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Lorentz Cobordism

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Abstract. A Lorentz cobordism between two (in general nondiffeomorphic) 3-manifolds M_0, M_1 is a pair (M, v), where M is a differentiable 4-manifold and v is a differentiable vector field on M, such that 1) the boundary of M is the disjoint union of M_0 and M_1, 2) v is everywhere nonzero, 3) v is interior normal on M_0 and exterior normal on M_1. Such a manifold M admits a Lorentz tensor with respect to which M_0 and M_1 are spacelike hypersurfaces; thus a Lorentz cobordism is a model of a portion of a spacetime in which "the topology of spacelike hypersurfaces is changing". We discuss the form that these changes can take, and give two methods for constructing a Lorentz cobordism between two nondiffeomorphic 3-manifolds. We comment on the possible relevance of Lorentz cobordism to the problem of gravitational collapse.

I. Introduction

Suppose we cut a "slab" out of a spacetime manifold by slicing along two disjoint spacelike hypersurfaces: we are left with a 4-manifold M with boundary $M_0 \cup M_1$, where M_0 and M_1 are disjoint 3-manifolds which are spacelike hypersurfaces with respect to the Lorentz structure [1] induced on M by that on our original spacetime. If we do this to the model spacetimes that have traditionally been studied in general relativity theory (except some of those with "singularities"?), the 3-manifolds M_0 and M_1 will be diffeomorphic; there will be no "change of topology". We consider here the case in which M_0 and M_1 are *not* diffeomorphic. We shall be interested in such things as the topology of the manifold M, the causal properties of Lorentz structures on it, the singularities of Lorentz structures on it, and, ultimately, the solution of Einstein's field equations on such an underlying manifold.

We shall, however, do things the other way round from the "cutting" operation imagined above. Rather, we shall start with two non-diffeomorphic 3-manifolds M_0, M_1, and then consider how we can construct a 4-manifold M and a (singular or nonsingular) Lorentz tensor g on M such that the boundary of M is the disjoint union of M_0 and M_1, with M_0 and M_1 spacelike with respect to g.

[1] We shall use the terms *Lorentz structure* and *Lorentz tensor* synonymously to mean a globally defined, second rank, symmetric tensor of Lorentz signature $(+, +, +, -)$.

We give two such constructions here, one of which yields singular Lorentz structures and the other of which yields nonsingular Lorentz structures. Detailed properties of these constructions, and explicit examples of them, will be treated in subsequent papers.

These manifolds are expected to exhibit causal anomalies: according to a theorem of Geroch [1], such a manifold must either be non-isochronous or contain closed timelike curves (or both). While one feels rather strongly that a realistic spacetime should not contain closed timelike curves, it is not entirely clear that spacetime must necessarily be isochronous. In any case, other considerations may render these anomalies physically innocuous. For example, in our singular case (Section IV), any smooth timelike curve along which the sense of time is reversed must pass through the singular point; perhaps no physical entity could survive such a trip.

Even if some of these manifolds are in principle acceptable for doing physics, one might not have been inclined until rather recently to take this possibility very seriously. It is our view that the recent results on gravitational collapse and singularities force us to now take it seriously; this is the primary motivation for the present work.

The "infinite density" singularity in terms of which one tends to think of gravitational collapse poses a serious challenge to general relativity. Of course, it is possible that general relativity will have to be altered somehow to meet this challenge, but before attempting this we would like to know whether the challenge can be met within the context of the theory we have now. The notion of changes in the topology of spacelike hypersurfaces seems to us very suggestive in this regard. For instance, is it possible that a Kruskal-type "wormhole" can, instead of pinching off into an infinite curvature singularity, be *severed* in a smooth fashion (meaning such that spacetime remains a differentiable manifold), after which the spacelike hypersurfaces consist of two unconnected pieces? So far as it goes, the answer to this question is yes. Whether this is of any relevance to gravitational collapse remains to be seen, but it is worth finding out.

The remainder of this paper takes the following form. In Section II we review briefly the cobordism theory of Wallace and Milnor, which is our main mathematical tool. Our problem is posed in Section III. Two solutions (there are undoubtedly others) are then given, a singular one in Section IV and a nonsingular one in Section VI. Section V is a digression on vector fields, and the Appendix contains a computation needed for Section VI.

Implicit in this work is, of course, the assumption that spacetime can be represented by a differentiable manifold; we consider this assumption to be an integral part of general relativity theory. Throughout the following, all manifolds will be assumed Hausdorff and paracompact. The word differentiable will always mean C^∞. We shall use the term closed manifold to mean compact manifold without boundary. For any manifold N, ∂N denotes the boundary of N, and the symbol $\simeq$ will occaisionally be used to denote diffeomorphism.

II. Cobordism and Morse Functions

The notion of cobordism seems to have been introduced by Thom [2], who studied the so-called cobordism ring of equivalence classes of cobordant manifolds. Subsequently, a cobordism theory[2] based on an extension of Morse theory was developed independently by Wallace [4] and Milnor [5]. We shall sketch here certain results from this theory; for further details the reader can consult Refs. [3–7]. In particular, the first three chapters of Ref. [6] contain an excellent, detailed exposition of the theory. Chapters 4, 5, and 6 of Ref. [7] provide a somewhat more elementary introduction to this material.

Let M_0, M_1 be $n-1$ dimensional differentiable manifolds. Then M_0 and M_1 are *cobordant* (or *cobounding*) if there exists an n-manifold M whose boundary is $M_0 \cup M_1$, with $M_0 \cap M_1 = \emptyset$. One says that M is a *cobordism* between M_0 and M_1. Before we can outline the Wallace-Milnor cobordism theory, we shall need a few results from Morse theory.

Suppose N is a differentiable manifold, $f: N \to R$ a differentiable function on N. A point $p \in N$ is called a *critical point* of f if the differential of f vanishes at p. In terms of local coordinates $\{x^1, \ldots, x^n\}$ in a neighborhood of p, this means $f_{,a}|_p = 0$ for $a = 1, \ldots, n$, where the comma denotes partial differentiation. If p is a critical point of f, one calls the point $f(p) \in R$ a *critical value* of f. A critical point p is called *nondegenerate* if the nullity of the Hessian of f at p is zero. In terms of local coordinates, this means that the matrix $f_{,ab}|_p$ is nonsingular. A critical point which is not nondegenerate is called *degenerate*.

A differentiable function which has no degenerate critical points is called a *Morse function*. There exist Morse functions on every differentiable manifold [8].

Let f be a Morse function on N, and let $N_c = f^{-1}(c)$; that is, N_c is the set of points $p \in N$ such that $f(p) = c$. If N_c contains a critical point of f it is called a *critical level* of f; otherwise it is called a *noncritical level* of f. A noncritical level N_c is a differentiable submanifold of N. If two noncritical levels have no critical level between them, they are diffeomorphic; in fact, if there are no critical values of f in the real interval $[c_1, c_2] \subset f(N)$, then the set of $p \in N$ such that $c_1 \leqq f(p) \leqq c_2$ is diffeomorphic to

[2] This theory is closely related to Smale's "handlebodies" [3].

$N_{c_1} \times [c_1, c_2]$ [8]. Any changes in the topology of the family N_c of level surfaces of f must take place as critical levels are crossed.

One can, in fact, give a detailed description of what happens when a critical level is crossed, using the following lemma of Morse [8]: Let p be a nondegenerate critical point of f. Then there is a local coordinate system $\{x^1, \ldots, x^n\}$ in a neighborhood U of p with $x^a(p)=0$ for all a and such that the identity

$$f(x)=f(p)-(x^1)^2-\cdots-(x^\lambda)^2+(x^{\lambda+1})^2+\cdots+(x^n)^2 \tag{1}$$

holds throughout U. One calls the integer λ the *(Morse) index* of the critical point p.

It is an immediate consequence of this lemma that nondegenerate critical points are isolated.

By a straightforward but rather lengthy study [6–8] of the orthogonal trajectories of the family N_c, using the representation of f provided by Morse's lemma, one can find out what is happening to the N_c when a critical level is crossed. The main result is the following.

Let N be a differentiable manifold of dimension n, f a Morse function on n, and N_c a critical level of f with one degenerate critical point p, of index λ, on it. Choose a and b, $a<c<b$, such that c is the only critical value of f in $[a, b]$. Then p has a neighborhood D^n in N, an n-cell whose boundary S^{n-1} is the union of three sets A, B, C. The set A is on N_a and is diffeomorphic to $D^{n-\lambda} \times S^{\lambda-1}$, the set B is on N_b and is diffeomorphic to $S^{n-\lambda-1} \times D^\lambda$, while C, lying between N_a and N_b, is diffeomorphic to $S^{n-\lambda-1} \times S^{\lambda-1} \times [a, b]$. Here a point (q, r) of A with $q \in \partial D^{n-\lambda}$ and $r \in S^{\lambda-1}$ is identified with (q, r, a) in C; and a point (q, r) of B with $q \in S^{n-\lambda-1}$ and $r \in \partial D^\lambda$ is identified with (q, r, b) in C. Moreover, if the cell D^n is removed from the part of N between N_a and N_b, the remainder is diffeomorphic to $(N_a - A) \times [a, b]$, where $(N_a - A) \times \{a\}$ corresponds to $N_a - A$, and $(N_a - A) \times \{b\}$ corresponds to $N_b - B$.

It is not difficult to get an intuitive feeling for this from Morse's representation of f. Note that the $\lambda-1$-sphere S_a specified by $(x^1)^2+\cdots+(x^\lambda)^2=c-a$, $x^{\lambda+1}=\cdots=x^n=0$ lies on N_a, and the $n-\lambda-1$-sphere S_b specified by $x^1=\cdots=x^\lambda=0$, $(x^{\lambda+1})^2+\cdots+(x^n)^2=b-c$ lies on N_b. Take a tubular neighborhood $D^{n-\lambda} \times S^{\lambda-1}$ in N_a of S_a. This is the set A. Its boundary is diffeomorphic to $S^{n-\lambda-1} \times S^{\lambda-1}$. Translating this boundary to N_b along the orthogonal trajectories of the family N_d yields the set C, which is a union of orthogonal trajectories. The intersection of C with N_b, which is again diffeomorphic to $S^{n-\lambda-1} \times S^{\lambda-1}$, is the boundary of a tubular neighborhood $S^{n-\lambda-1} \times D^\lambda$ in N_b of the $n-\lambda-1$-sphere S_b. This tubular neighborhood is the set B.

Thus, N_b is obtained from N_a by the following operation. A set $D^{n-\lambda} \times S^{\lambda-1}$ is removed from N_a. This leaves a manifold with boundary, the boundary being diffeomorphic to $S^{n-\lambda-1} \times S^{\lambda-1}$. A new manifold is obtained by identifying this boundary with the boundary of a set $S^{n-\lambda-1} \times D^{\lambda}$ (this latter boundary being also diffeomorphic to $S^{n-\lambda-1} \times S^{\lambda-1}$); the new manifold is N_b.

One says that in this operation a *spherical modification (*or *Morse surgery) of type* $\lambda - 1$ is performed on N_a.

Now we can get back to cobordism, in particular to the close relation between cobordism and spherical modifications pointed out by Wallace and Milnor. The main result is the following [4–6]. Let M_0, M_1 be two cobordant closed $n-1$-manifolds, and suppose the n-manifold M is a cobordism between them. Then there is a differentiable function f on M which has a finite number of critical points, none of them degenerate, and such that 1) $f(M) = [0, 1]$, 2) $f^{-1}(0) = M_0$, 3) $f^{-1}(1) = M_1$. Thus M_1 can be obtained from M_0 by a finite number of spherical modifications. One can, in fact, show that two closed manifolds are cobordant if and only if one can be derived from the other by a finite number of spherical modifications.

Given two $n-1$-manifolds M_0, M_1 and a sequence (this sequence is not unique) of spherical modifications which relate them, the above discussion of nondegenerate critical points gives a good characterization of the topology of a cobordism M between M_0 and M_1. Indeed, this characterization is enough to enable us to construct M [4–6].

Since we can always choose to do the modifications "one at a time", it will suffice to treat the case in which M_1 is obtained from M_0 by a single spherical modification, of type $\lambda - 1$, say. Remove a set $B_0 \simeq D^{n-\lambda} \times S^{\lambda-1}$ from M_0, and a set $B_1 \simeq S^{n-\lambda-1} \times D^{\lambda}$ from M_1. This can be done such that $M_0 - B_0 \simeq M_1 - B_1$. Form the set $(M_0 - B_0) \times [0, 1]$, and identify $M_0 - B_0$ with $(M_0 - B_0) \times \{0\}$ and $M_1 - B_1$ with $(M_0 - B_0) \times \{1\}$. Form the union $((M_0 - B_0) \times [0, 1]) \cup B_0 \cup B_1$, inserting B_0 and B_1 in $(M_0 - B_0) \times \{0\}$ and $(M_0 - B_0) \times \{1\}$ in accordance with the identification just made. The subset $(Fr B_0 \times [0, 1]) \cup B_0 \cup B_1$ in this space is an $n-1$-sphere. Adding a set D^n by identifying ∂D^n with this subset yields a topological manifold whose boundary is $M_0 \cup M_1$. In fact, the identification can be done in such a way[3] that this topological manifold is a differentiable manifold M which is, then, a cobordism between M_0 and M_1. One can also display[3] a differentiable function f on M with a single, nondegenerate critical point, with index λ, and such that $f(M) = [0, 1]$, $f^{-1}(0) = M_0$, and $f^{-1}(1) = M_1$.

The Wallace-Milnor theory is generally presented as a cobordism theory for closed manifolds. But note that it also tells us quite a lot about cobordism for noncompact, boundaryless manifolds. This is of interest from the point of view of applications to general relativity. In particular,

[3] For the details, see Ref. [5] or Ref. [6].

if two noncompact, boundaryless manifolds M_0, M_1 are related by a finite number of spherical modifications, then they are cobordant, and we can construct a cobordism between them as above. The only shortcoming in this approach to noncompact manifolds is that probably being related by a finite number of spherical modifications is not a necessary condition for two of them to be cobordant. But from the point of view of physics (certainly at this stage), it is enough to know that it is a sufficient condition.

III. Cobordism and Lorentz Structures

It is well known[4] that a differentiable manifold admits a Lorentz structure[1] if and only if it admits globally a nonsingular (meaning everywhere nonzero) vector field. In fact, let v be such a vector field. Then we can construct from it a Lorentz structure as follows. Every differentiable manifold admits globally a positive definite Riemannian metric[5]. Let h be such a one, and let u be the vector field obtained from v by normalizing it to unity with repect to the metric h. Then $g_{ab} = h_{ab} - c u_a u_b$, where the function c is everywhere greater than 1, is a Lorentz tensor. Of course, the differentiability class of g is the smallest of the differentiability classes of h, u, and c. Note also that u is timelike with repect to g.

Thus, a precise statement of the problem sketched in Section I is the following. Let M_0, M_1 be cobordant differentiable 3-manifolds without boundary. We seek a 4-manifold M and a vector field on M such that 1) the boundary of M is the disjoint union of M_0 and M_1, 2) v is everywhere nonzero, 3) v is interior normal on M_0 and exterior normal on M_1. According to the previous paragraph, M then admits a Lorentz structure with respect to which v is timelike (so that M_0 and M_1 are spacelike). We shall therefore call such a pair (M, v) a *Lorentz cobordism* between M_0 and M_1.

The existence of a Lorentz cobordism between any two closed 3-manifolds has been proved by Reinhart [11][6]. But we seek sufficiently explicit constructions to enable us to study the properties of Lorentz cobordisms mentioned in Section I, and we are interested in the noncompact case as well.

We shall consider the case in which M_1 can be obtained from M_0 by a single spherical modification. This is no loss in generality for the case

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[4] See, for instance, Ref. [9], p. 207.

[5] See, for example, Ref. [10], p. 126.

[6] Geroch, in Ref. [1], proves a similar but weaker theorem: he requires only that the vector field v is nowhere tangent to M_0 or M_1. Thus it could, for instance, be exterior on both M_0 and M_1. It seems to us that our requirement 3) is more in accord with what one has in mind physically.

in which M_0 and M_1 are compact, since then they are related by a finite number of spherical modifications. Our methods can be applied also to any two noncompact 3-manifolds which are related by a finite number of spherical modifications.

For the sake of intuitive feeling, we list here the possible spherical modifications of a 3-manifold. Recall that in a spherical modification of type $\lambda - 1$, a set $D^{4-\lambda} \times S^{\lambda-1}$ is cut out, and it is replaced by a set $S^{3-\lambda} \times D^{\lambda}$. Thus in three dimensions the possibilities are (listed by the type number $\lambda - 1$):

−1. The empty set is replaced by a 3-sphere. This may be relevant to the problem of the "initial singularity", but can be ignored if one's concern is cobordism.

0. The disjoint union of two 3-disks is replaced by a "tube" $S^2 \times D^1$. This could, for instance, correspond to the formation of a "wormhole".

1. A solid torus is replaced by another solid torus, but with the surfaces identified differently (for example, reversing the roles of meridians and parallels). It is difficult to have much intuitive feel for this. It is known [4] that any compact oriented 3-manifold can be obtained from the 3-sphere by modifications of this type.

2. A "tube" $D^1 \times S^2$ is replaced by the disjoint union of two 3-disks. This could, for instance, correspond to the severing of a "wormhole".

3. A 3-sphere is replaced by the empty set. This could have something to do with a "final singularity", but can be ignored so far as cobordism is concerned.

IV. A Singular Construction

Our requirement 2) of Section III, that the vector field v be nonsingular, is clearly a necessary condition for the Lorentz tensors constructed from v to be nonsingular. We shall drop this requirement in the present section, since we can then give a construction which is so simple that, as physicists, we should not ignore it: the resulting singularity in g (which is confined to a single well defined point) may prove to be physically innocuous.

Let M_0 be a (compact or noncompact) boundaryless 3-manifold, M_1 a manifold which is obtained from M_0 by a spherical modification of type $\lambda - 1$. According to the Wallace-Milnor cobordism theory outlined in Section II, we can construct a 4-manifold M, with boundary $M_0 \cup M_1$, and a differentiable function f on M such that $f(M) = [0, 1]$, $f^{-1}(0) = M_0$, $f^{-1}(1) = M_1$, and f has a single, nondegenerate critical point p of index λ on M with, say, $f(p) = \frac{1}{2}$.

Now the first vector field on M that one thinks of is $v_a = f_{,a}$. This is interior normal on M_0 and exterior normal on M_1. It is deficient with regard to the discussion of Section III only in being zero at the single point p. Nevertheless, it might be interesting to see what sort of Lorentz structures we get if we insist upon using this vector field.

From the Morse representation of f given in Section II (Eq. (1)), there exist local coordinates $\{x^a\}$ in a neighborhood U of p in terms of which v has the components $v_a = 2s_a x^a$ (no sum on a), where $s_a = -1$ for $a \leqq \lambda$ and $s_a = +1$ for $a > \lambda$. Let h be a Riemannian metric on M, and assume, for the sake of simplicity, that h is diagonal with respect to the coordinates $\{x^a\}$. Then normalizing v with respect to h yields the vector field

$$u_a = s_a x^a / (h^{bc} x^b x^c)^{\frac{1}{2}} \qquad \text{(no sum on } a)$$

in U. Now set

$$g_{ab} = h_{ab} - c u_a u_b = h_{ab} - c s_a s_b x^a x^b / h^{cd} x^c x^d \qquad \text{(no sum on } a, b)$$

where $c > 1$ everywhere.

This has, of course, a singularity – let us call it a *Morse singularity* – at p. The limit of g as we approach p from any definite direction exists, but it is in general different as we approach p from different directions. We feel that it is worth trying to make some physical sense of this state of affairs, but we shall not attempt to do so here.

V. Singular Vector Fields

We want to "get rid" somehow of the singularity in the gradient field of f, and to do so we shall have to know something about the singularities of vector fields in general. In this section we review the standard results; for more details, Ref. [12] can be consulted.

Let M, N be oriented n-manifolds without boundary, with M compact and N connected, and let $f: M \to N$ be a differentiable map. Let $df_p : TM_p \to TN_{f(p)}$ be the differential of f at the point $p \in M$ [7]. A point $p \in M$ is called a *regular point* of f if df_p is nonsingular, and a point $q \in N$ is called a *regular value* of f if $f^{-1}(q)$ contains only regular points. Note that (since M is assumed compact) if q is a regular value, then $f^{-1}(q)$ is a finite set.

Define $\operatorname{sgn} df_p$ to be $+1$ if df_p is orientation preserving, and -1 if df_p is orientation reversing. Define the *(Brouwer) degree* $\deg f$ to be

$$\deg f = \sum_{p \in f^{-1}(q)} \operatorname{sgn} df_p$$

where q is any regular value of f. In fact, this sum does not depend on the choice of q, so the degree is well defined.

Note that a diffeomorphism has degree $+1$ if it is orientation preserving, and degree -1 if it is orientation reversing.

[7] We use TM_p to denote the tangent space to M at p.

The importance of the Brouwer degree stems from the fact that it is a homotopy invariant: if two differentiable maps f, g from M to N are smoothly homotopic, then $\deg f = \deg g$. (The maps f, g are smoothly homotopic if there exists a differentiable map $F: M \times [0,1] \to N$, with $F(p,0) = f(p)$ and $F(p,1) = g(p)$ for all $p \in M$.) In particular, if M is connected and N is the n-sphere, the converse holds also, according to a theorem of Hopf: If M is a connected, oriented, boundaryless differentiable n-manifold, then two differentiable maps from M to S^n are smoothly homotopic if and only if they have the same Brouwer degree.

Now let v be a differentiable vector field on some n-manifold M, with an isolated zero at some $p \in M$. Let $v_a(x)$ be the components of v with respect to local coordinates $\{x^a\}$ in a neighborhood of p. Set $u_a(x) = v_a(x)/((v_1(x))^2 + \cdots + (v_n(x))^2)^{\frac{1}{2}}$. If we evaluate u on a small sphere centered at $x(p)$, we can regard $u_a(S^{n-1})$ as a (differentiable) mapping from S^{n-1} into S^{n-1}. The Brouwer degree of this map is called the *index* of the vector field v at the zero p. This definition is independent of the choice of local coordinates about p, and of the choice of (sufficiently small) sphere around p on which we evaluate u.

It is easy to see that if p is a nondegenerate critical point, with Morse index λ, of a differentiable function f, then the index of the gradient field $f_{,a}$ at p is $(-1)^\lambda$.

An important application of this concept of index is the Poincaré-Hopf Theorem: Let M be a closed differentiable manifold, v a differentiable vector field on M with isolated zeros. Then the sum of the indices of v at all its zeros is equal to the Euler number $\chi(M)$ of M.

VI. A Nonsingular Construction

In this section, we show how to construct a nonsingular Lorentz cobordism (M', v') from the singular Lorentz cobordism (M, v) of Section IV.

The general idea is the following. We remove from M an ε-ball D^4 around the critical point p, leaving an additional boundary component S^3. We also remove an ε-ball from an appropriately chosen closed 4-manifold N, leaving it with a boundary S^3. We then paste $M - D^4$ and $N - D^4$ together by identifying these two boundaries S^3; call the resulting manifold M'. If we have chosen N properly, we can extend v onto $N - D^4$ to get a nonsingular vector field v' on M'. We shall do this in such a way that M' is a differentiable manifold and v' is a differentiable vector field.

We now proceed to the details. Let (M, v) be as in Section IV, and let N be a closed differentiable 4-manifold, to be specified later. It is known that there exists a differentiable vector field, say w, on N which has a single, isolated singular point, say q [8]. From the Poincaré-Hopf Theorem (Section V) the index of w at q is equal to the Euler number $\chi(N)$ of N.

Remove an ε-ball around p in M, leaving a boundary component $V_1 \simeq S^3$; likewise cut out an ε-ball around q in N, leaving a boundary $V_2 \simeq S^3$. Call the manifolds that remain after these removals $\tilde{M}$ and $\tilde{N}$. Let $h: V_1 \to V_2$ be a diffeomorphism, and denote by $\tilde{M} \bigcup_h \tilde{N}$ the topological manifold obtained from $\tilde{M} \cup \tilde{N}$ by identifying each $x \in V_1$ with $h(x) \in V_2$. We put a differentiable structure on $\tilde{M} \bigcup_h \tilde{N}$ as follows [6, 14].

There exist neighborhoods U_1, U_2 of V_1, V_2 in $\tilde{M}, \tilde{N}$ and diffeomorphisms $g_1: V_1 \times (-1, 0] \to U_1$, $g_2: V_2 \times [0, 1) \to U_2$ such that $g_1(x, 0) = x$ for all $x \in V_1$ and $g_2(y, 0) = y$ for all $y \in V_2$. (One calls such a neighborhood U_1 a *product neighborhood*, or *collar neighborhood* of V_1.) Let j_1, j_2 be the inclusion maps $j_1: \tilde{M} \to \tilde{M} \bigcup_h \tilde{N}$, $j_2: \tilde{N} \to \tilde{M} \bigcup_h \tilde{N}$, and define a map $g: V_1 \times (-1, 1) \to \tilde{M} \bigcup_h \tilde{N}$ by

$$g(x, t) = j_1 \circ g_1(x, t) \qquad t \in (-1, 0]$$
$$g(x, t) = j_2 \circ g_2(h(x), t) \quad t \in [0, 1).$$

Now $\tilde{M} \bigcup_h \tilde{N}$ is covered by $j_1(\tilde{M} - V_1)$, $j_2(\tilde{N} - V_2)$, and $g(V_1 \times (-1, 1))$. We can define a differentiable structure on each of these three, and transfer them to $\tilde{M} \bigcup_h \tilde{N}$ *via* j_1, j_2, and g. With this differentiable structure, $\tilde{M} \bigcup_h \tilde{N}$ is a differentiable manifold, which we shall call M'; this differentiable manifold is of course a cobordism between M_0 and M_1.

Henceforth we shall refer to $U_1 \bigcup_h U_2 \subset M'$ as U. The map $g: V_1 \times (-1, 1) \to U$ is a diffeomorphism.

Now it remains to construct a nonsingular vector field on M'.

Suppose z is a vector field on $V_1 \times (-1, 1)$. Then we can map the restriction of z to $V_1 \times (-1, 0]$ onto the corresponding vector field z_1 on U_1 as follows:

$$z_1 = dg_1 \circ z \circ g_1^{-1}.$$

Similarly, the vector field z_2 on U_2 corresponding to the restriction of z to $V_1 \times [0, 1)$ is

$$z_2 = dg'_2 \circ z \circ g_2'^{-1},$$

where $g'_2(x, t) = g_2(h(x), t)$, $x \in V_1$, $t \in [0, 1)$. Let z_{1_0}, z_{2_0} denote the restrictions of z_1, z_2 to V_1, V_2:

$$z_{1_0} = dg_{1_0} \circ z_0 \circ g_{1_0}^{-1}$$
$$z_{2_0} = dg'_{2_0} \circ z_0 \circ g_{2_0}'^{-1},$$

[8] See, for instance, Ref. [13], p. 550.

where the subscripts zero on the right hand side denote the restrictions to $V_1 \times \{0\}$. We can then write

$$z_{2_0} = dg'_{2_0} \circ dg_{1_0}^{-1} \circ z_{1_0} \circ g_{1_0} \circ g'^{-1}_{2_0}.$$

Now we wish to join the vector field w on $\tilde{N}$ (defined at the beginning of this section) with the vector field v on $\tilde{M}$. Divide U_2 into two pieces $U'_2 = g_2(V_2 \times [0, \frac{1}{2}])$, $U''_2 = g_2(V_2 \times [\frac{1}{2}, 1))$, and denote $g_2(V_2 \times \{\frac{1}{2}\})$ by V_3. Of course $V_3 \simeq V_2$. Let w' be the restriction of w to $\tilde{N} - U'_2$. Then we can get a continuous nonsingular vector field on M' if we can get a continuous nonsingular vector field V on U'_2 which is equal to w' on V_3 and is equal to

$$v_{2_0} = dg'_{2_0} \circ dg_{1_0}^{-1} \circ v_0 \circ g_{1_0} \circ g'^{-1}_{2_0}. \tag{2}$$

on V_2, where v_0 is the restriction of v to V_1.

Recall that $V_3 \simeq V_2 \simeq S^3$. Normalizing w', v_{2_0} to unity, we can regard them as maps of S^3 into S^3. So the vector field V that we seek is nothing but a homotopy $V: S^3 \times [0, \frac{1}{2}] \to S^3$, between the maps v_{2_0} and w_0, where w_0 is the restriction of w' to V_3. From the theorem of Hopf quoted in Section V, a smooth homotopy between v_{2_0} and w_0 will exist if and only if, regarded as maps from S^3 to S^3, they have the same Brouwer degree. Note that the degree of w_0 is equal to the index of our original vector field w (on N) at the zero q, which is in turn equal to $\chi(N)$.

Thus, we proceed as follows. Given v_0, we compute v_{2_0} from Eq. (2). We then compute the degree of v_{2_0} regarded as a map from S^3 to S^3. We choose N to have Euler number $\chi(N) = \deg v_{2_0}$; then the degree of w_0 will be equal to the degree of v_{2_0}, so that a smooth homotopy V between v_{2_0} and w_0 exists. We therefore have a continuous, nonsingular vector field z' on M' : z' is equal to v on $\tilde{M}$, equal to w on $\tilde{N} - U'_2$, and equal to V on U'_2.

From Eq. (2), it is clear that $\deg v_{2_0}$ depends on v_0 only through the smooth homotopy class of v_0, that is only through the degree of v_0 regarded as a map from S^3 to S^3. If the spherical modification relating M_0 and M_1 is of type $\lambda - 1$, the degree of v_0 is $(-1)^\lambda$. So we need only consider the two cases where v_0 has degree $+1$ or -1. A sample calculation of the degree of v_{2_0} is done in the Appendix. It turns out that if v_0 has degree $+1$ (spherical modification of type -1, 1, or 3) than v_{2_0} has degree $+1$, and if v_0 has degree -1 (spherical modification of type 0 or 2) then v_{2_0} has degree $+3$.

So if the spherical modification relating M_0 and M_1 is of type -1, 1, or 3, we must choose N such that its Euler number is $+1$, for instance the four dimensional real projective space P^4. If the spherical modification is of type 0 or 2, we must choose N such that its Euler number is $+3$, for instance the two dimensional complex projective space CP^2 (regarded, of course, as a four dimensional real manifold).

We are almost done. The only problem is that our nonsingular vector field z' is continuous but (on V_2 and V_3) not necessarily differentiable, whereas we want a differentiable vector field. It is, however, not difficult to smooth z' out.

Under the map g^{-1}, z' induces a corresponding vector field on $V_1 \times (-1, 1)$; call it z. We shall smooth out z, and then map it back to U *via* g.

Using the Weierstrass approximation theorem, it is not difficult to show that any continuous mapping of a compact manifold into S^n can be uniformly approximated by a differentiable mapping. So z can be uniformly approximated by a differentiable vector field, say z''. Let $A(t)$ be a differentiable function which is equal to 1 for $t \in [-\frac{5}{8}, \frac{5}{8}]$ and equal to zero for $t \leqq -\frac{3}{4}$ and for $t \geqq \frac{3}{4}$. Define a vector field z''' on $V_1 \times (-1, 1)$ by

$$z'''(x, t) = A(t)\, z''(x, t) + (1 - A(t))\, z(x, t)$$

for $x \in V_1, t \in (-1, 1)$. Then z''' is differentiable everywhere on $V_1 \times (-1, 1)$, and is equal to z outside $V_1 \times (-\frac{3}{4}, \frac{3}{4})$. Moreover, since z'' is a uniform approximation to z, z''' is nowhere zero. Let w'' be the vector field on U which corresponds, under the map g, to z'''.

Define a vector field v' on M' as follows. Set v' equal to v on $\tilde{M} - U_1$, equal to w on $\tilde{N} - U_2$, and equal to w'' on U. Then v' is interior normal on M_0, exterior normal on M_1, and is differentiable and nonzero everywhere on M'. So (M', v') is the desired nonsingular Lorentz cobordism between M_0 and M_1.

It is unfortunate that the intuitively appealing spherical modifications of types 0 and 2 seem to be involving us in the study of so messy a 4-manifold as the two dimensional complex projective space. Perhaps a simpler 4-manifold with Euler number $+3$ can be found.

It would be very nice if we could first study some two dimensional Lorentz cobordisms between 1-manifolds which are related by a spherical modification. In this dimension there are in fact modifications analogous to the type 0 and 2 modifications on 3-manifolds. Unfortunately, one finds that in order to do the above construction in two dimensions one needs a closed 2-manifold with Euler number $+3$, and there does not exist such a manifold.

VIII. Conclusions

We have sketched the way in which the topology of spacelike hypersurfaces in spacetime might be changing (spherical modifications), and remarked that some of these changes (modifications of types 0 and 2) are not without intuitive appeal.

We have given two ways to construct a Lorentz cobordism. The detailed study of the cobordisms of Section VI would seem to be a formidable task. The cobordisms of Section IV are appealingly simple, but if we are to take them seriously as models of spacetime, we shall have to learn to live with their singularities.

It is our feeling that, if spacetime can be represented by a differentiable manifold at all, then *some* sort of Lorentz cobordism may well clarify what is going on in gravitational collapse.

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Appendix

For the construction of Section VI, we have to compute the index of v_{2_0} from that of v_0, using Eq. (2). This is straightforward, but one has to pay careful attention to the orientations.

The neighborhoods U_1 and U_2 are both diffeomorphic to $S^3 \times [1,2)$. For the sake of computational simplicity, we will not make any distinction between U_1, U_2 and their representations as $S^3 \times [1,2)$.

We can then represent the maps $g_1 : V_1 \times (-1,0] \to S^3 \times [1,2)$ and $g'_2 : V_1 \times [0,1) \to S^3 \times [1,2)$ as

$$g_1(x,t) = (x, 1-t) \qquad x \in V_1,\ t \in (-1,0]$$
$$g'_2(x,t) = (h(x), t+1) \qquad x \in V_1,\ t \in [0,1).$$

So Eq. (2) becomes

$$v_{2_0} = d(h(x), t+1)_0^{-1} \circ d(x, 1-t)_0 \circ v_0 \circ (x, 1-t)_0 \circ (h(x), t+1)_0^{-1}.$$

Now v_{2_0} depends on v_0 and on h. Since we are only interested in the degree of v_{2_0}, regarded as a map from S^3 to S^3, we can, as remarked in Section VI, pick any v_0 with index $(-1)^\lambda$. Likewise, we need only worry about the smooth homotopy class of h. This is $+1$ for an orientation preserving diffeomorphism and -1 for an orientation reversing diffeomorphism. So we need only do the computation for one orientation preserving h (the identity, say) and for one orientation reversing h (a reflection, say).

Let h be the identity. Then Eq. (3) says simply that v_{2_0} is obtained from v_0 by reversing its component normal to S^3. We shall compute the index of v_{2_0} for the case in which the index of v_0 is -1.

We can conveniently represent $S^3 \times [1,2)$ as the set of points in a Euclidean coordinate patch with $1 \leqq (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 < 2$, where the x^a are Euclidean coordinates. A v_0 with index -1 is given by $v_0 = (x^1, x^2, x^3, -x^4)$. (This would correspond to a modification of type 0.) Reversing the radial component of v_0, we find $v_{2_0} = ((1-f)\,x^1, (1-f)x^2, (1-f)x^3, (-1-f)x^4)$, where $f = 2((x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2)$.

Now regarding v_{2_0} as a map from S^3 to S^3, when we restrict the x^a to the sphere $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1$, we pick a value of v_{2_0} and hope it is a regular value. It happens that $v_{2_0} = (1, 0, 0, 0)$ is a regular value. Solving for the points x for which v_{2_0} has this value, we find that there are three such, which we shall call x_1, x_2, and x_3. They are $x_1^a = (-1,0,0,0)$, $x_{2,3}^a = (\frac{1}{2}, 0, 0, \pm (\frac{3}{4})^{\frac{1}{2}})$. Now we must check whether v_{2_0} preserves or reverses orientation at these three points.

An oriented set of coordinate neighborhoods on the sphere $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1$ can be specified as follows. For $x^1 > 0$, use x^2, x^3, x^4 as coordinates. For $x^1 < 0$, use $x^2, x^3, -x^4$ (the minus sign being necessary so that the two patches have like orientation). Define coordinates similarly on the sets $x^a > 0$ and $x^a < 0$, with $a = 2, 3, 4$.

We can use the last three components of v_{2_0} as coordinates in a neighborhood of $v_{2_0} = (1, 0, 0, 0)$ in the image space. In a neighborhood of x_1 we can use $x^2, x^3, -x^4$ as coordinates, and in a neighborhood of x_2 or x_3 we can use x^2, x^3, x^4. With these choices of coordinates, we calculate the determinant of dv_{2_0} at the three regular values x_1, x_2, x_3, and find that it is positive at all three regular values. So the index of v_{2_0} is $+3$.

References

1. Geroch, R. P.: J. Math. Phys. **8**, 782 (1967).
2. Thom, R.: Commun. math. Helv. **28**, 17 (1954).
3. Smale, S.: Ann. Math. **74**, 391 (1961).
4. Wallace, A. H.: Canad. J. Math. **12**, 503 (1960).
5. Milnor, J.: AMS Sym. Pure Math. III, 39 (1961).
6. — Lectures on the *h*-cobordism theorem. Princeton, N. J.: Princeton University Press 1965.
7. Wallace, A. H.: Differential topology: First steps. New York: W. A. Benjamin 1968.
8. Milnor, J.: Morse theory. Princeton, N. J.: Princeton University Press 1963.
9. Steenrod, N.: The topology of fibre bundles. Princeton, N. J.: Princeton University Press 1951.
10. Bishop, R. L., Crittendon, R. J.: Geometry of manifolds. New York: Academic Press 1964.
11. Reinhart, B. L.: Topology **2**, 173 (1963).
12. Milnor, J.: Topology from the differentiable viewpoint. Charlottesville, Va.: The University Press of Virginia 1965.
13. Alexandroff, P., Hopf, H.: Topologie I. Berlin: Springer 1935.
14. Munkres, J. R.: Elementary differential topology. Princeton, N. J.: Princeton University Press 1966.

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