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Connection between the Spectrum Condition and the Lorentz Invariance of $P(\phi)_2$

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Abstract. We prove that, for the $P(\phi)_2$ quantum field theory, the Wightman functions are Lorentz invariant if the energy-momentum spectrum lies in the forward light-cone. The ingredients of the proof are the following facts, established by Glimm and Jaffe: the field satisfies local commutativity, and also the estimates

$$\phi_V(f, t) \leqq \text{const} \|f\|_1 (H_V + I)$$
$$\pi_V(g, t) \leqq \|g\|_2 (H_V + I)$$

where V is a space cut-off, uniformly in V.

1. Introduction

Glimm and Jaffe [1] have proved that for the $P(\phi)_{2,V}$ theory (the self-interacting boson quantum field theory in two-dimensional space-time, with a polynomial interaction and a periodic box cut-off, V) the canonical conjugate field $\pi_V(g, t) \equiv \int \pi_V(x, t)\, g(x)\, dx$ satisfies the estimate

$$\pm \pi_V(g, t) \leqq \|g\|_2 (H_V + I) . \tag{1}$$

Here H_V is the Hamiltonian for the cut-off theory, and I is the identity operator. (There is a gap in the proof, in [1], of a similar estimate for $\nabla\phi_V$.) Furthermore [2] the field itself satisfies the estimate

$$\pm \phi_V(f, t) \leqq \text{const} \|f\|_1 (H_V + I) \tag{2}$$

where the constant is independent of V. These inequalities lead to bounds on vacuum expectation values of products of ϕ_V and π_V, showing that these expectation values are tempered distributions. Since the bounds are independent of V [2, 3] one obtains similar bounds for the smeared n-point Wightman distributions for the theory with no cut-offs. In particular, the Wightman function

$$W_n(z_1, \dots z_n) = (\Omega, \phi(z_1) \dots \phi(z_n)\, \Omega) \tag{3}$$

where $z_j \equiv (x_j, t_j)$, $j=1,2\ldots n$, are real two-vectors, is a tempered distribution invariant under translations, and so depends only on the $n-1$ differences $z_j - z_{j+1}$.

The field ϕ satisfies local commutativity, so that if $z_j - z_{j+1}$ is space-like, then

$$W_n(z_1, \ldots z_j, z_{j+1}, \ldots z_n) = W_n(z_1, \ldots z_{j+1}, z_j, \ldots z_n). \tag{4}$$

We say that the spectral condition holds if the simultaneous spectrum of energy and momentum lies in the forward light-cone:

$$P^0 \geqq \pm P^1. \tag{5}$$

The spectrum condition (5) has not yet been proved for the $P(\phi)_2$ theory. The work of Glimm and Jaffe [1, 3] towards a proof of (5) has a gap, pointed out by Fröhlich and Faris. In this paper we point out that the facts already established in [1, 3], and the spectrum condition, imply that the Wightman functions (3) are Lorentz invariant. If the vacuum is not unique, then the reduction theory of Borchers, Maurin and Brattelli [4] enables one to form the quotient Wightman theory over the centre, to obtain a theory with a unique vacuum. Thus the spectrum condition is the last remaining step in proving all the Wightman axioms.

The main step in the proof that (5) implies Lorentz invariance is to note that the spectrum condition (5), temperedness and local commutativity imply that W_n satisfies the hypotheses of the theorem on finite covariance of Bros, Epstein and Glaser [5], so that $W_n(z)$ is a finite covariant for each n. This means that W_n has the form

$$W_n(z) = \sum_{j+k \leqq N(n)} t_1^{j_1} \ldots t_n^{j_n} x_1^{k_1} \ldots x_n^{k_n} F_{(j)(k)}(z) \tag{6}$$

where $z=(z_1, \ldots z_n)$, $z_j=(x_j, t_j)$, and (j) and (k) are the ordered sets $(j_1, \ldots j_n)$, $(k_1, \ldots k_n)$ of non-negative integers, and $j=j_1+\cdots+j_n$, $k=k_1+\cdots+k_n$. For each (j), (k), $F_{(j)(k)}$ is a Lorentz invariant distribution, the boundary value (in the sense of $\mathscr{S}'$) of an invariant function holomorphic and one-valued in the "extended tube" in the $n-1$ difference vectors $z_j - z_{j+1}$ on which $W(z)$ depends. The form (6) is not unique, since invariant polynomial factors can be absorbed in $F_{(j)(k)}$. But there exists a unique least value of $N(n)$, called the *tensor rank* of W_n.

We can isolate the part of $W_n(z)$ with highest rank as follows. Let $z \mapsto z' = \Lambda z$ be a real Lorentz transformation with parameter λ:

$$\begin{aligned} t_j' &= \tfrac{1}{2}(\lambda + 1/\lambda)\, t_j + \tfrac{1}{2}(\lambda - 1/\lambda)\, x_j \\ x_j' &= \tfrac{1}{2}(\lambda - 1/\lambda)\, t_j + \tfrac{1}{2}(\lambda + 1/\lambda)\, x_j . \end{aligned} \tag{7}$$

In terms of the light-cone coordinates $u_j = t_j + x_j$, $v_j = t_j - x_j$ it becomes

$$u_j' = \lambda u_j, \qquad v_j' = v_j/\lambda. \tag{8}$$

Introduce the new function

$$W_n(\Lambda, z) = W_n(\Lambda z). \tag{9}$$

Since $F_{(j)(k)}(\Lambda z) = F_{(j)(k)}(z)$, we see that $W_n(\lambda, z)$ is a polynomial in λ and $1/\lambda$, whose coefficients are themselves finite covariants in z. We define $N^{\pm}(n)$ to be the degree of this polynomial in $\lambda^{\pm 1}$, and the coefficients of $\lambda^{N^+(n)}$ and $\lambda^{-N^-(n)}$ will be called the leading terms as $\lambda \to \infty, 0$, respectively.

$W(z)$ is analytic at space-like separated points, that is, points for which each difference $(x_i - x_j, t_i - t_j)$ is a space-like vector; this means that if $\{\mathcal{O}_1, \ldots, \mathcal{O}_n\}$ is a space-like separated collection of open sets in $\mathbb{R}^2$, then the functional $W: \mathcal{D}(\mathcal{O}_1) \times \cdots \times \mathcal{D}(\mathcal{O}_n) \to \mathbb{C}$ defined by $f = (f_1, \ldots f_n) \mapsto (\Omega, \phi(f_1) \ldots \phi(f_n)\, \Omega) = W_n(f)$ is given by an integral.

From now on, we assume the spectrum condition holds. The idea of our method is to use (1) to derive bounds on the vacuum expectation values of products $\pi(f_1) \ldots \pi(f_n)$ along space-like orbits of the Lorentz group in $\mathbb{R}^{2n}$, thus obtaining bounds for $N^{\pm}(n)$. These bounds are improved by applying the Schwarz inequality; analytic continuation and an application of temperedness shows that any non-invariant part of the function (3) must be a rational function. Use of (2) locally then shows that there can be no poles in the non-invariant part, which is thus a polynomial. Positivity implies that this polynomial is a constant, thus proving Lorentz invariance.

Cannon and Jaffe [6] have proved that the Lorentz group acts as an automorphism group of the observable algebra for the ϕ_2^4 theory, and this has been extended to the ϕ_2^{2n} and $P(\phi)_2$ theories by Rosen [7] and Klein [8]. However, it remains to be proved that these automorphisms are implemented by unitary operators, and the present paper is a step towards this.

2. Bounds on the Expectation Values of Products of π

Following the methods of Glimm and Jaffe [1, 3], the estimate (1) leads to

$$\pm \pi(f, t) = U(t)\, \pi(f, 0)\, U^*(t) \leqq U(t)\, \|f\|_2 (H + I)\, U^*(t) \leqq \|f\|_2 (H + I),$$

the inequality holding as matrix elements between vectors from a dense set. If $g \in \mathcal{D}(\mathbb{R})$, we can approximate $\int \pi(f, t)\, g(t)\, dt$ by a Riemann sum, to obtain $\pm \pi(f \otimes g) \leqq \|f\|_2 \, \|g\|_1 (H + I)$. So, putting $R = (H + I)^{-1}$, we obtain

$$\pm R^{\frac{1}{2}} \pi(f \otimes g)\, R^{\frac{1}{2}} \leqq \|f\|_2 \, \|g\|_1 . \tag{10a}$$

Similarly, (2) leads to

$$\pm R^{\frac{1}{2}} \phi(f \otimes g)\, R^{\frac{1}{2}} \leqq \text{const}\, \|f\|_1 \, \|g\|_1 . \tag{10b}$$

Write $W_\pi(f_1 \otimes g_1 \otimes \cdots \otimes f_n \otimes g_n)$ for $(\Omega, \pi(f_1 \otimes g_1) \ldots \pi(f_n \otimes g_n)\,\Omega)$. Then we get the bounds

$$\begin{aligned}
&|W_\pi(f_1 \otimes g_1 \otimes \cdots f_n \otimes g_n)| \\
&\quad = |(\Omega, R^{\frac{1}{2}}\pi(f_1 \otimes g_1)\, R^{\frac{1}{2}}(H+I)\, R^{\frac{1}{2}}\pi(f_2 \otimes g_2) \ldots R^{\frac{1}{2}}\pi(f_n \otimes g_n)\, R^{\frac{1}{2}}\Omega)| \\
&\quad \leqq \|f_1\|_2\, \|g_1\|_1\, \|(H+I)\, R^{\frac{1}{2}}\pi(f_2 \otimes g_2)\, R^{\frac{1}{2}}(H+I) \ldots R^{\frac{1}{2}}\pi(f_n \otimes g_n)\, R^{\frac{1}{2}}\Omega\| \\
&\quad \leqq \|f_1\|_2\, \|g_1\|_1\, \|R^{\frac{1}{2}}\pi(f_2 \otimes g_2)\, R^{\frac{1}{2}}(H+I)^2 \ldots R^{\frac{1}{2}}\pi(f_n \otimes g_n)\, R^{\frac{1}{2}}\Omega \\
&\qquad + R^{\frac{1}{2}}[(H+I), \pi(f_2 \otimes g_2)]\, R^{\frac{1}{2}}(H+I) \ldots R^{\frac{1}{2}}\pi(f_n \otimes g_n)\, R^{\frac{1}{2}}\Omega\| \\
&\quad \leqq \|f_1\|_2\, \|g_1\|_1\, \{\|f_2\|_2\, \|g_2\|_1\, \|(H+I)^2 \ldots R^{\frac{1}{2}}\pi(f_n \otimes g_n)\, R^{\frac{1}{2}}\Omega\| \\
&\qquad + \|f_2\|_2\, \|\dot{g}_2\|_1 \|(H+I) \ldots R^{\frac{1}{2}}\pi(f_n \otimes g_n) R^{\frac{1}{2}}\Omega\|\}\,.
\end{aligned}$$

Proceeding in the same way, we arrive at

$$|W_\pi(f_1 \otimes g_1 \ldots \otimes f_n \otimes g_n)| \leqq \prod_{i=1}^{n} \|f_i\|_2 |||g_1 \otimes \cdots \otimes g_n||| \tag{11}$$

where $|||\cdots|||$ is a sum of products of L_1-norms of the various g_i and their time derivatives, with at most $n-1$ derivatives in any term. The property of $|||\cdots|||$ that we need is

$$|||g(\lambda t_1) \otimes \cdots \otimes g(\lambda t_n)||| = O(\lambda^{-1}) \quad \text{as} \quad \lambda \to \infty\,. \tag{12}$$

Actually, the inequality (11) is first established for the periodic theory in a box of volume V; then, letting $V \to \infty$ through a subsequence, we obtain (11) for the theory without cut-offs. The requisite convergence is established in [3]. Let $W_\pi(\lambda, z) = W_\pi(\Lambda z)$, and let $N^{\pm}(0, n)$ be the tensor rank of W_π.

Lemma 1. *For any* n, $N^{\pm}(0, n) \leqq 3n/2 - 1$.

Proof. Let $G(z)$ be the leading term of W_π for $\lambda \to \infty$. Hermiticity of π implies that G is real at space-like points. Without loss in generality, we may assume that there exists a space-like point $\hat{z} = (\hat{x}_1, \hat{t}_1, \hat{x}_2, \hat{t}_2, \ldots \hat{x}_n, \hat{t}_n)$ such that $G(\hat{z}) = 4\delta > 0$. Then there exists a space-like neighbourhood $\mathcal{N} = \{z; |x_1 - \hat{x}_1| \leqq r, |t_1 - \hat{t}_1| \leqq r, \ldots, |t_n - \hat{t}_n| \leqq r\}$ such that $G \geqq 2\delta$ in $\mathcal{N}$. Since $\mathcal{N}$ is a closed region of analyticity of the coefficient of λ^j in $W_\pi(\lambda, z)$ for each j, these coefficients are bounded in $\mathcal{N}$, so $W_\pi(\lambda, z)$ is dominated by its leading term, $\lambda^{N^+(0,n)} G(z)$ as $\lambda \to \infty$. Hence there exists $\lambda_0 \geqq 1$ such that

$$W_\pi(\lambda, z) \geqq \delta \lambda^{N^+(0,n)} \quad \text{for all} \quad \lambda \geqq \lambda_0,\ z \in \mathcal{N}\,. \tag{13}$$

Therefore

$$W_\pi(z) = W_\pi(\lambda, \Lambda^{-1} z) \geqq \delta \lambda^{N^+(0,n)} \quad \text{for all} \quad \lambda \geqq \lambda_0,\ \Lambda^{-1} z \in \mathcal{N}\,. \tag{14}$$

Let $g \in \mathscr{D}(\Delta)$, $g(t) \geqq 0$, $\int g(t)\,dt = 1$, where $\Delta = [-\frac{1}{2}r, \frac{1}{2}r] \subset \mathbb{R}$, and let $f(x)$ be the characteristic function of Δ. Let

$$h_\lambda(x_1, t_1, \ldots, x_n, t_n) = \lambda^n f(\lambda x_1)\, f(\lambda x_2) \ldots f(\lambda x_n)\, g(\lambda t_1)\, g(\lambda t_2) \ldots g(\lambda t_n).$$

We now show that

$$\int W_\pi(z)\, h_\lambda(z - \Lambda \hat{z})\, dz \geqq \delta \lambda^{N^+(0,n)-n} \quad \text{if} \quad \lambda \geqq \lambda_0. \tag{15}$$

Since $\lambda \int g(\lambda t)\,dt = 1$ and $\int f(\lambda x)\,dx = \lambda^{-1}$, it is sufficient to prove that $W_\pi(z + \Lambda \hat{z}) \geqq \delta \lambda^{N^+(0,n)}$ on $\operatorname{supp} h_\lambda$. This would hold, by (14), if $\Lambda^{-1} z + \hat{z} \in \mathcal{N}$ if $z \in \operatorname{supp} h_\lambda$. But since $\lambda \geqq 1$ we have $|\pm \lambda \pm 1/\lambda| \leqq 2\lambda$, and from (7), if $z' = \Lambda^{-1} z$, $|t'_j| = \frac{1}{2}|(\lambda + 1/\lambda)\, t_j + (1/\lambda - \lambda)\, x_j| \leqq (|t_j| + |x_j|)\, \lambda \leqq r$ on $\operatorname{supp} h_\lambda$; similarly, $|x'_j| \leqq r$ on $\operatorname{supp} h_\lambda$. Thus $\Lambda^{-1} z + \hat{z} \in \mathcal{N}$ and (15) follows.

We obtain a bound on $N^+(0, n)$ by comparing (15) with (11) and (12). Since $\|f_i\|_2 = \lambda^{-\frac{1}{2}}$, Eq. (11) gives, using (12)

$$\int W_\pi(z)\, h_\lambda(z - \Lambda \hat{z})\, dz \leqq \lambda^{-\frac{1}{2}n} \||\lambda^n g(\lambda t_1) \ldots g(\lambda t_n)\|| = O(\lambda^{\frac{1}{2}n-1}). \tag{16}$$

(Note that translation by $-\Lambda \hat{z}$ does not alter the norms.) Comparing (16) with (15) gives $N^+(0, n) \leqq 3n/2 - 1$. In the same way, by considering $\lambda \to 0$ instead of $\lambda \to \infty$, we show that $N^-(0, n) \leqq 3n/2 - 1$. □

3. Consequences of the Schwarz Inequality

Let $W_{m,n}(\lambda, z) = (\Omega, \phi(\Lambda z_1) \ldots \phi(\Lambda z_m)\, \pi(\Lambda z_{m+1}) \ldots \pi(\Lambda z_{m+n})\, \Omega)$. Then $W_{m,n}$ has the unique expansion

$$W_{m,n}(\lambda, z) = \sum_{-N^- \leqq k \leqq N^+} \lambda^k W^k_{mn}(z)$$

where W^k_{mn} is a tempered distribution of tensor rank k, the boundary value of a function holomorphic in the extended tube, having a one-valued continuation into the union of the permuted extended tubes [9]. Because of local commutativity, $N^\pm(m, n)$ are independent of the order among ϕ and π.

Lemma 2. *For any $j \leqq m$, $k \leqq n$, we have*

$$N^\pm(m, n) \leqq \tfrac{1}{2}\{N^\pm(2m - 2j, 2n - 2k) + N^\pm(2j, 2k)\}.$$

Proof. Let $f_j \geqq 0$, $f_j \in \mathscr{D}(\mathbb{R}^2)$, $j = 1, 2, \ldots m + n$, be chosen with mutually space-like supports such that the real holomorphic function $W_{mn}(z)$ is of one sign, say > 0, on $\operatorname{supp}(f_1 \otimes \cdots \otimes f_{m+n})$. Let $f_j^\Lambda(z) = f_j\ (\Lambda z)$. Schwarz's inequality then gives

$$\begin{aligned} &|(\Omega, \phi(f_1^\Lambda) \ldots \phi(f_m^\Lambda)\, \pi(f_{m+1}^\Lambda) \ldots \pi(f_{m+n}^\Lambda)\, \Omega)| \\ &\leqq \|\phi(f_1^\Lambda) \ldots \phi(f_j^\Lambda)\, \pi(f_{m+1}^\Lambda) \ldots \pi(f_{m+k}^\Lambda)\, \Omega\| \,. \\ &\|\phi(f_{j+1}^\Lambda) \ldots \phi(f_m^\Lambda)\, \pi(f_{m+k+1}^\Lambda) \ldots \pi(f_{m+n}^\Lambda)\, \Omega\| \,. \end{aligned}$$

The right-hand side is

$$O(\lambda^{\pm N^{\pm}(2j,2k)})^{\frac{1}{2}} \,.\, O(\lambda^{\pm N^{\pm}(2m-2j,2n-2k)})^{\frac{1}{2}} \quad \text{as} \quad \lambda^{\pm 1} \to \infty \,.$$

The left-hand side has leading term $\lambda^{\pm N^{\pm}(m,n)}$ with non-zero coefficient, for some choice of $f_1, \ldots f_{m+n}$. Hence result. □

Lemma 3. $N^{\pm}(0,n) \leqq n$.

Remark. This is exactly the rank we would expect for an n^{th} time derivative of a scalar.

Proof. We first prove it for even n by induction. It is true for $n=2$ (Lemma 1). Write $N(n)$ for $N^{\pm}(0,n)$. Suppose then that $N(n-2)=n-2$, where n is even. Then by Lemma 2,

$$N(n) \leqq \tfrac{1}{2}\{N(2j) + N(2n-2j)\} \quad \text{for all} \quad j \,.$$

Choose $2j = n-2$. Then

$$\begin{aligned} N(n) &\leqq \tfrac{1}{2}\{N(n-2) + N(2n-n+2)\} \leqq \tfrac{1}{2}\{n-2+N(n+2)\} \\ &\leqq \tfrac{1}{2}\{n-2+\tfrac{1}{2}[N(2j)+N(2(n+2)-2j)]\} \\ &= 3(n-2)/4 + N(n+6)/4 \leqq 7(n-2)/8 + N(n+14)/8 \leqq \ldots \\ &\leqq n-2+N(n+2^{r+1}-2)/2^r \leqq n-2+\{3(n+2^{r+1}-2)/2-1\}/2^r \end{aligned}$$

by Lemma 1. Thus $N(n) \leqq n+1+3(n-2)/2^{r+1} - 1/2^r < n+2$ for large r. We now remark that $N(n)$ is even. For, the tempered distribution W_{0n} is the limit in $\mathscr{S}'$ of the vacuum expectation values for the theory with a box cut-off, and these approximate vacua are invariant under the PT transformation $z \to -z$. The even-ness of $W_{0n}(z)$ persists in the limit. Since $N(n)$ is thus an even integer $< n+2$, and n is even, we get $N(n) \leqq n$. If n is odd we apply Lemma 2 and use $N(2j) \leqq 2j$ to get the result. □

Lemma 4. *If for some even integer n, $N^{\pm}(n,0)=0$, then $N^{\pm}(n,2) \leqq 2$.*

Proof. By Lemma 2,

$$\begin{aligned} N(n,2) &\leqq \tfrac{1}{2}\{N(n,0)+N(n,4)\} = \tfrac{1}{2}N(n,4) \leqq N(n,8)/4 \leqq \cdots \leqq N(n,2^{r+1})/2^r \\ &\leqq \{N(2n,0)+N(0,2^{r+2})\}/2^{r+1} \leqq N(2n,0)/2^{r+1}+2 \,, \end{aligned}$$

by Lemma 3; letting $r \to \infty$, we get $N^{\pm}(n,2) \leqq 2$. □

4. The Analytic Structure of the Non-Invariant Terms

This and the next section are devoted to proving that if, for some n, $N^{\pm}(n-2,2) \leqq 2$, then $N^{\pm}(n,0)=0$. The idea is that if $W_{n-2,2}$, which is the second time derivative of $W_{n,0}$, is a second rank tensor, then $W_{n,0}$ must be a scalar. This is not obvious, since differentiation with respect to x_j and t_j can decrease the tensor rank, as well as increase it. Indeed, by Eq. (8), $\partial/\partial u_j$ decreases the tensor rank by one and $\partial/\partial v_j$ increases it, or vanishes. Since

$$(\Omega, \ddot{\phi}(z_1)\,\phi(z_2)\ldots\phi(z_n)\,\Omega) = -\sum_{j=2}^{n} (\Omega, \pi(z_1)\,\phi(z_2)\ldots\pi(z_j)\ldots\phi(z_n)\,\Omega)$$

the assumption $N(n-2,2) \leqq 2$ implies that every second derivative of W_{n0}, with respect to time, has rank $\leqq 2$. Let W^+ be the part of W_{n0} of highest rank, $N^+(n,0)$. Then $\frac{\partial^2 W_{n0}}{\partial t_i \partial t_j}$ contains the term $\frac{\partial^2 W^+}{\partial v_i \partial v_j}$ of rank $N^+(n,0)+2$, the remaining terms being of smaller rank. If $N^+(n-2,2) \leqq 2$ is assumed, we see that either $N^+(n,0)=0$ or $\frac{\partial^2 W^+}{\partial v_i \partial v_j}$ vanishes for all i,j. Thus if $N^+(n,0)>0$, W^+ has the form

$$W^+ = W_0^+(u) + \sum_{j=1}^{n} v_j W_j^+(u)\,, \qquad u=(u_1, \ldots u_n)\,.$$

Lemma 5. *If* $N^+(n-2,2) \leqq 2$, *then* W^+ *is a rational function, holomorphic if* $u_i \neq u_j$, $i=1,2,\ldots n; j=1,2,\ldots n$.

Proof. W_j^+ is analytic if $\operatorname{Im}(u_1-u_2)>0$, $\operatorname{Im}(u_2-u_3)>0, \ldots$ $\ldots \operatorname{Im}(u_{n-1}-u_n)>0$; with the variables in the other order, it is analytic if $\operatorname{Im}(u_1-u_2)<0$, $\operatorname{Im}(u_2-u_3)<0, \ldots \operatorname{Im}(u_{n-1}-u_n)<0$. The two functions coincide at a real point $(u_1, \ldots u_n)$ if there exists a real v, such that $(u_1, v_1; \ldots; u_n, v_n)$ is space-like. This is true if $u_i \neq u_j$; for then we may order the points $u_{i_1} < u_{i_2} < \cdots < u_{i_n}$, and choose the v's arranged in the other order, $v_{i_1} > v_{i_2} > \cdots > v_{i_n}$. This ensures that every difference $(u_i-u_j, v_i-v_j)=(z_i-z_j)$ is space-like: $(z_i-z_j)^2=(u_i-u_j)(v_i-v_j)<0$. Then, by the edge-of-the-wedge theorem ([10], Theorem 2–16) W_j^+ is holomorphic and one-valued in a $\mathbb{C}^n$-neighbourhood of the real axis, omitting the hyperplanes $u_i=u_j$. We now show that if $u_2, \ldots u_n$ are real, no two being equal, then $W_j^+(u_1; u_2, \ldots u_n)$ has an analytic continuation to every point in the u_1-plane, except the real points $u_1=u_j$, $j=2,3,\ldots n$. Suppose $u_{i_2} < u_{i_3} < \cdots < u_{i_n}$, and $\operatorname{Im} u_1 > 0$. Then a complex Lorentz transformation $u \to \lambda u$, where λ has a small negative imaginary part, leads us to a point for which $\operatorname{Im} u_1 > \operatorname{Im} u_{i_2} > \cdots > \operatorname{Im} u_{i_n}$. This lies in the forward tube corresponding to the permutation $(1,2,\ldots n) \to (1, i_2, \ldots i_n)$. The boundary value as $\operatorname{Im} u_1 \to 0$ is the $W^+(z)$ corresponding to this permutation. This is real if $u_1 \neq u_j$, $j=2,3,\ldots n$, since $W_{n0}(z)$ is real there. The Schwarz reflection principle then ensures that $W^+(z)$ has a one-valued continuation into the whole u_1-plane, omitting the points $u_1=u_j$, $j=2,3,\ldots n$.

The boundary value of W_j^+ is a distribution in u_1, so the singularities in u_1 are of finite order [10, Theorem 2–10]. Hence there exists a polynomial $\prod_{k=2}^{n} (u_1-u_k)^{n_k}$, such that $\prod_{j=1}^{n} (u_1-u_k)^{n_k} W^+(u_1, \dots u_n)$ is entire in u_1, for $u_2, \dots u_n$ in some neighbourhood of the real axis (namely, the neighbourhood given by the edge-of-the-wedge theorem). By the continuity theorem for functions of several complex variables, it is entire in u_1 whenever $u_2 \dots u_n$ are such that the function is analytic in u_1 for some u_1. Now, W_j^+ is the Laplace transform of a tempered distribution with support in the right half-space in each of its variables. It is therefore bounded by a polynomial in real and positive imaginary directions [10, Theorem 2–8]. By considering the boundary value from below in the u_1-plane, we similarly obtain a polynomial bound in the lower half-plane. Hence $\prod_{k=2}^{n} (u_1-u_k)^{n_k} W_j^+$ is a polynomial in u_1. The coefficients in this polynomial are functions of $u_2, \dots u_n$, holomorphic in the union of the permuted extended tubes in these variables. Proceeding in the same way, we conclude by induction that there exist numbers n_{ik}, $i, k = 1, 2, \dots n$, such that $\prod_{i=k}^{n} (u_i-u_k)^{n_{ik}} W_j^+$ is a polynomial. This holds for each j, so W^+ is a rational function, holomorphic unless $u_i = u_k$. □

Similarly, we show that W^-, the part of least rank, is a rational function of $z_1, \dots, z_n$, linear in the u's and holomorphic unless $v_i = v_j$ for some i, j.

5. Bounds on the Local Singularities of Wightman Functions

Let us consider W^+ as a function of $u_1 - u_2$, the other variables $v, u_2-u_3, \dots u_{n-1}-u_n$ being held fixed. Since it is a rational function, we can expand as a partial fraction

$$W^+(z) = \sum_{p \geqq 0} a_p (u_1-u_2)^p + \sum_{r=1}^{R} b_r/(u_1-u_2)^r + \sum_{\substack{j \geqq 3 \\ s}} c_{sj}/(u_1-u_j)^s . \quad (17)$$

Here, all the sums are finite, and a_p, b_r, c_{sj} are linear functions of v, and are rational functions of $u_2-u_3, u_3-u_4, \dots u_{n-1}-u_n$, analytic unless $u_i = u_j$ for some $i \neq j, i \geqq 2, j \geqq 2$. The idea is to isolate the worst singularity of $W^+(z)$ at $z_1 = z_2$, namely $b_R(u_1-u_2)^{-R}$, which will turn to be too singular to be allowed by (1) and (2) unless $R = 0$.

Let us choose $(\hat{z}_3, \dots \hat{z}_n) \in \mathbb{R}^{2n-4}$, mutually space-like and such that $|\hat{u}_j| > \varepsilon_1$ say, $j = 3, 4, \dots n$, and such that $b_R(v, u_2, \dots u_n)$ is, say, $\geqq \delta > 0$, in some neighbourhood $\mathcal{N}$ of $(\hat{z}_3, \dots \hat{z}_n)$ provided $|u_2| < \varepsilon_1$ and $|v_j| < \varepsilon_1$,

$j=1,2$. Let $G(z_3, \dots z_n) \in \mathcal{D}(\mathcal{N})$ and $\varepsilon > 0$, $\varepsilon < \varepsilon_1$ be chosen such that

$$|W^+(z_1, z_2; G)| > \tfrac{1}{2}\delta \|G\|_1 |u_1 - u_2|^{-R} \quad \text{for all} \quad z_1, z_2 \in B_\varepsilon$$

where

$$B_\varepsilon \equiv \{z_1, z_2; |u_1| < \varepsilon, |u_2| < \varepsilon, |v_1| < \varepsilon, |v_2| < \varepsilon\}.$$

This can always be done, since the leading term $b_R/(u_1 - u_2)^R$ dominates W^+ as $u_1 \to u_2$, and $b_R \geqq \delta$ on the support of G. Since

$$W^+ = \frac{1}{N!} \frac{\partial^N}{\partial \lambda^N} (\lambda^{N^-(n,0)} W(\lambda, z))$$

where $N = N^+(n, 0) + N^-(n, 0)$, we obtain the lower bound

$$\left| \frac{\partial^N}{\partial \lambda^N} \lambda^{N^-} W(\lambda; z_1, z_2; G) \right| > C|u_1 - u_2|^{-R} \quad \text{for all} \quad (z_1, z_2) \in B_\varepsilon, \tag{18}$$

where $C = \frac{1}{2}N!\delta\|G\|_1$. Now let $f: \mathbb{R} \to \mathbb{R}^+$ have its support in $[0, 1]$, and be such that $\int f(t)\, dt = 1$. Then for each value of the large parameter ϱ, define $f_\varrho(t) = \varrho f(\varrho t)$, and $h_\varrho(z_1, z_2) = f_\varrho(x_1)\, f_\varrho(t_1)\, f_\varrho(x_2 - 2/\varrho)\, f_\varrho(t_2 - 2/\varrho)$. The supports of h_ϱ in z_1 and z_2 are disjoint, and z_1 and z_2 cannot coincide with any z_j, $j \geqq 3$, if the latter lie in $\mathcal{N}$ and ϱ is large enough. On the support of h_ϱ, $|\varrho t_1| \leqq 1$, $|\varrho x_1| \leqq 1$, $|\varrho t_2 - 2| \leqq 1$ and $|\varrho x_2 - 2| \leqq 1$. Hence, on $\operatorname{supp} h_\varrho$,

$$\begin{aligned} |u_1 - u_2| &= |x_1 + t_1 - x_2 - t_2| \leqq |x_1| + |t_1| + |x_2| + |t_2| \\ &\leqq 1/\varrho + 1/\varrho + 3/\varrho + 3/\varrho \leqq 8/\varrho < \varepsilon \quad \text{if} \quad \varrho > \varrho_0 = 8/\varepsilon. \end{aligned}$$

Since $\operatorname{supp} h_\varrho \subset B_\varepsilon$, we have, from (18),

$$\left| \frac{\partial^N}{\partial \lambda^N} \lambda^{N^-} W(\lambda, h_\varrho \otimes G) \right| \geqq C(\varrho/8)^R. \tag{19}$$

The left hand side of (19) is independent of λ, so we may put $\lambda = 1$. We now get a contradiction with Eqs. (1) and (2), unless $R = 0$. By repeated differentiation, $\partial^N/\partial\lambda^N(\lambda^{N^-} W(\lambda, z))$ is a sum of terms

$$\int \prod_{i=1}^{j} (A_i x_1 + B_i t_1) \prod_{i=1}^{k} (C_i x_2 + D_i t_2) \dots \prod_{i=1}^{p} (E_i x_n + G_i t_n)$$

$$\left(\Omega, \frac{\partial^j \phi(z_1)}{\partial t_1^{j_0} \partial x_1^{j_1}} \frac{\partial^k \phi(z_2)}{\partial t_2^{k_0} \partial x_2^{k_1}} \cdots \frac{\partial^p \phi(z_n)}{\partial t_n^{p_0} \partial x_n^{p_1}} \Omega \right) h_\varrho(z_1, z_2) G(z_3, \dots z_n) dz. \tag{20}$$

Here, $A_i, \dots G_i$ are integers coming from the differentiation of t'_j, x'_j given by Eq. (7), and products of such expressions, several times with respect to λ, and then putting $\lambda = 1$. The important feature is that the j^{th} derivative of the field $\phi(z_1)$ carries a "small" factor $t_1^r x_1^s$, with $r + s = j = j_0 + j_1$; and the k^{th} derivative of the field $\phi(z_2)$ carries a "small" factor $t_2^y x_2^m$, with $y+m=k=k_0+k_1$. Let

$$f_1=\frac{\partial^{j_1}}{\partial x_1^{j_1}}(x_1^s f_\varrho(x_1)); \qquad g_1=\frac{\partial^{j_0}}{\partial t_1^{j_0}}(t_1^r f_\varrho(t_1));$$

$$f_2=\frac{\partial^{k_1}}{\partial x_2^{k_1}}(x_2^y f_\varrho(x_2-2/\varrho)); \quad g_2=\frac{\partial^{k_0}}{\partial t_2^{k_0}}(t_2^m f_\varrho(t_2-2/\varrho)).$$

Then (20) is bounded by a finite sum of terms of the form

$$\begin{aligned} &\mathrm{const}\,|(\Omega, \phi(f_1\otimes g_1)\,\phi(f_2\otimes g_2)\,\phi(z_3)\ldots\phi(z_n)\,\Omega)\,(G)| \\ &\quad \leqq \mathrm{const}\,\|R^{\frac{1}{2}}\phi(f_1\otimes g_1)\,R^{\frac{1}{2}}(H+I)\,R^{\frac{1}{2}}\phi(f_2\otimes g_2)\,R^{\frac{1}{2}}(H+I)\,R^{\frac{1}{2}}\ldots\Omega(G)\| \\ &\quad \leqq \mathrm{const}\,\|f_1\|_1\,\|g_1\|_1\{\|R^{\frac{1}{2}}\phi(f_2\otimes g_2)\,R^{\frac{1}{2}}(H+I)^2\ldots\Omega(G)\| \qquad (21)\\ &\qquad +\big\|R^{\frac{1}{2}}\dot{\phi}(f_2\otimes g_2)\,R^{\frac{1}{2}}(H+I)\,R^{\frac{1}{2}}\phi(z_3)\ldots\Omega(G)\big\|\} \\ &\quad \leqq \mathrm{const}\,\|f_1\|_1\,\|g_1\|_1\{C'\,\|f_2\|_1\,\|g_2\|_1+\|f_2\|_2\,\|g_2\|_1\} \end{aligned}$$

by Eq. (10), where the constant and C' are independent of ϱ. Now

$\|f_1\|_1=O(\varrho^{j_1-s})$, $\|f_2\|_1=O(\varrho^{k_1-y})$, $\|f_2\|_2=O(\varrho^{k_1-y+\frac{1}{2}})$, $\|g_1\|_1=O(\varrho^{j_0-r})$, $\|g_2\|_1=O(\varrho^{k_0-m})$,

as $\varrho\to\infty$. Thus (21) is bounded by

$$C\varrho^{j_1-s+j_0-r+k_0-m+k_1-y+\frac{1}{2}}=O(\varrho^{\frac{1}{2}}).$$

This contradicts (19) unless $R=0$, since R is an integer. Hence the part of W of highest rank has no poles in u_1-u_2. Since the labels 1 and 2 were arbitrary, we have proved:

Lemma 6. *If the spectrum condition holds, and* $N(n-2,2)\leqq 2$, *then* W_n^+ *is a polynomial.*

6. The Consequences of Positivity: the Proof Completed

The idea of this Section is that the tensor of highest rank, W^+, dominates the positivity condition on the Wightman functions, for large λ, and so satisfies positivity by itself. We also show that no polynomial other than a constant can satisfy the positivity condition, so that W^+ of Lemma 6 is a constant.

Lemma 7. *Let* $F(z_1,\ldots z_{2n})$ *be a symmetric polynomial of* $2n$ *two-vectors* $z_1,\ldots z_{2n}$, *satisfying*

a) $\int F(z_1,\ldots z_{2n})\,\bar{f}(z_n,\ldots z_1)\,f(z_{n+1},\ldots z_{2n})\,d^2z_1\ldots d^2z_{2n}\geqq 0,\ f\in\mathscr{D}$,

b) $F(z_1,\ldots z_{2n})=F(z_1+a,\ldots z_{2n}+a),\quad a\in\mathbb{R}^2$.

Then F is of degree zero.

Proof. Regard F as a function of $z_1 - z_2, \ldots z_{n-1} - z_n; z_{n+1} - z_{n+2}, \ldots \ldots z_{2n-1} - z_{2n}$, and the other two variables z_n and z_{n+1} in the combination $z_n - z_{n+1}$. Thus, for any $f \in \mathscr{D}(\mathbb{R}^{2(n-1)})$, and $h \in \mathscr{D}(\mathbb{R}^2)$, we have

$$\int F(\bar{f} \otimes f; z_n - z_{n+1}) \bar{h}(z_n) h(z_{n+1}) d^2 z_n d^2 z_{n+1} \geqq 0 .$$

So, for fixed f, $F(\bar{f} \otimes f; z_n - z_{n+1})$ is of positive type, but is a polynomial. Bochner's theorem then implies that it is independent of $z_n - z_{n+1}$. By polarization, $F(f \otimes g; z_n - z_{n+1})$ is independent of $z_n - z_{n+1}$. Hence $F(z_1 - z_2, \ldots z_{2n-1} - z_{2n})$ is independent of $z_n - z_{n+1}$ when $z_1 - z_2, \ldots \ldots z_{n-1} - z_n, z_{n+1} - z_{n+2}, \ldots z_{2n-1} - z_{2n}$ are held fixed. Regarding F now as a function of $z_1, \ldots z_{2n}$, the above result implies that $dF = \sum_1^{2n} \frac{\partial F}{\partial z_j} dz_j$ is zero if $dz_1 = \cdots = dz_n$, $dz_{n+1} = \cdots = dz_{2n}$, but no other restrictions are placed. Hence $\sum_1^n \frac{\partial F}{\partial z_j} = 0$. By symmetry, the sums of every n gradients of F are zero. For $n > 1$, this implies that any two gradients of F are equal, and hence that they vanish. For $n = 1$, Bochner's theorem gives the result immediately. □

Lemma 8. *If* $N^{\pm}(2n-2, 2) \leqq 2$, *then* $N^{\pm}(2n, 0) = 0$.

Proof. By Lemma 6, the part W^+ of W_{2n} of highest tensor rank is a polynomial. By Lemma 7, it is not positive semi-definite unless of zero degree. Hence there exists a function $f \in \mathscr{D}(\mathbb{R}^{2n})$ such that $W^+(\bar{f} \otimes f) = -\delta < 0$. Then $W^+(\bar{f}^\lambda \otimes f^\lambda) = -\lambda^N \delta$, where $N = N(2n, 0)$. Now, if $N > 0$, W^+ dominates W_{2n} as $\lambda \to \infty$, so that $W_{2n}(\bar{f}^\lambda \otimes f^\lambda) \to -\infty$ as $\lambda \to \infty$. This violates positivity. Hence $N^+(2n, 0) = 0$. Similarly, by considering W^- and letting $\lambda \to 0$, we prove that $N^-(2n-2, 2) \leqq 2$ implies $N^-(2n, 0) = 0$. □

Theorem. *If the spectrum condition holds, all the Wightman functions are Lorentz invariant.*

Proof. If n is even, we combine Lemmas 8 and 4, and apply induction, starting with $n = 1$. If n is odd, this result and Lemma 2, gives the result in that case too. □

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