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Dispersion Relations for the Vertex Function from Local Commutativity

II. Two-Dimensional Dispersion Relations

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Abstract. Representation formulas for the vertex function are derived which are valid for arbitrary complex values of two of the scalar variables inside the corresponding domain of holomorphy, while the third variable is evaluated in a neighbourhood of positive real values (the corresponding physical-region singularities). The analyticity domains always include at least the corresponding "axiomatic" Källén-Wightman domain.

Two alternative versions are given. In the first formula the occurring weight functions are the boundary values of the vertex function along the real axes of the scalar variables. The values of the arguments are such that the boundary values can be expressed in terms of the causal respectively time-ordered functions, which in a well-known way via reduction technique are related to the different form factors of the underlying field theory. In the second formula the weight functions are just the on-mass-shell matrix elements of the field operators which completely describe the dynamics of the theory and e.g. contain explicit information (in their support properties) on the mass spectrum of the field theory.

The assumptions behind the results are the very general physical assumptions of Källén and Wightman, i.e. essentially Lorentz-covariance, translation-invariance, "reasonable" mass- and energy-spectrum, and local commutativity. We further need some moderate integrability- and boundedness-properties of the vertex function. It is shown that the formalism can be extended to cover all cases of at most polynomial increase of the vertex function in asymptotic directions inside the Källén-Wightman holomorphy domain.

The kernel functions in the representation formulas can be explicitly seen to result in possible singularities only along the physical-region-cuts, i.e. from the contributions of the corresponding physical states, as well as along the more complicated Källén-Wightman boundary-surfaces of the holomorphy domain, which correspond to the so-called "anomalous cuts".

The kernel functions in the second version above are closely connected to simple perturbation theoretical functions.

1. Introduction

The notion of causality, in more or less sophisticated mathematical formulation, has had far-reaching implications on elementary particle physics.

Thus assumptions of causality, combined with some further physical hypotheses and formulated in field theoretical language, imply analyticity properties of different physical quantities such as e.g. scattering- and production-amplitudes, form factors etc. The analyticity properties can then be explored to relate the parameters of different processes to each other. The methods employed in that connection can usually be traced back to the use of (complex) contour-integrals of the analytic functions in question.

Well-known examples of such procedures are different integral representation formulas, e.g. the dispersion relations for the forward scattering amplitudes. As a rule the integrand of the representation formula in question can be factorised unambiguously into a "kernel function" and a "weight function"; or, anyhow, it can be written as a sum of terms with that general appearance. Then the kernel function will contain, broadly speaking, the analyticity properties derived from the above-mentioned general physical assumptions, while the weight function will contain the dynamical parameters of the specific process.

In the case of the forward dispersion relations the kernel function is as is well-known, actually a "Cauchy denominator" and the weight function is essentially the measurable cross section for the process.

It is further known that many physical quantities may be continued analytically as functions of more than one variable and that the functions may exhibit analyticity in large domains in the product space of the complex planes of these variables.

Due to the complications, met in the theory of analytic functions of several variables, the rather far-reaching implications of such analyticity properties have as of now only been partly available. For the case of the vertex functions, which via field theoretical reduction technique are related to different form factors, Källén and Wightman have explicitly constructed a domain of holomorphy [2]. The basic assumptions of their paper are essentially that the field theory in question should admit

1. *Lorentz-covariance* and translation-invariance.
2. A *"reasonable" mass spectrum.*
3. *Causality* in the form of *local commutativity.*

The third assumption is of fundamental importance both because of its necessity for the interpretation of the reduction formulas and for its far-reaching implications.

In an earlier paper, hereafter called I [1], we have used the results of Källén and Wightman [2] to construct representation formulas, "one-dimensional dispersion relations", for particular values of the arguments of the vertex function. The domain of validity of these representation formulas can be described as neighbourhoods of the "axiomatic" boundary of the analyticity domain, i.e. the above-mentioned Källén-Wightman domain (hereafter called D_{KW}). To be specific the vertex function is considered close to the intersection between one of the physical-region-cuts, i.e. close to positive real values of one of the conventional scalar variables, and one of the "complicated" Källén-Wightman boundary surfaces. The physical-region-cuts occur as reminiscences of the corresponding two-point-function boundaries (cf. Section 7). The more complicated Källén-Wightman boundaries are often referred to as "anomalous cuts".

The dispersion relations constructed in that way are "one-dimensional" in the sense that the occurring integrals are effectively one-dimensional. In this case, too, the kernel functions are Cauchy denominators, while the weight functions are for the momentum-space function the on-mass-shell matrix elements of the field operators. In that way both the detailed dynamics and e.g. the mass spectrum of the field theory enter directly into the representation formulas.

In this paper we will extend the results of paper I to more general situations.

Thus the vertex function will be described by representation formulas, "two-dimensional dispersion relations", which are valid for *arbitrary* complex values of two of the three conventional scalar variables. The third variable is, just as in paper I, evaluated just above or just below the corresponding physical region cut, i.e. close to an arbitrary – but fixed – real value.

The term "two-dimensional" is used to indicate that the formulas contain integrals along different two – (real) – dimensional "contour surfaces" in the product of the complex planes of the conventional scalar variables of the vertex functions. We use the Lorentz squares of the three external energy momentum vectors (fulfilling energy-momentum conservation) as momentum space variables and the Lorentz squares of the coordinate-differences between the field-points as the corresponding coordinate-space variables. We will in this paper give several alternative versions of the two-dimensional dispersion relations.

The first sections, i.e. Sections 2–5, are devoted to the derivation of a "basic" representation formula, in the sense that the resulting relation contains all the information that can be inferred from the above-mentioned "axiomatic" analyticity properties of the vertex function. The occurring kernel functions in the integral representations can be explicitly seen to exhibit possible singularities only along the above-mentioned physical-region-cuts, i.e. for positive real values of the scalar variables, as well as along the "anomalous cuts", i.e. the above-mentioned more complicated boundary surfaces of the Källén-Wightman domain.

The weight functions in the "basic" representation formula are combinations of different boundary values of the vertex function along the real axes of its arguments. The occurring boundary values are actually physical in the sense that in all cases of interest the momentum-space boundary values can, e.g., be expressed in terms of the causal (i.e. the retarded and the advanced) or the time-ordered functions. These (distribution valued) functions may then via conventional reduction technique be related to the different matrix elements of the operators of the theory. Such connections are discussed in some detail in Section 4 of paper I.

It should, however, be noted that the momentum-space weight functions in the "basic" formula do not explicitly exhibit e.g. the mass spectrum of the theory. This is so because the causal and time-ordered functions in their property of "interpolating quantities" between the different matrix elements of the theory, are non-vanishing also for values of the arguments, which do not correspond to a physical process.

In Section 7 it is shown how to exchange the above-mentioned weight functions to the same kind of weight functions, i.e. the on-mass-shell matrix elements of the field operators, that are met in the formulas of paper I. The properties of the weight functions in that way describe the dynamics in detail and we note especially that their support properties mirror the mass spectrum of the theory. Some straight-forward integration then results in a "new" dispersion relation and this representation formula is also two-dimensional, in the same sense as above. The occurring kernel functions turn out to be closely related to different (simple) perturbation theoretical functions. This is made evident in Section 8 where some further comments and extensions are included.

We would like to stress the great similarity between the above-described results for vertex functions, i.e. three-point functions, and the well-known Källén-Lehmann [11] representation formula for two-point functions. In both cases, the occurring weight functions are the on-mass-shell matrix elements of the field operators and the kernel functions are simple perturbation theoretical functions.

We have made some simplifying assumptions on the "asymptotic" behaviour of the vertex function in connection with all the formulas mentioned so far. Thus the vertex function is supposed to fulfil certain boundedness conditions in different directions "around infinity" in the variable space (inside the holomorphy domain) in order that the occurring integrals should converge. In Section 6 it is shown that the same kind of representation formulas can be derived as long as the vertex function is asymptotically at most polynomially increasing. Such relations are of the kind usually called "subtracted dispersion relations".

Due to the similarity between the properties of the momentum space vertex function (here called G) and the corresponding coordinate-space vertex function (F) the same kind of relations are valid in both cases.

We will for convenience use momentum-space concepts in the derivations of the formulas and only in the end comment on some differences in case the corresponding coordinate-space notions are used.

We will further just as in paper I only consider a scalar field theory in order to simplify the notations as much as possible. On the other hand this "model theory" will in order to spell out the most general situation for the three-point function contain three different scalar fields.

2. The Methods used in Deriving the Dispersion Relations

In this section we will give a brief survey of the methods employed in the following sections, Sections 3–5.

The dispersion formulas will be derived in two steps. In each step we will rely upon the explicit appearance and the properties of the *axiomatic holomorphy domain D_{KW} constructed by Källén and Wightman*. In Section 2 of paper I we have briefly discussed the appearance of the boundaries of the domain D_{KW} and both in the original paper [2] and elsewhere [3] there exist extensive pictorial descriptions of it.

The first step of the derivation will be to give representation formulas which describe the vertex functions along different parts of the above-mentioned axiomatic boundary. To that end we will use very similar methods to those employed in paper I. Because the domain D_{KW} exhibits a complete symmetry between the three scalar variables of the vertex function [2], it is only necessary to investigate a few specific cases in detail. The remaining results can then be achieved by obvious permutation of indices. We will here in some detail investigate the case when the third variable (Z_3) is kept fixed close to the corresponding physical region cut (the "Z_3-cut"). The boundary region that we are interested in is then a hyper-surface of three real dimensions that actually divides the two-(complex)-dimensional (i.e. four-real-dimensional) domain of variation of the two remaining scalar variables into two (disjoint) parts. The situation is depicted in Figs. 1 and 2 where we have shown the boundary of the Källén-Wightman domain D_{KW} for two arbitrary but fixed (complex) values of the first scalar variable (Z_1) in the complex plane of the second variable (Z_2). The boundary then appears as the two curves marked (I) and (II) in these figures. In Figs. 1 and 2 we have further marked out another curve, denoted (III). The curve (III) consists in both cases of parts of a circle with the radius R.

It is clear that by a suitable combination of the curves (I), (II) and (III) we can in all cases create a "Cauchy-contour" $C(R, Z_1)$ with the following properties:

(i) The curve $C(R, Z_1)$ is closed and continuous and only encircles points inside the corresponding analyticity domain of the vertex function.

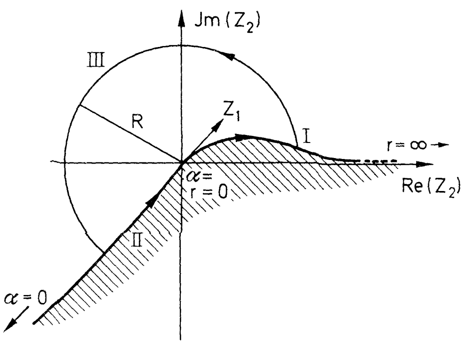

Fig. 1

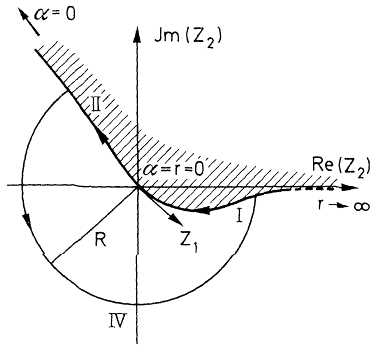

Fig. 2

Thus in the case of arbitrary but fixed value of the variable Z_1 in the upper (Fig. 1) or the lower (Fig. 2) half complex plane, only such values of the variable Z_2 which are inside the domain D_{KW} (the unshaded areas) are encircled.

(ii) In the limit when the radius R tends to infinity, the limiting curve $C(\infty, Z_1)$ in particular encircles *all* such points.

(iii) The curve $C(R, Z_1)$ is oriented in a positive sense by means of the arrows indicated in the figures.

Then the second step in deriving the dispersion formulas is to *represent the vertex function for arbitrary values of the variable Z_2 inside* the respective curve $C(R, Z_1)$ by means of a complex contour-integral along $C(R, Z_1)$. To that end, we only need the values of the vertex function G along the curves (I), (II) and (III). In the special case when we can neglect the contribution from the curve (III) in the limit when R tends to infinity we deduce that we only need the values of G along the curves (I) and (II), i.e. just the values of G along the "finite" parts of the Källén-Wightman boundary. These values are, however, the same as those which are achieved in the above-mentioned "first step". We can consequently by combining the integral relations from the first and the second step achieve a resulting formula for the vertex function in which only the boundary values on the real axes of its arguments occur.

Some details for the first step in the program is given in Section 3, and the corresponding discussion for the second step occurs in Section 4.

The above-described formalism is, however, unsatisfactory in two different ways. We first of all note that a certain asymmetry between the arguments Z_1 and Z_2 results out of the very method of derivation. Such an asymmetry must of course due to the above-mentioned actual symmetry of the domain D_{KW} be only fictitious. Secondly, we note that the resulting formulas actually seem to imply the existence of possible singularities not only along the positive real axis of the variable Z_1 (i.e. the corresponding physical-region cut) but also for *negative* real values of the same variable, irrespective of the values of the variable Z_2. According to the results of Källén and Wightman there are no such singularities and a closer investigation tells us that these anomalies also stem from the method of derivation.

In Section 5 it is shown that the "symmetrisation procedure" which solves the first-mentioned difficulty also results in a final formula for the vertex function which explicitly exhibits the "correct" analyticity properties.

3. The Vertex Function along the Boundary of the Källén-Wightman Domain

In this section we will give a few details of the first step in the derivation of the dispersion relations.

We will show how to represent the vertex function for values of the arguments in the neighbourhoods of the boundaries of the domain D_{KW}. The resulting dispersion integrals will contain the boundary values of the vertex function solely along the real axes of the variables. The methods which are employed are very similar to the ones exhibited in some detail in paper I.

In paper I we used the results of Källén and Wightman to prove that the four functions $\Gamma^{\pm(3)}$ and $\gamma^{\pm(3)}$ are analytic in the whole complex plane of the variable Z_1, except for the real axis:

$$\begin{aligned}\Gamma^{\pm(3)}(Z_1; r, \zeta_3) &= G\left(Z_1, r + \frac{r\zeta_3}{r - Z_1}, \zeta_3 \pm i\varepsilon\right) \quad r > 0,\ \zeta_3 > 0 \\ \gamma^{\pm(3)}(Z_1; \alpha, \beta, \zeta_3) &= G\left(Z_1, \beta\zeta_3 - \frac{\beta}{\alpha} Z_1, \zeta_3 \pm i\varepsilon\right) \quad \alpha + \beta = 1, \alpha > 0, \beta > 0, \zeta_3 > 0\end{aligned} \tag{1}$$

(cf. Sections 3A and 3B of paper I).

The functions $\Gamma^{\pm(3)}$ and $\gamma^{\pm(3)}$ are equal to the vertex function G evaluated for particular values of the three scalar arguments.

The third argument of the vertex function is in all cases above chosen to be close to positive real values, in particular just above respectively just below the corresponding physical region-cut (the infinitesimal quantity ε will be chosen to be positive).

The second argument is further chosen in such a way, that we actually consider the vertex function just inside the boundaries of the axiomatic analyticity domain D_{KW} derived by Källén and Wightman.

In paper I it was pointed out that the choice in Eq. (1) is such that the vertex function is, e.g., evaluated just "above" (inside D_{KW}) the intersection between the F'_{12}-surfaces respectively the $\mathscr{F}$-surface of Källén and Wightman and the Z_3-cut in case we consider the function $\Gamma^{+(3)}$ in the lower respectively upper half complex ζ_1-plane. For the function $\Gamma^{-(3)}$ the situation is the same but with the roles of the upper and lower half complex ζ_1-planes reversed. Finally the functions $\gamma^{\pm(3)}$ describe in a similar way the vertex function close to the intersection between the same physical region-cut and the "axiomatic" singularity surfaces called F'_{13} and F'_{23} by Källén and Wightman. In all cases the limit $\varepsilon\to 0$ corresponds to the values of G just on the boundary. The above-described analyticity properties of the functions $\Gamma^{\pm(3)}$ and $\gamma^{\pm(3)}$ can because of Cauchy's theorem be employed to carry through different contour-integrals. Thus the following formulas are e.g. valid for the functions $\Gamma^{\pm(3)}$ (cf. Eq. (11) of paper I):

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\zeta_1}{\zeta_1-Z_1}\,d\zeta_2\,\delta\left(\zeta_2-r-\frac{r\zeta_3}{r-\zeta_1}\right)G(\zeta_1+i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)&\\ =\Gamma^{\pm(3)}(Z_1,r,\zeta_3)\quad \operatorname{Im}Z_1>0&\\ =0\qquad\qquad\qquad \operatorname{Im}Z_1<0\,.&\end{aligned}\tag{2}$$

We have in connection with Eq. (2) assumed that the limiting relations

$$L_n^{\pm}=\lim_{R\to\infty}\int_0^{\pi}\frac{d\Theta}{R^n}\,G\left(Re^{i\Theta},r-\frac{r\zeta_3}{R}e^{-i\Theta},\zeta_3\pm i\varepsilon\right)=0\tag{3}$$

are fulfilled for $n=0$. This is, however, not a severe restriction. According to the construction in connection with Eqs. (18) and (19) in paper I similar formulas, subtracted dispersion relations, can be derived as long as the limit relations in Eq. (3) are fulfilled for a finite integer n. The corresponding formulas do not contribute much more than a certain notational complexity to the situation and we will here for simplicity only discuss the case when Eq. (3) is fulfilled for $n=0$. A more general case is discussed in Section 6.

If the parameter r in Eq. (2) is varied inside the "allowed region" from $r=0$ to $r=\infty$ Eq. (2) represents the vertex function along the curve marked I in Fig. 1. This is the relevant boundary of the domain D_{KW} for the case $\operatorname{Im}Z_1>0$, $\operatorname{Im}Z_2>0$. The argument Z_2 is then defined by the relation (cf. Eq. (1)):

$$(Z_1-r)(Z_2-r)+r\zeta_3=0\,;\qquad r>0\tag{4}$$

which is, actually, the analytic expression for the curve I in Figs. 1 and 2. With a similar limit-assumption as in Eq. (3), the following formulas are valid for the functions $\gamma^{\pm(3)}$ (cf. Eq. (26) of paper I):

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\zeta_1}{\zeta_1-Z_1}d\zeta_2\,\delta\left(\zeta_2-\beta\zeta_3+\frac{\beta}{\alpha}\zeta_1\right)G(\zeta_1+i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)&\\ =\gamma^{\pm(3)}(Z_1;\alpha,\beta,\zeta_3)\qquad \operatorname{Im}Z_1>0&\\ =0\qquad\qquad\qquad\qquad \operatorname{Im}Z_1<0\,.&\end{aligned}\tag{5}$$

If the parameters α and β are varied (fulfilling the conditions $\alpha+\beta=1$, $\alpha>0$, $\beta>0$ (cf. Eq. (1))), Eq. (5) describes the vertex function along the curve marked II in Fig. 1, i.e. the corresponding relevant boundary of D_{KW} for the case $\operatorname{Im}Z_1>0$, $\operatorname{Im}Z_2<0$. The argument Z_2 is then defined by

$$\alpha\beta\zeta_3-\alpha Z_2-\beta Z_1=0;\quad \alpha>0,\ \beta>0\quad \alpha+\beta=1\,.\tag{6}$$

Eq. (6) is the analytic expression for the curve II in Figs. 1 and 2. Eqs. (2) and (5) give a complete description of the vertex function along the relevant boundaries of D_{KW} for the case $\operatorname{Im}Z_1>0$, i.e. for the situation depicted in Fig. 1.

We note that the integrals in Eqs. (2) and (5) on the other hand *vanish* in case $\operatorname{Im}Z_1<0$, because there is in that case no pole-contribution from the interior of the integration curve.

Completely equivalent formulas can be developed for the case $\operatorname{Im}Z_1<0$, i.e. for the case depicted in Fig. 2.

The formulas corresponding to Eq. (2), i.e. to values of the arguments along the curve I in Fig. 2, are (cf. Eq. (12) of paper I):

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\zeta_1}{\zeta_1-Z_1}d\zeta_2\,\delta\left(\zeta_2-r-\frac{r\zeta_3}{r-\zeta_1}\right)G(\zeta_1-i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)\\ &=-\Gamma^{\pm}(Z_1,r,\zeta_3)=-G(Z_1,Z_2,\zeta_3\pm i\varepsilon)\ \operatorname{Im}Z_1<0;Z_2=r+\frac{r\zeta_3}{r-Z_1}\\ &=0\qquad\qquad\qquad\qquad\qquad \operatorname{Im}Z_1>0\,.\end{aligned}\tag{7}$$

The formulas for the vertex function along the curve II in Fig. 2 are in that case (cf. Eq. (5) of this paper and Eq. (26) of paper I):

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\zeta_1}{\zeta_1-Z_1}d\zeta_2\,\delta\left(\zeta_2-\beta\zeta_3+\frac{\beta}{\alpha}\zeta_1\right)G(\zeta_1-i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)\\ &=-\gamma^{\pm}(Z_1,\alpha,\beta,\zeta_3)=-G(Z_1,Z_2,\zeta_3\pm i\varepsilon),\ \operatorname{Im}Z_1<0,Z_2=\beta\zeta_3-\frac{\beta}{\alpha}Z_1\\ &=0\qquad\qquad\qquad\qquad\qquad \operatorname{Im}Z_1>0\,.\end{aligned}\tag{8}$$

We note the appearance of the minus signs in Eqs. (7) and (8), which stem from the fact that all complex integrals must be performed in the "positive sense". The weight functions occurring in the formulas above are all of them physical in the sense that they can be expressed in terms of the causal and time-ordered boundary values of the vertex functions. This is proved in some detail in Section 4 of paper I, where the well-known connections between these boundary values and reduction formalism is briefly touched upon.

4. The Vertex Function for Arbitrary Values of the Arguments Inside the Källén-Wightman Domain

The second step in the derivation of the dispersion relations is to represent the vertex function for arbitrary values of the complex variables Z_1 and Z_2 inside the domain D_{KW} by means of its values along the boundary surfaces, i.e. by means of the functions $\Gamma^{\pm(3)}$ and $\gamma^{\pm(3)}$ of Section 3.

We will start by fixing the variable Z_1 in an arbitrary position in the upper half plane. Such a situation is described in the complex plane of Z_2 in Fig. 1. We note that by varying the parameter α from the value $\alpha=0$ ($\beta=1$) to the value $\alpha=1$ ($\beta=0$) the curve II is traversed from infinite values of Z_2 up to the origin. This defines a direction which coincides with the positive orientation according to the arrows in Fig. 1, i.e. the requirements of property (iii) in Section 2. Similarly the curve I is traversed in the corresponding positive direction by varying the parameter r from the value $r=0$ to the value $r=\infty$. We may now introduce the Cauchy-contour $C(R,Z_1)$ described in Section 2 with the properties (i)–(iii). With a self-explanatory notation we write for an arbitrary value of Z_2 inside the curve $C(R,Z_1)$ when $\operatorname{Im} Z_1>0$:

$$G^+ = G_R^{+(\mathrm{I})} + G_R^{+(\mathrm{II})} + G_R^{+(\mathrm{III})}\,. \tag{9}$$

We will here assume that the contribution from the circle III can be neglected in the limit when the corresponding radius R tends to infinity (cf. property (ii)):

$$G_\infty^{+(\mathrm{III})}=0\,. \tag{10}$$

In Section 6 it is shown how a more general situation can be incorporated inside the formalism.

Then we may write in the limit $R \to \infty$

$$G_\infty^{+\mathrm{I}} = \frac{1}{2\pi i} \int_0^\infty dr \left(1 - \frac{\zeta_3 Z_1}{(r-Z_1)^2}\right) \left(\frac{1}{r + \dfrac{r\zeta_3}{r-Z_1} - Z_2}\right) \Gamma^{\pm(3)}(Z_1; r, \zeta_3)$$

$$= \frac{1}{(2\pi i)^2} \iint d\zeta_1\, d\zeta_2\, K^{\mathrm{I}}(\zeta_1,\zeta_2,\zeta_3; Z_1, Z_2)\, G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 \pm i\varepsilon), \quad (11)$$

$$G_\infty^{+\mathrm{II}} = \frac{1}{2\pi i} \int_0^1 d\alpha \left\{-\zeta_3 + \frac{Z_1}{\alpha^2}\right\} \int_0^1 d\beta \frac{\delta(1-\alpha-\beta)}{\beta\zeta_3 - \dfrac{\beta}{\alpha} Z_1 - Z_2} \gamma^{\pm(3)}(Z_1; \alpha, \beta, \zeta_3)$$

$$= \frac{1}{(2\pi i)^2} \iint d\zeta_1\, d\zeta_2\, K^{\mathrm{II}}(\zeta_1,\zeta_2;\zeta_3; Z_1, Z_2)\, G(\zeta_1 + i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 \pm i\varepsilon). \quad (12)$$

We have here in order to simplify the notation introduced the kernel-functions K^{I} and K^{II} defined by:

$$K^{\mathrm{I}} = \int_0^\infty dr\, \delta((r-\zeta_1)(r-\zeta_2) + r\zeta_3) \frac{(r-Z_1) - \dfrac{\zeta_3 Z_1}{r-Z_1}}{((r-Z_1)(r-Z_2) + r\zeta_3)}$$

$$\cdot \left[\frac{(\zeta_1 - r)\,\Theta(\zeta_1 - r)}{\zeta_1 - Z_1} + \frac{(r-\zeta_1)\,\Theta(r-\zeta_1)}{\zeta_1 - Z_1}\right], \quad (13)$$

$$K^{\mathrm{II}} = \iint_0^1 d\alpha\, d\beta\, \delta(1-\alpha-\beta)\, \delta(\alpha\beta\zeta_3 - \alpha\zeta_2 - \beta\zeta_1)$$

$$\cdot \frac{Z_1 - \alpha^2\zeta_3}{\alpha\beta\zeta_3 - \alpha Z_2 - \beta Z_1} \frac{1}{\zeta_1 - Z_1}. \quad (14)$$

In the second lines of Eqs. (11) and (12) we have introduced the representationformulas in Eqs. (2) and (5) for the functions $\Gamma^{\pm(3)}$ and $\gamma^{\pm(3)}$.

We note that the right-hand side of Eq. (9) defined by Eqs. (10), (11) and (12) actually *vanishes* unless $\mathrm{Im}(Z_1) > 0$ and the couple (Z_1, Z_2) is chosen inside the corresponding Källén-Wightman domain D_{KW} (i.e. the unshaded region of Fig. 1). The reason is that unless both of these conditions are fulfilled there is *no* polecontribution from the interior of the corresponding "contour-surfaces" (cf. the second lines of Eqs. (2) and (5)):

Consequently we may write

$$G^+(Z_1, Z_2; \zeta_3 \pm i\varepsilon) = G(Z_1, Z_2; \zeta_3 \pm i\varepsilon) \quad \text{if} \quad \mathrm{Im}\, Z_1 > 0 \quad \text{and} \quad (Z_1, Z_2) \in D_{KW}$$
$$= 0 \quad \text{in other cases}. \quad (15)$$

Very similar formulas can be developed for the case when $\operatorname{Im} Z_1 < 0$. A typical such situation is described by Fig. 2. We note, however, that in order to keep the positive orientation for the corresponding Cauchy-contour $C(R, Z_1)$ the curves I and II must be traversed in the *opposite* direction compared to the situation depicted in Fig. 1. The formulas corresponding to Eqs. (9), (11) and (12) are then for the case $\operatorname{Im} Z_1 < 0$

$$G^{(-)} = G_\infty^{-(\mathrm{I})} + G_\infty^{-(\mathrm{II})} \tag{16}$$

$$\begin{aligned} G_\infty^{-(\mathrm{I})} &= -\frac{1}{2\pi i} \int_0^\infty dr \left(1 - \frac{\zeta_3 Z_1}{(r-Z_1)^2}\right) \frac{1}{\left(r + \dfrac{r\zeta_3}{r-Z_1} - Z_2\right)} \Gamma^{\pm(3)}(Z_1; r, \zeta_3) \\ &= \frac{1}{(2\pi i)^2} \iint d\zeta_1\, d\zeta_2\, K^{\mathrm{I}}(\zeta_1, \zeta_2; \zeta_3; Z_1, Z_2)\, G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 \pm i\varepsilon) \end{aligned} \tag{17}$$

$$\begin{aligned} G_\infty^{-(\mathrm{II})} &= -\frac{1}{2\pi i} \int_0^1 d\alpha \left\{-\zeta_3 + \frac{Z_1}{\alpha^2}\right\} \int_0^1 d\beta \frac{\delta(1-\alpha-\beta)}{\beta\zeta_3 - \dfrac{\beta}{\alpha} Z_1 - Z_2} \gamma^{\pm(3)}(Z_1; \alpha, \beta, \zeta_3) \\ &= \frac{1}{(2\pi i)^2} \iint d\zeta_1\, d\zeta_2\, K^{\mathrm{II}}(\zeta_1, \zeta_2; \zeta_3; Z_1, Z_2)\, G(\zeta_1 - i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 \pm i\varepsilon). \end{aligned} \tag{18}$$

We have also in this case assumed that the contribution from the curve (III) can be neglected in the limit when the radius R tends to infinity (cf. Eq. (10))

$$G_\infty^{-(\mathrm{III})} = 0\,. \tag{19}$$

We note the appearance of the minus signs in the first lines of Eqs. (17) and (18) as compared to Eqs. (11) resp. (12). These minus signs are due to the fact mentioned above that the integration curves in order to keep the positive orientation have to be traversed in the opposite direction compared to the earlier case. Due to the minus signs in Eqs. (7) and (8) there is, however, a compensation in the second line of Eqs. (17) and (18). In that way the only difference between Eqs. (17) and (18) respectively Eqs. (11) and (12) is the appearance of boundary values of the vertex function G with different signs of the limiting imaginary parts. That is, however, enough in order for a relation similar to Eq. (15) to be valid also for the function $G^{(-)}$ defined by Eqs. (16) and (19):

$$\begin{aligned} G^{(-)}(Z_1, Z_2, \zeta_3 \pm i\varepsilon) &= G(Z_1, Z_2, \zeta_3 \pm i\varepsilon) \quad \text{if} \quad \operatorname{Im} Z_1 < 0;\ (Z_1, Z_2) \in D_{KW} \\ &= 0 \quad \text{in other cases}\,. \end{aligned} \tag{20}$$

From Eqs. (16) and (20) we finally deduce that we may write for the vertex function G for all (non-real) values of the arguments Z_1 and Z_2 inside the domain D_{KW}, i.e. the axiomatic analyticity domain of the vertex function, when the third argument is close to real positive values:

$$\begin{aligned}
G(Z_1, Z_2, \zeta_3 \pm i\varepsilon) &= G^{(+)}(Z_1, Z_2, \zeta_3 \pm i\varepsilon) + G^{(-)}(Z_1, Z_2, \zeta_3 \pm i\varepsilon) \\
&= \left(\frac{1}{2\pi i}\right)^2 \iint d\zeta_1\, d\zeta_2 [K^{\mathrm{I}}(\zeta_1, \zeta_2; \zeta_3; Z_1, Z_2)\,(G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 \pm i\varepsilon) \\
&\quad + G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 \pm i\varepsilon)) + K^{\mathrm{II}}(\zeta_1, \zeta_2; \zeta_3; Z_1, Z_2) \\
&\quad \cdot (G(\zeta_1 + i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 \pm i\varepsilon) + G(\zeta_1 - i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_2 \pm i\varepsilon))]\,.
\end{aligned} \tag{21}$$

5. The Symmetries and the Singularities of the Dispersion Relation

In Eq. (21) we have exhibited the vertex function by means of integrals containing on the one hand combinations of (physical) boundary values of the vertex function and on the other hand the two kernel functions, K^{I} and K^{II} which are defined in Eq. (13) respectively Eq. (14).

The method of derivation has, however, introduced a certain asymmetry between the indices 1 and 2, i.e. between the occurrence of the arguments Z_1 and Z_2, respectively the integration variables ζ_1 and ζ_2. This asymmetry, which appears in the expressions for the kernel functions K^{I} and K^{II}, but not in connection with the occurring boundary values of the vertex function, is of course only fictitious.

There is a further anomaly in the expressions for K^{I} and K^{II} in the fact that the "Cauchy denominator" $(\zeta_1 - Z_1)^{-1}$ can be seen to occur also for negative values of the integration variable ζ_1. Consequently the integrals of Eq. (21) may exhibit singularities also for *negative* values of the variable Z_1 irrespective of the values of the variable Z_2.

In this section we will show that the kernel functions K^{I} and K^{II} can be rewritten as the sums of several terms. One of these terms for each of the kernel functions exhibits explicitly both the expected symmetry between the indices 1 and 2 as well as the "correct" analyticity properties. The remaining terms will be proved to give vanishing contributions to the representation formula in Eq. (21).

We will investigate this problem in some detail for the kernel function K^{I}. To that end we note the following equalities which stem from a partial-fraction-expansion in the variable Z_1:

$$\begin{aligned}
&\frac{r-\zeta_1}{\zeta_1-Z_1}\,\frac{r-Z_1-\dfrac{\zeta_3 Z_1}{r-Z_1}}{(r-Z_1)(r-Z_2)+r\zeta_3}=\frac{1}{r-Z_1}\\
&+\frac{(r-\zeta_1)\left[r-\zeta_1-\dfrac{\zeta_1\zeta_3}{r-\zeta_1}\right]}{(r-\zeta_1)(r-Z_2)+r\zeta_3}\,\frac{1}{\zeta_1-Z_1}\\
&+\frac{Z_2-r}{\zeta_2-Z_2}\,\frac{r-Z_2-\dfrac{\zeta_3 Z_2}{r-Z_2}}{(r-Z_1)(r-Z_2)+r\zeta_3}=\frac{r-Z_2-\dfrac{\zeta_3 Z_2}{r-Z_2}}{(r-Z_1)(r-Z_2)+r\zeta_3}\left(\frac{\zeta_2-r}{\zeta_2-Z_2}-1\right)\\
&+\frac{2r+\zeta_3-\zeta_1-\zeta_2}{(\zeta_1-Z_1)(\zeta_2-Z_2)}+\frac{1}{r-Z_1}\,.
\end{aligned}\tag{22}$$

To go from the second to the third line in Eq. (22) we have repeatedly made use of the relation between the integration variables ζ_1 and ζ_2 coming from the δ-function in the integrand of Eq. (13):

$$r\zeta_3+(r-\zeta_1)(r-\zeta_2)=0\,.\tag{23}$$

We further note the equality:

$$\frac{r-Z_1-\dfrac{Z_1\zeta_3}{r-Z_1}}{(r-Z_1)(r-Z_2)+r\zeta_3}=\frac{2r+\zeta_3-Z_1-Z_2}{(r-Z_1)(r-Z_2)+r\zeta_3}-\frac{1}{r-Z_1}\tag{24}$$

as well as a similar one with index 1 exchanged to index 2.

By a combined use of equalities like the ones occurring in Eqs. (22) and (24) we can write the integrand in the defining Eq. (13) for the kernel function K^{I} as a sum of three terms, S^{I}, R_1^{I} and R_2^{I}:

$$\begin{aligned}
&\frac{r-Z_1-\dfrac{Z_1\zeta_3}{r-Z_1}}{(r-Z_1)(r-Z_2)+r\zeta_3}\left[\frac{\zeta_1-r}{\zeta_1-Z_1}\,\Theta(\zeta_1-r)\,\Theta(r-\zeta_2)\right.\\
&\qquad\qquad\left.+\frac{r-\zeta_1}{\zeta_1-Z_1}\,\Theta(r-\zeta_1)\,\Theta(\zeta_2-r)\right]\\
&=S^{\mathrm{I}}+R_1^{\mathrm{I}}+R_2^{\mathrm{I}}\,.
\end{aligned}\tag{25}$$

The quantity S^{I} is then defined by

$$\begin{aligned}
S^{\mathrm{I}}=&\frac{\Theta(\zeta_1-r)\,\Theta(r-\zeta_2)}{\zeta_1-Z_1}\left[1+\tfrac{1}{2}(\zeta_1+Z_1-2r)\frac{2r+\zeta_3-Z_1-Z_2}{(r-Z_1)(r-Z_2)+r\zeta_3}\right]\\
&+\frac{\Theta(r-\zeta_1)\,\Theta(\zeta_2-r)}{\zeta_2-Z_2}\left[1+\tfrac{1}{2}(\zeta_2+Z_2-2r)\frac{2r+\zeta_3-Z_1-Z_2}{(r-Z_1)(r-Z_2)+r\zeta_3}\right].
\end{aligned}\tag{26}$$

We note that the term S^{I} is obviously symmetric under the permutation of indices 1 and 2. The term S^{I} further, because of the occurring step functions, exhibits possible singularities only for non-negative real values of the arguments Z_1 and Z_2 and for values of the arguments on the boundary curve I of Eq. (4).

The remaining two terms R_j^{I}, $j=1,2$ in Eq. (25) are defined by:

$$R_1^{\mathrm{I}} = \left(\tfrac{1}{2}\,\frac{2r+\zeta_3-Z_1-Z_2}{(r-Z_1)(r-Z_2)+r\zeta_3} - \frac{1}{r-Z_1}\right) \cdot [\Theta(\zeta_1-r)\,\Theta(r-\zeta_2) - \Theta(r-\zeta_1)\,\Theta(\zeta_2-r)]\,, \tag{27}$$

$$R_2^{\mathrm{I}} = \frac{2r+\zeta_3-\zeta_1-\zeta_2}{(\zeta_1-Z_1)(\zeta_2-Z_2)}\,\Theta(\zeta_2-r)\,\Theta(r-\zeta_1)\,. \tag{28}$$

We will now show that the integral in Eq. (21) actually does not get any non-vanishing contribution from $R_j^{\mathrm{I}}, j=1,2$. We start by investigating R_1^{I}. We note that the quantity R_1^{I} only depends upon the integration variables ζ_1 and ζ_2 through the step functions in the last parenthesis of Eq. (27). The contribution from R_1^{I} to the integral in Eq. (21) will be called J_1^{I} and we get:

$$\begin{aligned} J_1^{\mathrm{I}} = {} & \frac{1}{(2\pi i)^2}\int_0^\infty dr\left[\tfrac{1}{2}\,\frac{2r+\zeta_3-Z_1-Z_2}{(r-Z_1)(r-Z_2)+r\zeta_3} - \frac{1}{r-Z_1}\right] \\ & \cdot \iint d\zeta_1\,d\zeta_2\,\delta((r-\zeta_1)(r-\zeta_2)+r\zeta_3)\,\{G(\zeta_1+i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon) \\ & + G(\zeta_1-i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)\,[\Theta(\zeta_1-r)\,\Theta(r-\zeta_2)-\Theta(r-\zeta_1)\,\Theta(\zeta_2-r)]\,. \end{aligned} \tag{29}$$

In a few earlier papers [4–6] we have investigated the consequencies of certain boundedness properties for the vertex function combined with the Källén-Wightman analyticity properties. The results can be formulated in terms of relations of the kind usually called sum rules. Comparing Eq. (29) to the sum rules of Eqs. (14) and (15) of Ref. [4] (cf. Eq. (A 1) of appendix A in Ref. [5]) as well as to the sum rules of Eqs. (26) and (27) of Ref. [6] (cf. the remarks made in connection with Eq. (16) of I) we note that the sum rules imply the vanishing of the integral J_1^{I} under the same boundedness conditions that make Eq. (21) into a well-defined expression.

The contribution from the quantity R_2^{I} to the same integral vanishes for different reasons and partly due to a compensating term from the kernel function K^{II}. We note that the integration domain for ζ_1 and ζ_2 in connection with R_2^{I} in Eq. (28) can be described as the intersection between the surface in Eq. (23) (because of the δ-function) and the two (non-overlapping) domains D_2 and T_1:

$$\begin{aligned} &D_2: \zeta_3 > 0,\ \zeta_1 > 0,\ \zeta_2 \geqq (\sqrt{\zeta_3} + \sqrt{\zeta_1})^2 \\ &\mathrm{T}_1: \zeta_3 > 0,\ \zeta_1 < 0,\ \zeta_2 > 0\,. \end{aligned} \tag{30}$$

We may then replace the step functions $\Theta(\zeta_2 - r)\,\Theta(r - \zeta_1)$ in the expression for R_2^{I} in Eq. (28) by the sum of the characteristic functions for the domains D_2 and T_1 which we will call $\chi(D_2)$ respectively $\chi(T_1)$.

The integration variable r occurs in connection with R_2^{I} in the δ-function with the argument of Eq. (23) and in the combination $(2r + \zeta_3 - \zeta_1 - \zeta_2)$.

We then get as contribution to the integral for K^{I} in Eq. (13) from the integrand R_2^{I}:

$$\begin{aligned} \int_0^\infty dr\, \delta((r-\zeta_1)(r-\zeta_2) + r\zeta_3) R_2^{\mathrm{I}} = &\int_0^\infty dr [\chi(D_2) + \chi(T_1)] \\ &\cdot \frac{2r + \zeta_3 - \zeta_1 - \zeta_2}{(\zeta_1 - Z_1)(\zeta_2 - Z_2)} \delta((r-\zeta_1)(r-\zeta_2) + r\zeta_3) \\ = &\frac{\chi(T_1)}{(\zeta_1 - Z_1)(\zeta_2 - Z_2)}\,. \end{aligned} \tag{31}$$

To reach this result we note that the two roots of Eq. (23) with respect to the variable r, $r_\pm$, which may contribute according to the δ-function, fulfil

$$\begin{aligned} 2r_\pm + \zeta_3 - \zeta_1 - \zeta_2 &= \pm\sqrt{\lambda(\zeta)} \\ \lambda(\zeta) = \zeta_1^2 + \zeta_2^2 + \zeta_3^2 &- 2\zeta_1\zeta_2 - 2\zeta_1\zeta_3 - 2\zeta_2\zeta_3\,. \end{aligned} \tag{32}$$

In the integration range D_2 both roots contribute because in D_2

$$r_+ > 0, \quad r_- > 0 \tag{33}$$

but in the integration range T_1 only the root r_+ can contribute because in T_1

$$r_+ > 0, \quad r_- < 0\,. \tag{34}$$

Due to the well-known fact that the boundary values of the vertex functions are the same above and below the *negative* real axis of the arguments we conclude that in the integration range T_1, where the variable ζ_1 is negative according to Eq. (30), the following equality is valid:

$$\begin{aligned}&\chi(T_1)\{G(\zeta_1+i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)+G(\zeta_1-i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)\}\\ &=\chi(T_1)\{G(\zeta_1-i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)+G(\zeta_1+i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)\}\,.\end{aligned} \tag{35}$$

The combination of boundary values in the last line of Eq. (35) is, however, the same combination that occurs together with the kernel function K^{II} in Eq. (21).

An investigation of the kernel function K^{II} along the same lines as we have done for the kernel function K^{I} above will reveal a contribution to the integration range T_1 which apart from a *negative* sign is equal to the one in Eq. (31).

We may, consequently, deduce from this fact and from the equality in Eq. (35) that the sum of these contributions vanishes. Consequently, the only non-vanishing contribution to the kernel function K^{I} comes from the term S^{I} in Eq. (26).

The corresponding "symmetric" non vanishing contribution to the kernel function K^{II} is S^{II}:

$$\begin{aligned}&S^{\mathrm{II}}(\zeta_1,\zeta_2;\zeta_3;\alpha,\beta;Z_1,Z_2)\\ &=\frac{[\Theta(\zeta_1)\,\Theta(-\zeta_2)+\frac{1}{2}\,\Theta(\zeta_1)\,\Theta(\zeta_2)]}{\zeta_1-Z_1}\;\frac{Z_1-\alpha^2\zeta_3}{\alpha\beta\zeta_3-\alpha Z_2-\beta Z_1}\\ &+\frac{[\Theta(\zeta_2)\,\Theta(-\zeta_1)+\frac{1}{2}\,\Theta(\zeta_1)\,\Theta(\zeta_2)]}{\zeta_2-Z_2}\;\frac{Z_2-\beta^2\zeta_3}{\alpha\beta\zeta_3-\alpha Z_2-\beta Z_1}\,.\end{aligned} \tag{36}$$

We note that according to the step functions in Eq. (36) the function S^{II} may exhibit singularities only for values of the arguments Z_1 and Z_2 along the positive real axes as well as along the boundary curve II of Eq. (6). The resulting representation formula for the vertex function G is then finally:

$$\begin{aligned}G(Z_1,Z_2;\zeta_3\pm i\varepsilon)=&\frac{1}{(2\pi i)^2}\iint d\zeta_1\,d\zeta_2\left\{\int_0^\infty dr\,\delta(r\zeta_3+(r-\zeta_1)(r-\zeta_2))\right.\\ &\cdot S^{\mathrm{I}}(\zeta_1,\zeta_2;\zeta_3;r;Z_1,Z_2)\,[G(\zeta_1+i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)\\ &+G(\zeta_1-i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)]\\ &+\iint_0^1 d\alpha\,d\beta\,\delta(1-\alpha-\beta)\,\delta(\alpha\beta\zeta_3-\alpha\zeta_2-\beta\zeta_1)\,S^{\mathrm{II}}(\zeta_1,\zeta_2;\zeta_3;\alpha,\beta;Z_1,Z_2)\\ &\left.\cdot\,[G(\zeta_1+i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3\pm i\varepsilon)+G(\zeta_1-i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3\pm i\varepsilon)]\right\}.\end{aligned} \tag{37}$$

6. Subtracted Dispersion Relations

We have implicitly assumed certain asymptotic boundedness properties of the vertex function in connection with all the formulas derived so far. Thus the vertex function must vanish "sufficiently fast" in asymptotic directions (*inside* the domain D_{KW}, however) in order that e.g. Eq. (3) should be fulfilled with the power $n=0$. Similar requirements are also implied by our neglect of the contributions from the integrals along the curve III to Eqs. (9) and (16) in the limit when the circle radius $R\to\infty$ (cf. Eqs. (10) and (19)).

We will now show that the formalism can with minor changes be extended to cover more general asymptotic behaviour of the vertex function G.

To that end we consider the set of related functions g_{n_1+1,n_2+1} defined for all positive integers n_1 and n_2 by

$$\begin{aligned}
&g_{n_1+1,n_2+1}(Z_1,Z_2,\zeta_3\pm i\varepsilon;\xi_1,\xi_2)\\
&\quad=\left(\frac{\partial}{\partial\xi_1}\right)^{n_1}\left(\frac{\partial}{\partial\xi_2}\right)^{n_2} g_{11}(Z_1,Z_2,\zeta_3\pm i\varepsilon;\xi_1,\xi_2)\\
&g_{11}(Z_1,Z_2,\zeta_3\pm i\varepsilon;\xi_1,\xi_2)=(G(Z_1,Z_2,\zeta_3\pm i\varepsilon)\\
&\quad-G(\xi_1,Z_2,\zeta_3\pm i\varepsilon)-G(Z_1,\xi_2,\zeta_3\pm i\varepsilon)\\
&\quad+G(\xi_1,\xi_2,\zeta_3\pm i\varepsilon))(Z_1-\xi_1)^{-1}(Z_2-\xi_2)^{-1}\,.
\end{aligned}\tag{38}$$

It is convenient to extend the definitions in Eq. (38) to all non-negative integers by

$$\begin{aligned}
&g_{00}(Z_1,Z_2,\zeta_3\pm i\varepsilon;\xi_1,\xi_2)=G(Z_1,Z_2,\zeta_3\pm i\varepsilon)\\
&g_{01}(Z_1,Z_2,\zeta_3\pm i\varepsilon;\xi_1,\xi_2)=(G(Z_1,Z_2,\zeta_3\pm i\varepsilon)-G(Z_1,\xi_2,\zeta_3\pm i\varepsilon))\\
&\qquad\qquad\times(Z_2-\xi_2)^{-1}\\
&g_{10}(Z_1,Z_2,\zeta_3\pm i\varepsilon;\xi_1,\xi_2)=(G(Z_1,Z_2,\zeta_3\pm i\varepsilon)-G(\xi_1,Z_2,\zeta_3\pm i\varepsilon))\\
&\qquad\qquad\times(Z_1-\xi_1)^{-1}\,.
\end{aligned}\tag{39}$$

We will consider the functions g_{n_1,n_2} only for values of the "subtraction points" ξ_1 and ξ_2 on the *negative* real axis. The reason for this choice is that when one of the arguments (in our case the third argument) is fixed close to real positive values and one of the arguments is fixed and negative, then the vertex function, considered as a function of the remaining third scalar variable, is analytic in the whole complex plane, cut along the positive real axis. This is a consequence of the analyticity properties proved by Källén and Wightman [2].

The following two properties of the function g_{n_1+1,n_2+1} are then immediate results from the defining Eq. (38) and the remark made above:

1. If the parameters (ξ_1, ξ_2) are arbitrary negative numbers then for all positive integers n_1 and n_2 the functions g_{n_1,n_2} are analytic in the variable Z_1 and Z_2 in the same domain as the vertex function G, i.e. a kind of "generalized vertex functions".

2. In the case when the vertex function G is at most polynomially increasing in asymptotic directions inside the analyticity domain, then by a suitable choice of (i.e. for sufficiently large) integers n_1 and n_2 the related function g_{n_1,n_2} can be made to vanish in the same directions.

Consequently, we deduce that even when the dispersion relations for G do not exist due to lack of convergence of the integrals, the representation formulas may have a meaning for one of the related functions g_{n_1,n_2} with the integers n_1 or (and) n_2 larger than zero. We will from now on assume that this is the case and that we may represent the function g_{n_1,n_2} by an application of the formula in Eq. (37). We further note the following recursion relations, that can by straight-forward algebra be proved to be valid for all non-negative integers n_1 and n_2:

$$\begin{aligned} g_{n_1,n_2}(Z_1, Z_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) &= g_{n_1,n_2}(\xi_1, Z_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) \\ &+ g_{n_1,n_2}(Z_1, \xi_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) - g_{n_1,n_2}(\xi_1, \xi_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) \\ &+ (Z_1 - \xi_1)(Z_2 - \xi_2) g_{n_1+1,n_2+1}(Z_1, Z_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) \end{aligned} \tag{40}$$

(we note the similarities between Eq. (40) and Eq. (38)).

Due to the remark made after Eq. (39) and property 1 above, we conclude that e.g. the function $g_{n_1,n_2}(\xi_1, Z_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) \equiv \hat{g}_{n_1,n_2}(Z_2)$ is analytic in the whole complex plane of the variable Z_2, cut along the positive real axis, if the parameters ξ_1 and ξ_2 are negative. This property can be used to represent the function $\hat{g}_{n_1,n_2}$ by means of a complex contour-integral in a similar way as we have done repeatedly in Sections 2–5. Neglecting for the moment all questions of convergence we may write:

$$\hat{g}_{n_1,n_2}(Z_2) = \frac{1}{2\pi i} \int \frac{d\zeta_2}{\zeta_2 - Z_2} \left(\hat{g}_{n_1,n_2}(\zeta_2 + i\varepsilon) - \hat{g}_{n_1,n_2}(\zeta_2 - i\varepsilon)\right). \tag{41}$$

The occurring weight function in the integral of Eq. (41) has actually a simple interpretation in terms of the vertex function G. To see that in some detail we will start to consider the case $n_1 = n_2 = 1$, and from that case the general situation becomes evident.

The discontinuity of the function $\hat{g}_{11}(Z_2)$ across the positive real axis of the variable Z_2 is easily seen to be

$$\begin{aligned} &g_{11}(\xi_1, \zeta_2 + i\varepsilon, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) - g_{11}(\xi_1, \zeta_2 - i\varepsilon, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2) \\ &= \frac{1}{\zeta_2 - \xi_2} \frac{\partial}{\partial \xi_1} (G(\xi_1, \zeta_2 + i\varepsilon, \zeta_3 \pm i\varepsilon) - G(\xi_1, \zeta_2 - i\varepsilon, \zeta_3 \pm i\varepsilon)). \end{aligned} \tag{42}$$

The discontinuity is consequently, except for the (non-singular) denominator, a derivative of the discontinuity of the vertex function itself across the same physical-region-cut. In this case two of the three arguments of the vertex function take on positive values in the limit, while the third one is negative. According to the considerations in Section 4 of paper I the indicated boundary values are physical in the sense that they can be expressed in terms of the causal respectively the time-ordered functions. The discontinuity in Eq. (42) has actually a simple expression in terms of the matrix elements of the underlying field theory. The result for the general case with arbitrary integers n_1 and n_2 can be immediately derived from Eq. (42) by a suitable number of derivatives (cf. Eq. (38)).

In case Eq. (41) should need a "larger convergence power" in the integrands the corresponding procedure can instead be carried out for e.g. the function $\hat{g}_{n_1 N_2}$ with a sufficiently large integer N_2. In that case the following recursion relation, valid for all non-negative integers m_1 and m_2, is of interest:

$$\hat{g}_{m_1, m_2}(Z_2) = \hat{g}_{m_1, m_2}(\xi_2) + (Z_2 - \xi_2)\, \hat{g}_{m_1, m_2 + 1}(Z_2). \tag{43}$$

By means of Eq. (42) and if necessary a repeated use of Eq. (43) the function $g_{n_1, n_2}(\xi_1, Z_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2)$ in the recursion relation of Eq. (40) can be written as the sum of a polynomial in the variable Z_2 and a dispersion-integral only containing physical values of the vertex function.

The same procedure can be performed to express the function $g_{n_1, n_2}(Z_1, \xi_2, \zeta_3 \pm i\varepsilon; \xi_1, \xi_2)$ in Eq. (40) in terms of a corresponding polynomial and a similar dispersion relation.

It is now clear that by a repeated use of Eq. (40) and the procedure described above we may represent even a polynomially increasing vertex function G for arbitrary complex values of the arguments Z_1 and Z_2. The resulting expression is the sum of a polynomial in the variables Z_1 and Z_2, together with "one-dimensional dispersion relations" as in Eq. (41) and "two-dimensional dispersion relations" as in Eq. (37). This is the general representation formula for all vertex functions which are at most polynomially increasing in asymptotic directions.

7. Dispersion Relations with On-Mass-Shell Matrix Elements as Weight Functions

The representation formula of Eq. (37) does in principle exhibit all the analyticity properties of the vertex functions which are derived in the paper by Källén and Wightman.

Thus, the kernel functions in these relations contain possible singularities only for values of the arguments along the boundary surfaces called I and II of Eqs. (4) and (6) respectively, as well as along the positive real axes of the variables Z_1 and Z_2. We note, especially, that these latter singularities, which are generally known as "physical-region-cuts", do seemingly start at the origins, i.e. at $Z_j=0\ j=1,2$.

Such physical-region-singularities do, broadly speaking, correspond to contributions from the different physical states of the theory. In that way the statement that the physical-region-cuts start in the origins is a reflection of the generality in the Källén-Wightman spectrum condition, i.e. the sole requirement that the occurring masses and energies in the theory are non-negative numbers.

In a theory containing no zero-mass particles we would, however, expect that the corresponding physical-region singularities start with an isolated pole-singularity corresponding to the one-particle-state situated at the square of the particle mass. Only "further up", i.e. further along the positive real axis would branch-points for the physical-region cuts occur, corresponding to the thresholds from the different scattering states. There is, consequently, a "*mass gap*" in the general case. This property is, however, not obvious in the formula of Eq. (37) because the weight functions in the dispersion relations, i.e. the different boundary values of the vertex function G are in general non-vanishing also for values of the arguments that do not correspond to physical states.

In paper I we have shown however, that particular combinations of such boundary values have these physical and dynamical support properties.

In this section we will show how to express the boundary values required in connection with Eq. (37) in terms of the on-mass-shell matrix elements of the field operators which do explicitly exhibit the mass spectrum of the field theory. We will be satisfied, just as in paper I, to discuss a theory which contains three scalar fields A, B and C and we will not specify any dynamical relations between the fields.

More general cases as e.g. spinor-, vector- and higher tensor-fields, can be discussed in a similar way but with a corresponding increase in the complexity of the notation.

There are two particular on-mass-shell matrix elements of interest in connection with the investigations in paper I, i.e. the quantities

$G_B(a_1, a_2, a_3)$ and $G_A(a_1, a_2, a_3)$ defined by

$$\begin{aligned}
&G_B(-p_1^2, -(p_1+p_3)^2, -p_3^2) \\
&= (2\pi)^6 \sum_{|n\rangle, |m\rangle} \{\Theta(-p_1)\,\Theta(p_3)\,\delta(p_1+p_n)\,\delta(p_3-p_m) \\
&\quad \cdot \langle 0|A|n\rangle \langle n|B|m\rangle \langle m|C|0\rangle + \Theta(p_1)\,\Theta(-p_3)\,\delta(p_1-p_n)\,\delta(p_3+p_m) \\
&\quad \cdot \langle 0|C|m\rangle \langle m|B|n\rangle \langle n|A|0\rangle\}\,,
\end{aligned} \tag{44}$$

$$\begin{aligned}
&G_A(-(p_2+p_3)^2, -p_2^2, -p_3^2) \\
&= (2\pi)^6 \sum_{|n\rangle, |m\rangle} \{\Theta(-p_2)\,\Theta(p_3)\,\delta(p_2+p_n)\,\delta(p_3-p_m) \\
&\quad \cdot \langle 0|B|n\rangle \langle n|A|m\rangle \langle m|C|0\rangle + \Theta(p_2)\,\Theta(-p_3)\,\delta(p_2-p_n)\,\delta(p_3+p_m) \\
&\quad \cdot \langle 0|C|m\rangle \langle m|A|n\rangle \langle n|B|0\rangle\}\,.
\end{aligned} \tag{45}$$

We note that the two terms occurring inside the respective parenthesis $\{\ldots\}$ are CPT – conjugate terms and that they both occur because of the well-known CPT – invariance properties of a scalar field theory fulfilling the physical assumptions of Källén and Wightman mentioned in Section 1. The indicated sums in Eqs. (44) and (45) should be performed over complete sets of states $|n\rangle$ and $|m\rangle$ with energy-momentum-vectors p_n and p_m respectively. We use the notion of discrete sums in connection with such states because in order to avoid the well-known difficulties implied by Haag's theorem the "conventional method" with a finite quantisation volume with periodic boundary conditions is introduced.

The step function with a vector argument is used to indicate that the vector belongs to the forward light-cone.

The occurrence of such step functions in Eqs. (44) and (45) is a reflection of the above-mentioned spectrum conditions of the theory.

The reason for the use of negative-light-cone vectors is our wish to keep the symmetrical energy-momentum conservation relation:

$$\sum_{j=1}^{3} p_j = 0\,. \tag{46}$$

We note that the vector $\pm p_3$ (with the sign chosen in such a way that the energy component is positive) occurs only as the energy-momentum vector of a state with the quantum numbers of the C-field in both Eqs. (44) and (45). In the same way the vectors $\pm p_1$ and $\pm p_2$ are related to the field A and the field B in Eq. (44) respectively Eq. (45). We therefore deduce that the distribution-valued functions $G_A(a_1, a_2, a_3)$ and $G_B(a_1, a_2, a_3)$ have support *only* for such values of the mass variables (a_2, a_3) and (a_1, a_3) respectively that correspond to physical states with the "correct" quantum numbers.

The third scalar variable, i.e. the variable a_1 in connection with G_A and the variable a_2 in connection with G_B, corresponds to a momentum transfer. There is consequently an upper limit of variation for physical values, e.g. in connection with G_B we have the inequality $a_2 \leqq (\sqrt{a_1} - \sqrt{a_3})^2$. In paper I the discontinuity across the positive real axis of the third variable of the vertex function is represented in terms of the above-described matrix elements G_A and G_B. The formulas are valid for values of the arguments Z_1 and Z_2 along the boundary surfaces I and II of Eqs. (4) and (6).

For values of the arguments Z_1 and Z_2 along the curve I we can write according to Eq. (69) of paper I:

$$\begin{aligned}\delta G^{\mathrm{I}}(Z_1, Z_2, a_3) &\equiv G(Z_1, Z_2, a_3 + i\varepsilon) - G(Z_1, Z_2, a_3 - i\varepsilon)\\ &= 2\pi i \int da_1\, da_2\, \delta(r a_3 + (r - a_1)(r - a_2)) \Big\{ G_B(a_1, a_2, a_3) \tfrac{1}{2} \frac{a_1 + Z_1 - 2r}{a_1 - Z_1}\\ &\quad \times \Theta(a_1 - r)\Theta(r - a_2) + G_A(a_1, a_2, a_3) \tfrac{1}{2} \frac{a_2 + Z_2 - 2r}{a_2 - Z_2} \Theta(a_2 - r)\Theta(r - a_1)\Big\}.\end{aligned} \tag{47}$$

Correspondingly we can write for values of the arguments Z_1 and Z_2 along the curve II:

$$\begin{aligned}\delta G^{\mathrm{II}}(Z_1, Z_2, a_3) &\equiv G(Z_1, Z_2, a_3 + i\varepsilon) - G(Z_1, Z_2, a_3 - i\varepsilon)\\ &= 2\pi i \int da_1\, da_2\, \delta(\alpha a_2 + \beta a_1 - \alpha\beta a_3) \Big\{ G_B(a_1, a_2, a_3) \frac{\alpha}{a_1 - Z_1} \Theta(a_1)\\ &\quad + G_A(a_1, a_2, a_3) \frac{\beta}{a_2 - Z_2} \Theta(a_2)\Big\}.\end{aligned} \tag{48}$$

We will now investigate the corresponding difference for *arbitrary* values of the arguments Z_1 and Z_2 inside the Källén-Wightman analyticity domain for the vertex function. This quantity will be called ΔG and from Eq. (37) a representation formula for ΔG can be constructed with

$$\begin{aligned}\Delta G &= G(Z_1, Z_2, \zeta_3 + i\varepsilon) - G(Z_1, Z_2, \zeta_3 - i\varepsilon)\\ &= \frac{1}{(2\pi i)^2} \iint d\zeta_1\, d\zeta_2 \Big\{ \int_0^\infty dr\, \delta(r\zeta_3 + (r - \zeta_1)(r - \zeta_2))\\ &\quad \cdot [G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 + i\varepsilon) - G(\zeta_1 + i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 - i\varepsilon)\\ &\quad + G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 + i\varepsilon) - G(\zeta_1 - i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 - i\varepsilon)]\\ &\quad \cdot S^{\mathrm{I}}(\zeta_1, \zeta_2; \zeta_3; r; Z_1, Z_2) + \iint_0^1 d\alpha\, d\beta\, \delta(1 - \alpha - \beta)\, \delta(\alpha\beta\zeta_3 - \alpha\zeta_2 - \beta\zeta_1)\\ &\quad \cdot [G(\zeta_1 + i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 + i\varepsilon) - G(\zeta_1 + i\varepsilon', \zeta_2 - i\varepsilon'', \zeta_3 - i\varepsilon)\\ &\quad + G(\zeta_1 - i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 + i\varepsilon) - G(\zeta_1 - i\varepsilon', \zeta_2 + i\varepsilon'', \zeta_3 - i\varepsilon)]\\ &\quad \cdot S^{\mathrm{II}}(\zeta_1, \zeta_2; \zeta_3; \alpha, \beta; Z_1, Z_2)\Big\}.\end{aligned} \tag{49}$$

The main observation is now that the weight functions, i.e. the expressions inside the square brackets in Eq. (49) can be expressed in terms of the matrix elements G_A and G_B by means of Eqs. (47) and (48).

This is so because the required combinations of boundary values in Eq. (49) are easily seen to be:

$$\begin{aligned}
&\delta(r\zeta_3+(r-\zeta_1)(r-\zeta_2))\,[\delta G^{\mathrm{I}}(\zeta_1+i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3)+\delta G^{\mathrm{I}}(\zeta_1-i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3)]\\
&=\delta(r\zeta_3+(r-\zeta_1)(r-\zeta_2))4\pi i\int da_1\,da_2\,da_3\,\delta(a_3-\zeta_3)\,\delta(ra_3+(r-a_1)(r-a_2))\\
&\cdot\left\{G_B(a_1,a_2,a_3)\tfrac{1}{2}\frac{a_1+\zeta_1-2r}{(a_1-\zeta_1)_P}\,\Theta(a_1-r)\,\Theta(r-a_2)+G_A(a_1,a_2,a_3)\right.\\
&\left.\cdot\tfrac{1}{2}\frac{a_2+\zeta_2-2r}{(a_2-\zeta_2)_P}\,\Theta(r-a_1)\,\Theta(a_2-r)\right\}
\end{aligned}\tag{50}$$

respectively

$$\begin{aligned}
&\delta(\alpha\beta\zeta_3-\alpha\zeta_2-\beta\zeta_1)\,[\delta G^{\mathrm{II}}(\zeta_1+i\varepsilon',\zeta_2-i\varepsilon'',\zeta_3)+\delta G^{\mathrm{II}}(\zeta_1-i\varepsilon',\zeta_2+i\varepsilon'',\zeta_3)]\\
&=\delta(\alpha\beta\zeta_3-\alpha\zeta_2-\beta\zeta_1)4\pi i\int da_1\,da_2\,da_3\,\delta(a_3-\zeta_3)\,\delta(\alpha\beta a_3-\beta a_1-\alpha a_2)\\
&\cdot\left\{G_B(a_1,a_2,a_3)\frac{\alpha}{(a_1-\zeta_1)_P}\,\Theta(a_1)+G_A(a_1,a_2,a_3)\frac{\beta}{(a_2-\zeta_2)_P}\,\Theta(a_2)\right\}.
\end{aligned}\tag{51}$$

To get the results of Eqs. (50) and (51), the following well-known relations for the principal-value distribution are used repeatedly:

$$\tfrac{1}{2}\left\{\frac{1}{a-\zeta-i\varepsilon}+\frac{1}{a-\zeta+i\varepsilon}\right\}=\frac{1}{(a-\zeta)_P}. \tag{52}$$

If the results of Eqs. (50) and (51) are introduced into Eq. (49), the integrals over the variables ζ_1 and ζ_2 can be carried out. During that process a few formally undefined expressions occur, stemming from the lack of sufficient damping in some integrals. A repeated use of relations similar to the ones in Eq. (29) results, however, in the vanishing of all such expressions. There is a further problem connected to the use of Eqs. (50) and (51). In the proof in Section 5 (Eqs. (30)–(35)) that the kernel function R_2^{I} gives a vanishing contribution to the representation integrals, some specific assumptions are made. Thus among other things we have actually implicitly assumed that the occurring boundary values of the vertex function depend only on the scalar variables ζ_1, ζ_2 and ζ_3 and that they are *independent* of the integration parameters r, α and β.

The expressions for the boundary values occurring in Eqs. (50) and (51) do seemingly not have these properties and therefore the above-mentioned proof does not function. It is, however, evident that the whole question is rather a problem of whether certain changes of orders of integration are permissible or not. The right to perform such changes can be seen to be directly connected to different assumptions on the properties of the matrix elements G_A and G_B. We will, however, not discuss the relations between G_A and G_B which are implied by the use of Eqs. (50) and (51) in this publication.

The resulting formula for the function ΔG is a two-dimensional dispersion relation in the same sense as above with the matrix elements G_A and G_B as weight functions:

$$\Delta G(Z_1, Z_2, \zeta_3) = \frac{1}{(2\pi i)^2} \int da_1\, da_2 \{Q_A(Z_1, Z_2, \zeta_3; a_1, a_2)\, G_A(a_1, a_2, \zeta_3) + Q_B(Z_1, Z_2, \zeta_3; a_1, a_2)\, G_B(a_1, a_2, \zeta_3)\}\,. \tag{53}$$

The kernel functions Q_A and Q_B in Eq. (53) turn out to be completely symmetric in the sense that

$$Q_A(Z_1, Z_2, \zeta_3; a_1, a_2) = Q_B(Z_2, Z_1, \zeta_3; a_2, a_1)\,. \tag{54}$$

The explicit expression for e.g. the kernel function Q_B is conveniently divided into three terms. Each one of the terms is related to one of the different boundary surfaces of the analyticity region and we will in a self-explanatory notation call them Q_B^{I}, Q_B^{II} and Q_B^{cut}:

$$Q_B = Q_B^{\mathrm{I}} + Q_B^{\mathrm{II}} + Q_B^{\mathrm{cut}}\,, \tag{55}$$

$$\begin{aligned} Q_B^{\mathrm{I}} = {} & \frac{4\pi i}{a_1 - Z_1} \int_0^\infty dr\, \delta((r-a_1)(r-a_2) + r\zeta_3) \\ & \cdot \left\{1 + \tfrac{1}{2}(a_1 + Z_1 - 2r) \frac{2r + \zeta_3 - Z_1 - Z_2}{(r - Z_1)(r - Z_2) + r\zeta_3}\right\} \\ & \cdot \log\left(\frac{r\zeta_3}{(r - Z_1)(r - Z_2)}\right) \Theta(a_1 - r)\, \Theta(r - a_2)\,, \end{aligned} \tag{56}$$

$$\begin{aligned} Q_B^{\mathrm{II}} = {} & \frac{2\pi i\, \Theta(a_1)}{a_1 - Z_1} \iint_0^1 d\alpha\, d\beta\, \delta(1 - \alpha - \beta)\, \delta(\alpha a_2 + \beta a_1 - \alpha\beta\zeta_3) \\ & \cdot \frac{Z_1 - \alpha^2 \zeta_3}{\alpha\beta\zeta_3 - \alpha Z_2 - \beta Z_1} \left\{\log\left(\frac{\alpha\beta\zeta_3 - \alpha Z_2}{-\beta Z_1}\right) - \log\left(\frac{\alpha\beta\zeta_3 - \beta Z_1}{-\alpha Z_2}\right)\right\}, \end{aligned} \tag{57}$$

$$\begin{aligned} Q_B^{\mathrm{cut}} = {} & \frac{2\pi i}{a_1 - Z_1} \cdot \frac{1}{a_2 - Z_2} \left[2 \int_0^\infty dr\, \delta((r - a_1)(r - a_2) + r\zeta_3)\, \Theta(a_1 - r)\, \Theta(r - a_2)\right. \\ & \cdot (a_1 + a_2 - \zeta_3 - 2r) \log\left(\frac{r - Z_2}{r - a_2}\right) + \iint_0^1 d\alpha\, d\beta\, \delta(1 - \alpha - \beta) \\ & \cdot \delta(\alpha a_2 + \beta a_1 - \alpha\beta\zeta_3)\, \Theta(a_1) \left(\alpha\zeta_3 - \frac{a_1}{\alpha}\right) \times \left(\log\left(\frac{-Z_2}{-a_2}\right)\right. \\ & \left.\left. + \log\left(\frac{\alpha\beta\zeta_3 - \alpha Z_2}{\beta a_1}\right)\right)\right]. \end{aligned} \tag{58}$$

From a comparison between Eq. (56) and Eq. (26) respectively between Eq. (57) and Eq. (36) we deduce that the main difference between the kernel functions of Eq. (37) and the kernel function of Eq. (53) is the occurrence of some logarithmic factors, as well as the "new" term Q_B^{cut}.

The results of Eq. (58) are based upon the formulas in Eq. (37) and these contain intrinsic assumptions on the behaviour of the vertex function in different asymptotic directions. "Subtracted" versions of Eq. (53), which are valid for more general asymptotic behaviour of the vertex function can be developed in the same way as in Section 6. We will, however, not give any explicit examples because such formulas will in order to cover the general case contain such a complicated notation that they will only be of interest as a curiosity. For a particular case, however, the derivation is straight forward along the lines of Section 6.

8. Concluding Remarks

1. We note that there is a completely different and somewhat more abstract way to derive such representation formulas for the vertex function that we have given in Eq. (37) in this paper.

In the mathematical literature there is a procedure called the Bergman-Weil integral formulas which is applicable to the specific case under investigation.

The conditions for the applicability of that procedure are that the domain of holomorphy of the function should be bounded by analytic hyper surfaces and that the vertex function should exhibit certain boundedness properties in asymptotic directions. The first condition is actually fulfilled for the Källén-Wightman domain and particular examples of analytic hypersurfaces are the boundary surfaces I (Eq. (4)), II (Eq. (6)) and the physical-region-cuts, i.e. the positive real axes of the variables. The second condition is of a similar nature as the one which was used as simplifying assumption in Sections 2–5, i.e. that the vertex function must actually vanish sufficiently fast around infinity.

2. By an explicit use of the above-mentioned Bergman-Weil formalism [8] Källén and Toll [9] have given a representation formula for the vertex function which is valid when *all the three* scalar variables are arbitrary complex numbers (inside the domain of holomorphy, of course). These authors have given several alternative versions of the representation formulas. In one of these versions, the kernel function of the integral formula is the perturbation theoretical function corresponding to the so-called Mercedes graph [9]. Subsequent investigation has shown that this particular perturbation theory function can actually be written as a superposition of functions corresponding to simple perturbation theoretical graphs [10]. One of the unsolved problems in that connection

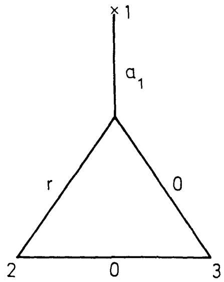

Fig. 3

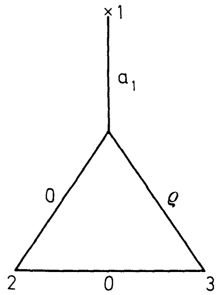

Fig. 4

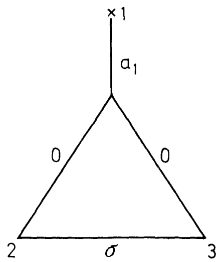

Fig. 5

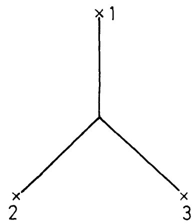

Fig. 6

is, however, that the weight functions in these alternative versions of the Källén-Toll results have a rather distant and clumsy relation to the original functions.

It is interesting to note that in our dispersion relations where the weight functions are known to be the on-mass-shell matrix elements of the field operators, i.e. Eq. (53), the occurring kernel functions i.e. the functions Q_A and Q_B can likewise be written as the sum (Eq. (55)) of functions related to perturbation theory functions. Thus, the term Q_B^{I} in Eq. (56) is closely connected to the perturbation theory function corresponding to the graph of Fig. 3. In the same way we find (by making the change of variables $\zeta_3 \alpha = \varrho$ and $\zeta_3 \beta = \sigma$ respectively) that the two terms in Q_B^{II} of Eq. (57) are similarly related to the graphs of Figs. 4 and 5 respectively.

The contribution which we have called Q_B^{cut} in Eq. (57) is in turn related to the "Tripod graph" of Fig. 6.

3. It is immediately clear that the results of Eq. (37) respectively Eq. (53) can be generalised to the analogous case in which we consider e.g. the discontinuity across the physical-region-cut for the variable Z_1 with the variables Z_2 and Z_3 arbitrary complex. The permutations of indices, necessary to find the results for that case from the derived one,

is straight-forward (note in that connection the Remark 2 in Section 6 of paper I), and we will not give the details.

4. Representation formulas for the coordinate-space vertex function F, similar to the ones derived above for the momentum-space function G, can be derived in the same way. The main difference is that in the equation corresponding to Eq. (53) the occurring weight functions are non-vanishing for all non-negative values of the integration variables. Thus, even in theories only containing particles with non-zero mass, we expect to find "light-cone singularities" in connection with the coordinate-space function.

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