c$, we have $z_1 = z_2 = \mp g^{-\frac{1}{2}}(0)$, whereas for $a_c^2 < a^2 \leqq 1$ they are real and different. For $-1 \leqq a < -a_c$ we have $g^{-\frac{1}{2}}(0) < z_1 \leqq 1$ and $g^{-1}(0) \leqq z_2 < g^{-\frac{1}{2}}(0)$, whereas for $a_c < a \leqq 1$ $-g^{-\frac{1}{2}}(0) < z_1 < -g^{-\frac{1}{2}}(0)$ and $-1 \leqq z_2 < -g^{-\frac{1}{2}}(0)$.

The region $\mathscr{Z}(g)$ consists thus of the disc $|z| < g^{-\frac{1}{2}}(0)$ except the intervals $-g^{-\frac{1}{2}}(0) < z \leqq -g^{-1}(0)$ and $g^{-1}(0) \leqq z < g^{-\frac{1}{2}}(0)$.

6. Dependence of $\mathscr{L}(g, z)$ on $|s(\tau)|$

The shape of the region $\mathscr{L}(g, z)$ depends through the values of $g(z)$ on the detailed information one possesses on $|s(\tau)|$, so it has to change if one changes this information. Suppose we have, instead of (2.3), a more detailed knowledge

$$|f(\tau)| \leqq |s_i(\tau)|\,, \tag{6.1}$$

the function $|s_i(\tau)|$ satisfying

$$|s_i(\tau)| \leqq |s(\tau)| \qquad \text{(almost everywhere)} \tag{6.2}$$

with the sign of inequality valid on a set of positive length. We denote by $g_i(z)$ and $\mathscr{L}(g_i, z)$ the analogs, for $|s_i(\tau)|$, of $g(z)$ and $\mathscr{L}(g, z)$, and look for the comparison of $\mathscr{L}(g, z)$ and $\mathscr{L}(g_i, z)$. It is evident that $\mathscr{L}(g_i, z) \subset \mathscr{L}(g, z)$. What we want to show is that, if the condition we have imposed on the inequality (6.2) holds, the boundary of $\mathscr{L}(g_i, z)$ lies completely inside $\mathscr{L}(g, z)$. In order to give the proof we use the more detailed notation

$$f(g; z) = g(z)\frac{g^{-1}(0) + z\chi(z)}{1 + g^{-1}(0)\, z\chi(z)}, \tag{6.3}$$

$$f(g_i; z) = g_i(z)\frac{g_i^{-1}(0) + z\xi(z)}{1 + g_i^{-1}(0)\, z\xi(z)}, \tag{6.4}$$

for the functions f obtained with the informations (2.3) and (6.1), respectively. It then consists in showing that $f(g_i; z)$ can be written as

$$f(g_i; z) = g(z)\frac{g^{-1}(0) + z\chi(g_i; z)}{1 + g^{-1}(0)\, z\chi(g_i; z)} \tag{6.5}$$

with $\chi(g_i; z) \in \mathscr{L}(z)$, and that $\chi(g_i; z)$ corresponds to *inner* points of $\mathscr{L}(z)$ if $\xi(z)$ is a *boundary* point of $\mathscr{L}(z)$.

We have, indeed, $\chi(g_i; z)$ completely defined by

$$\frac{g^{-1}(0) + z\chi(g_i; z)}{1 + g^{-1}(0)\, z\chi(g_i; z)} = g_i(z)\, g^{-1}(z)\frac{g_i^{-1}(0) + z\xi(z)}{1 + g_i^{-1}(0)\, z\xi(z)}. \tag{6.6}$$

Taking the limit $z \to \tau$ for the absolute value of (6.6) we get on the r.h.s. $|s_i(\tau)|\, |s(\tau)|^{-1}$, which proves that $\chi(g_i; z)$ is an inner point of $\mathscr{L}(z)$.

An immediate consequence of importance for comparison with experiment are the (strict) inequalities

$$g(x)\frac{g^{-1}(0) + |x|}{1 + g^{-1}(0)\,|x|} > g_i(x)\frac{g_i^{-1}(0) + |x|}{1 + g_i^{-1}(0)\,|x|}, \tag{6.7}$$

$$g(x)\frac{g^{-1}(0) - |x|}{1 - g^{-1}(0)\,|x|} < g_i(x)\frac{g_i^{-1}(0) - |x|}{1 - g_i^{-1}(0)\,|x|}, \tag{6.8}$$

for the upper and lower bounds of $f(x)$ at $z = x$ (real), and the inequalities

$$g'(0)\, g^{-1}(0) + g(0) - g^{-1}(0) > g_i'(0)\, g_i^{-1}(0) + g_i(0) - g_i^{-1}(0), \tag{6.9}$$

$$g'(0)\, g^{-1}(0) - g(0) + g^{-1}(0) < g_i'(0)\, g_i^{-1}(0) - g_i(0) + g_i^{-1}(0) \tag{6.10}$$

for the upper and lower bounds of $f'(0)$, which follow from (6.3) and (6.4).

7. Conclusions

We have shown that if one possesses information only on the modulus of the pion electromagnetic form factor on the cut $t \geqq 4m^2$, in the form of upper and lower bounds (1.3), it turns out that the bound $|S(t)|$ alone proves useful for the derivation of information on $F(t)$ inside the analyticity region, whereas the lower bound $|I(t)|$ is irrelevant. In particular, one derives the same information from the function $|S(t)|$ in both cases where it represents the exact value of $|F(t)|$ at $t \geqq 4m^2$ or only an upper bound. The information consists in the indication of the region ($\mathscr{R}(S, t)$, the conformal image in the cut t-plane of $\mathscr{L}(g, z)$) which is covered by the values at a given t of all functions $F(t)$ compatible with $F(0) = 1$ and $|I(t)| \leqq |F(t)| \leqq |S(t)|$ $(t \geqq 4m^2)$. For real values of t it leads to upper and lower bounds for $F(t)$. It provides also (upper and lower) bounds for $F'(0)$. The irrelevance of $|I(t)|$ consists in the fact that, although it affects the set of admissible functions $F(t)$, it is of no consequence on the region $\mathscr{R}(S, t)$.

The information implies that in the region around $t = 0$ which is the conformal image in the cut t-plane of $\mathscr{L}(g)$ there can be no zeros of $F(t)$.

We mention that these results, being obtained by extremal methods, are the best ones under the stated assumptions.

Finally we have shown that if we have an improved upper bound of $|F(t)|$ $(t \geqq 4m^2)$ given by a function $|S_i(t)|$ ($\leqq |S(t)|$, but inequivalent with it) the corresponding region $\mathscr{R}(S_i, t)$ is strictly included in $\mathscr{R}(S, t)$, i.e. they have no common boundary points. For real values of t this implies that the corresponding upper (lower) bounds both for $F(t)$ and $F'(0)$ are strictly lower (higher) for $|S_i(t)|$ than for $|S(t)|$.

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