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The Equation Curl $W_\mu(x) = 0$ in Quantum Field Theory*

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Abstract. In one time and arbitrarily many space dimensions we obtain necessary and sufficient conditions for the existence of a local operator solution of the equation $\partial_\nu \omega = W_\nu$. Here the given local fields W_ν satisfy $\partial_\mu W_\nu - \partial_\nu W_\mu = 0$ and the spectrum of the two point function $(\Omega, W_\mu(x) W_\nu(y)\Omega)$ is assumed to have a mass gap.

For the mathematical treatment of local field theoretic models involving pseudovector or vector fields the curl of which vanishes[1] it is important to know whether these fields are gradients of local, pseudoscalar or scalar primitive fields respectively and whether the primitive fields are relatively local to the rest of the underlying fields of the model. For the pseudoscalar current: $\bar{\psi}\gamma^5\gamma^\mu\psi$: where ψ denotes a free massive Dirac field in one time and one space dimension the primitive field can be expressed in terms of creation and annihilation operators and the questions just raised can be decided by rather tedious computations [1]. This particular primitive field plays an important role in the solution of the Federbush model [2, 1]. In models that are only partially solvable e.g. for the derivative coupling of a massless, neutral, pseudoscalar particle to a charged spinor field [3] general citeria are needed. We therefore pose the following problem: Let a local Wightman theory in 1 time and n space dimensions be given: $\{\mathscr{H}; U(a,\Lambda); \phi_\alpha(x), W_\mu(x)$ $\alpha = 1, \ldots, l$ $\mu = 0, 1, \ldots, n\}$ [4]. Let the linear domain of definition D common to the operators

$$\phi_\alpha(\varphi), \phi_\alpha^*(\varphi) \subset \phi_\alpha(\bar{\varphi})^*, W_\mu(f) \quad \text{where} \quad \varphi, f \in \mathscr{S}(R^{1+n}),$$

invariant under the application of these field operators and $U(a,\Lambda)$ and containing the unique vacuum state Ω be just the set D_1 of quasilocal states (i.e. those states which can be obtained by smearing monomials

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1 In one time and one space dimension these are exactly the local models with conserved vector or pseudovector currents.

in the fields with test functions from $\mathscr{S}$, forming linear combinations and applying the resulting "quasilocal" operators to Ω). Let us assume that there exists a mass $m_0 > 0$ such that states with mass smaller than m_0 do not contribute to the two-point function

$$(\Omega, W_\mu(x)\, W_\nu(y)\Omega) \tag{0.1}$$

of the local, hermitian pseudovector fields $W_\mu(x)$.[2] Let the curl of $W_\mu(x)$ vanish

$$\partial_\mu W_\nu(x) - \partial_\nu W_\mu(x) = 0\,. \tag{0.2}$$

What can we conclude in this general framework about the existence of a primitive pseudoscalar field $\omega(x)$ with $W_\mu(x) = \partial_\mu \omega(x)$, its locality and its relative locality?

Before we formulate the answer to this question in a theorem we would like to point out a fundamental difference between the cases $n=1$ and $n>1$. For $n=1$ the vanishing of the curl of the local pseudovector field $W_\nu(x)$ implies the existence of a local, conserved vector current $V^\mu(x)$ defined by

$$V(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W(x) \tag{0.3}$$

which in general implies the existence of a non-trivial conserved charge Q [5]. The charge operator Q can be expressed by a line integral involving W_μ. In contrast, for $n>1$, the analogous construction never leads to a genuine conserved charge.

The second part of this statement requires a proof: Let e be an arbitrary space-like vector of Minkowski length -1: $e^2 = -1$. Let $f(x)$ be a test function in $\mathscr{S}(R^{1+n})$ arbitrary apart from the condition $\int dx\, f(x) = 1$. Let the symbol B stand for an arbitrary quasilocal operator. We define the operator Q_e^f on the dense set D_1 of quasilocal states $B\Omega$ by

$$Q_e^f B\Omega = \int_{-\infty}^{+\infty} ds\, e^\mu [\textstyle\int dx\, f(x)\, W_\mu(x - se), B]\,\Omega\,, \qquad Q_e^f \Omega = 0\,. \tag{0.4}$$

In order to prove that Q_e^f actually defines an operator, it is sufficient to show that the integral in (0.4) converges strongly in the Hilbert space $\mathscr{H}$ and that $Q_e^f B\Omega = 0$ whenever $B\Omega = 0$ (the consistency requirement).

[2] It is for the sake of simplicity that we restrict ourselves to hermitian pseudovector fields: hermitian vector fields can be treated similarly, non-hermitian fields are decomposed into their hermitian parts.

We start by showing the strong convergence.

$$\Psi_e^f(s) = e^\mu [\int dx\, f(x)\, W_\mu(x-se), B]\,\Omega \tag{0.5}$$

is for all e, f and s a vector in $\mathscr{H}$. We form the scalar product

$$(\Psi_e^f(s), \Psi_e^f(t)) \tag{0.6}$$

which is easily shown to be a function simultaneously continuous in s and t. Hence the norm squared

$$(\Psi_e^f(s), \Psi_e^f(s)) = \|\Psi_e^f(s)\|^2 \tag{0.7}$$

depends continuously on s. Due to the quasilocal character of B, $\|\Psi_e^f(s)\|^2$ decreases rapidly of $|s|\to\infty$. The continuity and decrease guarantee the strong convergence. Thus $Q_e^f B\Omega$ defines a vector in $\mathscr{H}$ which we will show is always the null vector, and thus a separate demonstration of the consistency requirement is superfluous.

First, we shall prove that $Q_e^f B\Omega$ does not depend on f provided $f\in\mathscr{S}(R^{1+n})$ and $\int dx\, f(x)=1$. To this end, let f_1 and f_2 be test functions in $\mathscr{S}(R^{1+n})$ subject to the condition $\int dx\, f_1(x)=1=\int dx\, f_2(x)$ and arbitrary otherwise. Let e be an arbitrary space-like vector with $e^2=-1$. Finally, let B and B' be any two quasilocal operators. We claim that

$$\int_{-\infty}^{+\infty} ds\, e^\mu [\int dx\, g(x)\, W_\mu(x-se), B]\,\Omega = 0 \tag{0.8}$$

for $g(x)=f_1(x)-f_2(x)\in\mathscr{S}(R^{1+n})$, $\int dx\, g(x)=0$.

Indeed it suffices to show that

$$\int_{-\infty}^{+\infty} ds\, e^\mu F_\mu^g(se) = 0 \tag{0.9}$$

where $F_\mu^\varphi(y)$ is defined by

$$\begin{aligned} F_\mu^\varphi(y) &= \int dx\, \varphi(x+y)\,(B'\Omega, [W_\mu(x), B]\Omega) \\ &= (B'\Omega, [\int dx\, \varphi(x)\, W_\mu(x-y), B]\Omega) \end{aligned} \tag{0.10}$$

for an arbitrary test function $\varphi\in\mathscr{S}(R^{1+n})$. For $e^2=-1$, $F_\mu^g(se)\in\mathscr{S}(R^1)$ as a function of s. Because $g\in\mathscr{S}(R^{1+n})$ and $\int dx\, g(x)=0$, g can be represented in the following form

$$g(x) = \sum_{\nu=0}^{n} \partial_\nu h^\nu(x), \qquad h^\nu(x)\in\mathscr{S}(R^{1+n}) \qquad \nu=0,1,\ldots,n \tag{0.11}$$

and we may conclude (summation convention!)

$$
\begin{aligned}
\int_{-\infty}^{+\infty} ds\, e^{\mu} F^{g}_{\mu}(se) &= \lim_{S\to\infty} \int_{-S}^{+S} ds\, e^{\mu} F^{g}_{\mu}(se) \\
&= -\lim_{S\to\infty} \int_{-S}^{+S} ds\, e^{\mu}(B'\Omega, [\textstyle\int dx\, h^{\nu}(x+se)\, \partial_{\nu} W_{\mu}(x), B]\Omega) \\
&= -\lim_{S\to\infty} \int_{-S}^{+S} ds (B'\Omega, [\textstyle\int dx\, e^{\mu} h^{\nu}(x+se)\, \partial_{\mu} W_{\nu}(x), B]\Omega) \qquad (0.12) \\
&= +\lim_{S\to\infty} \int_{-S}^{+S} ds \frac{d}{ds} (B'\Omega, [\textstyle\int dx\, h^{\nu}(x+se)\, W_{\nu}(x), B]\Omega) \\
&= \lim_{S\to\infty} \{F^{h^{\nu}}_{\nu}(Se) - F^{h^{\nu}}_{\nu}(-Se)\} = 0\,.
\end{aligned}
$$

Next, let f be an arbitrary test function from $\mathscr{S}(R^{1+n})$ subject to the condition $\int dx\, f(x) = 1$. For $n > 1$, to every space-like vector e with $e^2 = -1$, there exists an orthogonal vector $e^{\perp}$ with $(e^{\perp})^2 = -1$:

$$
e \cdot e^{\perp} = 0\,. \tag{0.13}
$$

$f_{\sigma}(x) = f(x - \sigma e^{\perp})$ is a test function in $\mathscr{S}(R^{1+n})$ concentrated around the point $\sigma e^{\perp}$ which satisfies the condition $\int dx\, f_{\sigma}(x) = 1$. We consider the expression

$$
(B'\Omega, Q^{f_{\sigma}}_{e} B\Omega) = \int_{-\infty}^{+\infty} ds\, e^{\mu}(B'\Omega, [\textstyle\int dx\, f(x + se - \sigma e^{\perp})\, W_{\mu}(x), B]\Omega) \tag{0.14}
$$

which was just shown to be independent of σ. It follows from the quasi-locality of B and the orthogonality of e and $e^{\perp}$ that

$$
e^{\mu}(B'\Omega, [\textstyle\int dx\, f(x + se - \sigma e^{\perp})\, W_{\mu}(x), B]\Omega) \in \mathscr{S}(R^2) \tag{0.15}
$$

as a function of s and σ. This implies that

$$
\int_{-\infty}^{+\infty} ds\, e^{\mu}(B'\Omega, [\textstyle\int dx\, f(x - \sigma e^{\perp})\, W_{\mu}(x - se), B]\Omega) \in \mathscr{S}(R^1) \tag{0.16}
$$

as a function of σ. In view of its σ-independence we conclude

$$
(B'\Omega, Q^{f}_{e} B\Omega) = (B'\Omega, Q^{f_{\sigma}}_{e} B\Omega) = 0\,. \tag{0.17}
$$

Since the quasilocal states form a dense set in $\mathscr{H}$ we obtain $Q^{f}_{e} B\Omega = 0$ for all quasilocal operators B i.e.,

$$
Q^{f}_{e} = 0 \quad \text{q.e.d.}^{3}
$$

[3] An alternative proof of $Q^{f}_{e} = 0$ for $n > 1$ consists of showing that in this case Q^{f}_{e} does not depend on e as long as $e^2 = -1$. From this it follows, in particular, that $Q^{f}_{e} = Q^{f}_{-e} = -Q^{f}_{e}$ i.e. $Q^{f}_{e} = 0$.

We introduce the following notations: Let $\{a, \Lambda\}$ be an element of the orthochronous Poincaré group $\mathscr{P}^{\uparrow}$. We set $\{a, \Lambda\} f(x) = f(\Lambda^{-1}(x-a))$.

Let the energy-momentum spectrum of the theory $\{\mathscr{H}; U(a, \Lambda); \phi_\alpha(x), W_\mu(x)\ \alpha = 1, \ldots, l\ \mu = 0, 1, \ldots, n\}$ have a partial particle structure i.e. apart from the point 0 associated solely with the vacuum state let the spectrum of the mass operator of the theory contain further isolated points corresponding to particles. Then, by D_0^{ex} we denote the set of "non-overlapping asymptotic states" [6, 7].

The previously indicated difference between the cases $n=1$ and $n>1$ enters into the formulation of the following theorem valid in our general framework.

Theorem. *For $n>1$ there exists a field $\omega(x)$ with the following properties:*

(a) *After smearing with an arbitrary test function f in $\mathscr{S}(R^{1+n})$, $\omega(f)$ is defined on D_1 and maps D_1 into D_1; if $\Phi, \Psi \in D_1$, then $(\Phi, \omega(f)\Psi)$ is a tempered distribution regarded as a functional of f.*

(b) *$\omega(x)$ is hermitian i.e. $\omega(f)^* = \omega(\bar{f})$ on D_1;*

(c) *$\omega(x)$ is pseudoscalar i.e. for all $\{a, \Lambda\} \in \mathscr{P}_+^{\uparrow}$*

$$U(a, \Lambda)\, \omega(f)\, U(a, \Lambda)^{-1} = \omega(\{a, \Lambda\} f) \text{ on } D_1$$

$$U(I_n)\, \omega(f)\, U(I_n)^{-1} = -\omega(\{I_n\}) \text{ on } D_1$$

where I_n denotes the reflection on the hyperplane $x_n = 0$;

(d) *$\omega(x)$ is local;*

(e) *$\omega(x)$ is relatively local with respect to the fields $\phi_\alpha, \phi_\beta^*\ \alpha, \beta = 1, \ldots, l$;*

(f) *$\omega(x)$ is a primitive field for the pseudovector field $W_\mu(x)$ i.e.*

$$(\partial_\mu \omega)(f) = W_\mu(f) \text{ on } D_1 .$$

For $n=1$ the analogous statements are true if and only if $Q=0$. For $n=1$, $Q \neq 0$ the analogue of the statements (a)–(d), (f) *and* (e) *replaced by*

$$[\omega(x), \phi_\alpha^{(*)}(y)] = \tfrac{1}{2}\varepsilon(x^1 - y^1)\,[Q, \phi_\alpha^{(*)}(y)] \quad \text{for} \quad (x-y)^2 < 0$$

hold true if and only if $[Q, W_\mu(x)] = 0$.

If in addition $(\Omega, \omega(x)\Omega) = 0$ is required then the field $\omega(x)$ is unique.

Remark. If the energy-momentum spectrum of the theory $\{\mathscr{H}; U(a, \Lambda); \varphi_\alpha(x), W_\mu(x)\ \alpha = 1, \ldots, l\ \mu = 0, 1, \ldots, n\}$ has a partial particle structure we may replace D_1 in the statements of the theorem by $\hat{D}_1$ where $\hat{D}_1$ is the set of all states which can be obtained by smearing an arbitrary monomial in the fields $\phi_\alpha, \phi_\beta^*$ and ω with testfunctions from $\mathscr{S}$, applying it after extension by continuity to vectors in D_0^{ex} and forming the linear span of the resulting states and the states in D_1. $\hat{D}_1$ is invariant under the application of the (extended) fields ϕ_α, ϕ_β^*, W_μ, ω and under the application of $U(\mathrm{a}, \Lambda)$ and Q.

We shall give a constructive proof.

a) Construction of the Field $\omega(x)$

Let e be an arbitrary space-like vector of Minkowski-length $-1: e^2 = -1$. We define fields $\omega_e(x)$ on quasilocal states $B\Omega$ as follows,

$$\begin{aligned}\omega_e(f)B\Omega = \frac{1}{2}\int_{-\infty}^{+\infty} ds\, \varepsilon(s)\, e^\mu [\textstyle\int dx\, f(x)\, W_\mu(x-se), B]\Omega \\ + B\int dx' \{\textstyle\int dx\, f(x)\, \bar{D}(x-x')\}\, \partial'^\mu W_\mu(x')\Omega \quad \text{for} \quad f \in \mathscr{S}(R^{1+n})\end{aligned} \tag{a.1}$$

where $\bar{D}$ is the time symmetric Green's function of the wave equation and $\varepsilon(s)$ denotes the antisymmetric step function. It follows from the argument beginning with Eq. (0.5) that the first term of the r.h.s. defines a vector in $\mathscr{H}$. In order that Eq. (a.1) actually defines an operator it suffices to show that $\int dx' \{\int dx\, f(x)\, \bar{D}(x-x')\}\, \partial'^\mu W_\mu(x')\Omega$ is contained in D_1 and that $\omega_e(f)\, B\Omega = 0$ whenever $B\Omega = 0$. From the definition of $\bar{D}$ it follows that

$$\mathscr{F}_{x'}\{\textstyle\int dx\, f(x)\, \bar{D}(x-x')\}\,(k) = -P\,\frac{\tilde{f}(k)}{k^2}\,, \tag{a.2}$$

where P denotes the principal value. Let $g(k^2)$ be a real C^∞-function with $|g(k^2)| \leqq 1$ and

$$g(k^2) = \begin{cases} 1 & \text{for} \quad k^2 > \frac{2}{3}m_0^2 \\ 0 & \text{for} \quad k^2 < \frac{1}{3}m_0^2 \end{cases}.$$

We define $\mathring{f}(x')$ by

$$\mathring{f}(x') = \mathscr{F}_k^{-1}\{g(k^2)\, \mathscr{F}_y\{\textstyle\int dx\, f(x)\, \bar{D}(x-y)\}\,(k)\}\,(x')\,. \tag{a.3}$$

$\mathring{f}(x')$ is contained in $\mathscr{S}(R^{1+n})$. Because of our assumption on the spectrum of the two-point function $(\Omega, W_\mu(x)\, W_\nu(y)\Omega)$ we may conclude

$$\begin{aligned}&\textstyle\int dx' \{\int dx\, f(x)\, \bar{D}(x-x')\}\, \partial'^\mu W_\mu(x')\Omega \\ &\qquad = \textstyle\int dx'\, \mathring{f}(x')\, \partial'^\mu W_\mu(x')\Omega = C^f\Omega\end{aligned} \tag{a.4}$$

where $C^f = \int dx\, \mathring{f}(x')\, \partial'^\mu W_\mu(x')$ is a quasilocal operator itself with $C^{\bar{f}*} = C^f$ on D_1. Hence, $C^f\Omega \in D_1$ and $\omega_e(f)B\Omega$ defines a vector in $\mathscr{H}$. We note the following relations

$$(\Omega, BC^f\Omega) = \frac{1}{2}\int_{-\infty}^{+\infty} ds\, \varepsilon(s)\, e^\mu(\Omega, B\textstyle\int dx\, f(x)\, W_\mu(x-se)\Omega)\,, \tag{a.5}$$

$$(\Omega, C^f B\Omega) = \frac{1}{2}\int_{-\infty}^{+\infty} ds\, \varepsilon(s)\, e^\mu(\Omega, \textstyle\int dx\, f(x)\, W_\mu(x-se)B\Omega)\,. \tag{a.5'}$$

$(\Omega, B\int dx\, f(x)\, e^\mu W_\mu(x-se)\Omega)$ is contained in $\mathscr{S}(R^1)$ as a function of s. For a proof we apply the argument of Ref. [8] for a strong decrease of the truncated Wightman functions for large space-like separation of the arguments in theories with a mass gap. The assumption we made on the two point function $(\Omega, W_\mu(x)\, W_\nu(y)\Omega)$ is already sufficient for our purposes. Thus the r.h.s. of (a.5) makes sense. Using the mass gap assumption once more we find

$$\begin{aligned} e^\mu(\Omega, B\textstyle\int dx\, f(x)\, W_\mu(x-se)\Omega) \\ &= -(\Omega, B\textstyle\int dx\, e^\mu\{\partial^\nu \mathring{f}(x)\}\, \partial_\nu W_\mu(x-se)\Omega) \\ &= (\Omega, B\textstyle\int dx\{e^\mu \partial_\mu \partial^\nu \mathring{f}(x+se)\}\, W_\nu(x)\Omega) \\ &= \frac{d}{ds}(\Omega, B\textstyle\int dx\{\partial^\nu \mathring{f}(x)\}\, W_\nu(x-se)\Omega)\,. \end{aligned} \tag{a.6}$$

Inserting this identity into the integral on the r.h.s. of (a.5) we obtain the desired equality sign

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{+\infty} ds\, \varepsilon(s)\, e^\mu(\Omega, B\textstyle\int dx\, f(x)\, W_\mu(x-se)\Omega) \\ &= -(\Omega, B\textstyle\int dx\{\partial^\nu \mathring{f}(x)\}\, W_\nu(x)\Omega) \\ &= (\Omega, BC^f\Omega)\,. \end{aligned} \tag{a.7}$$

Relation (a.5′) is proved analogously.

Next, we shall show that the consistency requirement is satisfied i.e. $B\Omega=0$ implies $\omega_e(f)B\Omega=0$. Since $\omega_e(f)B\Omega$ defines a vector in $\mathscr{H}$ and since the set of quasilocal states is dense in $\mathscr{H}$ it is sufficient to prove that for all quasilocal operators B'

$$(B'\Omega, \omega_e(f)B\Omega)=0 \quad \text{if} \quad B\Omega=0\,.$$

Using (a.5) and (a.5′) this follows immediately from

$$\begin{aligned} (B'\Omega, \omega_e(f)B\Omega) &= \frac{1}{2}\int_{-\infty}^{+\infty} ds\, \varepsilon(s)\, ([\textstyle\int dx\, \bar{f}(x)\, e^\mu W_\mu(x-se), B']\,\Omega, B\Omega) \\ &\quad + (B'C^{\bar{f}}\Omega, B\Omega) = (\omega_e(\bar{f})B'\Omega, B\Omega)\,. \end{aligned} \tag{a.8}$$

Hence $\omega_e(f)$ defines an operator which by the last equation satisfies $\omega_e(f)^* = \omega_e(\bar{f})$ on D_1.

We claim that $\omega_e(f)$ maps D_1 into D_1.[4] For a proof it is sufficient to show that

$$\tfrac{1}{2}\textstyle\int ds\, \varepsilon(s) \int dx\, f(x)\, [e^\mu W_\mu(x-se), B] \tag{a.9}$$

[4] The author is indebted to Prof. H. Araki who pointed out to him that by proving this claim the original theorem which involved extension by continuity could be sharpened and the proof considerably simplified.

is a quasilocal operator. Since B is a linear combination of smeared monomials in the field operators we may assume without loss of generality that B has the following form

$$\int \prod_{r=1}^{m} d^{1+n} y_{(r)} \varphi(y_{(1)}, \ldots, y_{(m)}) A_1(y_{(1)}) \ldots A_m(y_{(m)}) \tag{a.10}$$

where A_r stands for one of the fields ϕ_α, ϕ_β^*, W_ν and where $\varphi \in \mathscr{S}(R^{m(1+n)})$.

Let $\chi(t)$ and $\hat{\chi}(t)$ be two real, infinitely differentiable functions with $|\chi(t)| \leqq 1$, $\chi(t)=0$ for $|t|>2$, $\chi(t)=1$ for $|t|<1$ and $|\hat{\chi}(t)| \leqq 1$, $\hat{\chi}(t)=1$ for $t<-2$, $\hat{\chi}(t)=0$ for $t>-1$. We introduce two sequences of continuous mappings $\mathscr{S} \to \mathscr{S}$ by

$$f(x) \to f^{(j)}(x) = \frac{1}{2} \int_{-\infty}^{+\infty} ds\, \varepsilon(s)\, \chi\left(\frac{s}{j+1}\right) f(x+se) \qquad j=0,1,\ldots$$

and

$$f(x) \to f^{(j,j-1)}(x) = f^{(j)}(x) - f^{(j-1)}(x) \qquad j=1,2,\ldots\,.$$

Expression (a.9) rewritten in this notation becomes

$$\lim_{J\to\infty} [e^\mu W_\mu(f^{(J)}), B]\,. \tag{a.11}$$

Invoking locality we find

$$\begin{aligned}
[e^\mu W_\mu(f^{(J)}), B] &= e^\mu \int dx \int \prod_{r=1}^{m} dy_{(r)} f^{(J)}(x)\, \varphi(y_{(1)}, \ldots, y_{(m)}) \\
&\quad \cdot \left[1 - \prod_{r=1}^{m} \hat{\chi}((x-y_{(r)})^2)\right] W_\mu(x) \prod_{r=1}^{m} A_r(y_{(r)}) \\
&\quad - e^\mu \int \prod_{r=1}^{m} dy_{(r)} \int dx\, \varphi(y_{(1)}, \ldots, y_{(m)})\, f^{(J)}(x) \\
&\quad \cdot \left[1 - \prod_{r=1}^{m} \hat{\chi}((x-y_{(r)})^2)\right] \prod_{r=1}^{m} A_r(y_{(r)}) \cdot W_\mu(x) \\
&= \int dx \int \prod_{r=1}^{m} dy_{(r)} f^{(0)}(x)\, \varphi(y_{(1)}, \ldots, y_{(m)}) \\
&\quad \cdot \left[1 - \prod_{r=1}^{m} \hat{\chi}((x-y_{(r)})^2)\right] e^\mu W_\mu(x) \prod_{r=1}^{m} A_r(y_{(r)}) \\
&\quad + \int dx \int \prod_{r=1}^{m} dy_{(r)} \sum_{j=1}^{J} \left\{ f^{(j,j-1)}(x)\, \varphi(y_{(1)}, \ldots, y_{(m)}) \right. \\
&\quad \left. \cdot \left[1 - \prod_{r=1}^{m} \hat{\chi}((x-y_{(r)})^2)\right]\right\} e^\mu W_\mu(x) \prod_{r=1}^{m} A_r(y_{(r)}) \\
&\quad - \cdots\,.
\end{aligned} \tag{a.12}$$

We observe that for any natural number L

$$\left\{(j^2)^L f^{(j,j-1)}(x)\,\varphi(y_{(1)},\ldots,y_{(m)})\left[1-\prod_{r=1}^{m}\hat{\chi}((x-y_{(r)})^2)\right]\right\}_{j=1,2,3,\ldots}$$

forms a null-sequence in $\mathscr{S}(R^{(1+m)(1+n)})$ if f and φ are test functions in $\mathscr{S}(R^{1+n})$ and $\mathscr{S}(R^{m(1+n)})$ respectively. Hence

$$f^{(0)}(x)\,\varphi(y_{(1)},\ldots,y_{(m)})\left[1-\prod_{r=1}^{m}\hat{\chi}((x-y_{(r)})^2)\right]$$
$$+\sum_{j=1}^{J} f^{(j,j-1)}(x)\,\varphi(y_{(1)},\ldots,y_{(m)})\left[1-\prod_{r=1}^{m}\hat{\chi}((x-y_{(r)})^2)\right]$$

converges to a test function $\psi(x,y_{(1)},\ldots,y_{(m)})$ in $\mathscr{S}(R^{(1+m)(1+n)})$ as J tends to infinity. This implies that expression (a.9) may be written as a difference of two quasilocal operators

$$\int dx \int \prod_{r=1}^{m} dy_{(r)}\,\psi(x,y_{(1)},\ldots,y_{(m)})\,e^\mu W_\mu(x)\prod_{r=1}^{m} A_r(y_{(r)})$$

and $$\int \prod_{r=1}^{m} dy_{(r)} \int dx\,\psi(x,y_{(1)},\ldots,y_{(m)})\prod_{r=1}^{m} A_r(y_{(r)})\cdot e^\mu W_\mu(x)$$

which is the result we set out to prove.

Regard the matrix element $(\Phi,\omega_e(f)\Psi)$ as a functional of f where $\Phi,\Psi\in D_1$. As a weak limit (in $\mathscr{S}'$) of tempered distributions:

$$(\Phi,\omega_e(f)\Psi)=\lim_{J\to\infty}(\Phi,\omega_e(f^{(J)})\Psi)\,, \tag{a.13}$$

it defines a tempered distribution itself according to a well-known theorem [9]. It follows from the nuclear theorem [9] that the vacuum expectation values involving an arbitrary combination of the fields ω_e, ϕ_α, ϕ_β^* and W_μ define tempered distributions.

The proof of statement (a) of the theorem will be accomplished after showing that $\omega_e(x)$ actually is independent of e i.e. that all fields $\omega_e(x)$ with $e^2=-1$ define one and the same operator-valued distribution $\omega(x)$.

To prove this point we propose to establish the following identity for all quasilocal operators B and B' entering into the definition of F

$$\frac{1}{2}\int_{-\infty}^{+\infty} ds\,\varepsilon(s)\,e^\mu F_\mu^f(se)=\frac{1}{2}\int_{-\infty}^{+\infty} ds\,\varepsilon(s)\,F_n^f(se_{(n)}) \tag{a.14}$$

where e is an arbitrary space-like vector with $e^2=-1$, f is an arbitrary test function in $\mathscr{S}(R^{1+n})$ and where $e_{(n)}$ denotes the unit vector in the direction of the positive nth coordinate axis. The identity (a.14) can be derived by using the following properties of F^f_μ:

(1) $F^f_\mu(y)$ is continuously differentiable and curl $F^f_\mu(y)=0$.

(2) In the topology of $\mathscr{S}(R^1)\, s^N F^f_\mu(se)$ is uniformly bounded in the interior of $\{e\,|\,e^2<0\}$. This is true for any natural number N.

b) $\omega(x)$ is a Hermitian Field cf. Eq. (a.8)

c) $\omega(x)$ is a Pseudoscalar Field

The proof is elementary. It uses the fact that $\omega_e(x)$ does not depend on e as long as e is space-like with $e^2=-1$.

d) Locality of $\omega(x)$

Let B and B' be two arbitrary quasilocal operators and let f and g be any two test functions in $\mathscr{S}(R^{1+n})$, the supports of which are space-like separated. One must control the convergences in the following manipulations. Setting $e^\mu \int dx\, f(x)\, W_\mu(x-se)=eW*f(se)$,

$$\begin{aligned}
&(\omega(\bar f)B'\Omega, \omega(g)B\Omega)-[f\leftrightarrow g]\\
&\quad=\tfrac{1}{4}\int\int ds\,dt\,\varepsilon(s)\,\varepsilon(t)\,([eW*\bar f(se), B']\Omega, [eW*g(te), B]\Omega)\\
&\qquad+\tfrac{1}{2}\int ds\,\varepsilon(s)\,([eW*\bar f(se), B']\Omega, BC^g\Omega)\\
&\qquad+\tfrac{1}{2}\int dt\,\varepsilon(t)\,(B'C^{\bar f}\Omega, [eW*g(te), B]\Omega)\\
&\qquad+(B'C^{\bar f}\Omega, BC^g\Omega)-[f\leftrightarrow g]\\
&\quad=\tfrac{1}{4}\int\int ds\,dt\,\varepsilon(s)\,\varepsilon(t)\,\{(\Omega, B'^*eW*f(se)\,eW*g(te)B\Omega)\\
&\qquad-(\Omega, B'^*B\Omega)\,(\Omega, eW*f(se)\,eW*g(te)\Omega)\} \qquad\qquad (\text{d.1})\\
&\qquad+\cdots\\
&\qquad+\cdots\\
&\qquad+\cdots\\
&\qquad+\tfrac{1}{2}\int ds\,\varepsilon(s)\,\{(\Omega, B'^*eW*f(se)BC^g\Omega)\\
&\qquad+(\Omega, C^{\bar f *}B'^*eW*g(se)\,B\Omega)\}\\
&\qquad-(\Omega, C^f B'^* BC^g\Omega)-[f\leftrightarrow g]\,.
\end{aligned}$$

In passing to the last line relations (a.5) and (a.5′) have been used and we have taken into account the fact that the curled brackets as functions of their arguments are absolutely integrable [8]. Consequently, the order of integration is arbitrary and we obtain

$$\begin{aligned}
&(\omega(\bar{f})B'\Omega, \omega(g)B\Omega)-[f\leftrightarrow g] \\
&\quad=\tfrac{1}{4}\int\int ds\,dt\,\varepsilon(s)\,\varepsilon(t)\,\{(\Omega, B'^*[eW*f(se), eW*g(te)]B\Omega) \\
&\quad\quad-(\Omega, B'^*B\Omega)(\Omega,[eW*f(se), eW*g(te)]\Omega)\} \\
&\quad\quad+(\Omega, B'^*B\Omega)(\Omega,[C^f, C^g]\Omega)\,.
\end{aligned} \tag{d.2}$$

Now, to any pair f, g with space-like separated support there exists a decomposition converging in the topology of $\mathcal{S}$

$$f(x)\,g(y)=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} f_i(x)\,g_j(y) \quad \text{with} \quad f_i,\,g_j\in\mathcal{S}(R^{1+n}) \quad \text{for} \quad i,j=1,2,\ldots \tag{d.3}$$

and space-like vectors e_{ij} with $(e_{ij})^2=-1$ $i,j=1,2,\ldots$ such that for all $x\in\operatorname{supp} f_i$, $y\in\operatorname{supp} g_j$ and all $\lambda\in[0,\infty[$

$$(x-y-\lambda e_{ij})^2<0\,.$$

Hence for $n>1$

$$\begin{aligned}
&(\omega(\bar{f})B'\Omega, \omega(g)B\Omega)-[f\leftrightarrow g] \\
&\quad=-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^{\infty} ds\int_{-\infty}^{0} dt\{(\Omega, B'^*[e_{ij}W*f_i(se_{ij}), e_{ij}W*g_j(te_{ij})]B\Omega) \\
&\quad\quad-(\Omega, B'^*B\Omega)(\Omega,[e_{ij}W*f_i(se_{ij}), e_{ij}W*g_j(te_{ij})]\Omega)\} \\
&\quad\quad-(\Omega, B'^*B\Omega)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{-T}^{0} dt(C^{\bar{f}_i}\Omega, e_{ij}W*g_j(te_{ij})\,\Omega) \\
&\quad\quad+(\Omega, B'^*B\Omega)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{-T}^{0} dt(\Omega, e_{ij}W*g_j(te_{ij})C^{f_i}\Omega)+o(1) \\
&\quad=-(\Omega, B'^*B\Omega)\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_{-T}^{0} dt\int_0^{\infty} ds(\Omega,[e_{ij}W*f_i(se_{ij}), \\
&\quad\quad e_{ij}W*g_j(te_{ij})]\,\Omega)+o(1)
\end{aligned} \tag{d.4}$$

as T^{-1} tends to zero, whence we infer

$$(\omega(\bar{f})\,B'\Omega, \omega(g)B\Omega)-[f\leftrightarrow g]=0\,. \tag{d.5}$$

This implies

$$[\omega(f), \omega(g)]\,\Phi=0 \quad \text{for} \quad \Phi\in D_1\,. \tag{d.6}$$

For $n=1$ we may conclude the validity of Eq. (d.6) under the assumption

$$[Q, W_\mu(x)]=0\,. \tag{d.7}$$

From an argument analogous to the one given in Section a), Q is known to map D_1 into D_1.

e) Relative Locality of $\omega(x)$

Let $A_r(x)$ be $\phi_r(x)$ for $1 \leqq r \leqq l$, $\phi^*_{r-l}(x)$ for $l+1 \leqq r \leqq 2l$ and $W_{r-2l-1}(x)$ for $2l+1 \leqq r \leqq 2l+n+1$. Using the notation of the preceding section we obtain

$$
\begin{aligned}
&(\omega(\bar{f})B'\Omega, A_r(g)B\Omega) - (A_r(g)^* B'\Omega, \omega(f)B\Omega) \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{2} \int_{-\infty}^{+\infty} ds\, \varepsilon(s)\, (\Omega, B'^*[e_{ij}W * f_i(se_{ij}), A_r(g_j)]B\Omega) \qquad \text{(e.1)} \\
&= \begin{cases} 0 & \text{for } n>1 \\ -\dfrac{1}{2}\displaystyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (\operatorname{sgn} e^1_{ij})\, (\Omega, B'^*[Q, A_r(g_j)]B\Omega) \int dx\, f_i(x) & \text{for } n=1 \end{cases} \\
&= \begin{cases} 0 & \text{for } n>1 \\ \displaystyle\int dx \int dy\, f(x)\, g(y) \frac{\varepsilon(x^1-y^1)}{2} (B'\Omega, [Q, A_r(y)]B\Omega) & \text{for } n=1 \end{cases}
\end{aligned}
$$

f) $\omega(x)$ is a Primitive Field

Let f be an arbitrary test function in $\mathscr{S}(R^{1+n})$ and B an arbitrary quasilocal operator

$$
\begin{aligned}
\int dx\, f(x)\, \partial_\mu \omega(x) B\Omega &= [\int dx\, f(x)\, W_\mu(x), B]\Omega - BC^{\partial_\mu f}\Omega \\
&= [\int dx\, f(x)\, W_\mu(x), B]\Omega + B\int dx' \{\partial'^\nu \partial'_\nu \mathring{f}(x')\}\, W_\mu(x')\Omega \\
&= \int dx\, f(x)\, W_\mu(x) B\Omega\,,
\end{aligned}
$$

i.e.

$$-\omega(\partial_\mu f) = W_\mu(f) \quad \text{on } D_1\,.$$

Uniqueness of the Operator Solution

The uniqueness of the local primitive field $\omega(x)$ (on D_1) required to be relatively local or to satisfy the commutation relation (e.1) follows from Schur's lemma and the normalization $(\Omega, \omega(x)\Omega) = 0$.

If the energy-momentum spectrum of the theory $\{\mathscr{H}; U(a, \Lambda); \phi_\alpha(x), W_\mu(x)\ \alpha = 1, \ldots, l\ \mu = 0, 1, \ldots, n\}$ has a partial particle structure, it suffices to carry over Hepp's argument [6] to the present problem in order to prove the remark following the statement of the theorem. In the case of one time and one space dimension we observe that the operator-valued fields $[Q, \phi_\alpha(x)]$, $[Q, \phi^*_\beta(x)]$ are local and relatively local with respect to $\phi_\gamma(x)$, $[Q, \phi_\gamma(x)]$, $\phi^*_\delta(x)$, $[Q, \phi^*_\delta(x)]$, $W_\mu(x)$ $\alpha, \ldots, \delta = 1, \ldots, l$, $\mu = 0, 1$, whence it follows that after Hepp's extension the charge operator has again the property

$$Q\hat{D}_1 \subset \hat{D}_1 .$$

Up to now the theorem is proved for $n>1$. For $n=1$ the conditions "$Q=0$" and "$[Q, W_\mu(x)]=0$" have been shown to be sufficient. We now prove the necessity of the conditions 1) "$Q=0$", 2) "$[Q, W_\mu(x)]=0$".

The charge operator Q is defined by [5]

$$Q B\Omega = \int_{-\infty}^{+\infty} ds[\int dx\, f(x+se_{(1)})\, W_1(x), B]\Omega, \quad Q\Omega=0$$

where $f\in\mathscr{S}(R^2)$ and $\int dx\, f(x)=1$.

1. Let $\omega(x)$ satisfy $W_\mu(x)=\partial_\mu\omega(x)$ and be local and relatively local to $\phi_\alpha(x)$ and $\phi^*_\beta(x)$ $\alpha, \beta=1,2,\ldots,l$. We conclude

$$\begin{aligned} QB\Omega &= -\int_{-\infty}^{+\infty} ds[\int dx\, \partial_1 f(x+se_{(1)})\, \omega(x), B]\Omega \\ &= -\int_{-\infty}^{+\infty} ds \frac{d}{ds}[\int dx\, f(x+se_{(1)})\, \omega(x), B]\Omega = 0 . \end{aligned}$$

2. Let $\omega(x)$ satisfy $W_\mu(x)=\partial_\mu\omega(x)$ and be local. We conclude

$$\begin{aligned} [Q, W_\mu(g)]B\Omega &= \int_{-\infty}^{+\infty} ds \frac{d}{ds}[\int dx\, f(x+se_{(1)})\, \omega(x), \omega(\partial_\mu g)]B\Omega \\ &= 0 . \end{aligned}$$

Thus, in addition to the sufficiency, the necessity of the conditions is also proved.

In conclusion we give two examples in one time and one space dimension.

1. Let $\psi(x)$ be a free Dirac field with mass $m>0$, $\gamma^5=\gamma^0\cdot\gamma^1$. $W_\mu(x)=:\tilde{\psi}(x)\,\gamma^5\gamma_\mu\psi(x):$ is a hermitian, local pseudovector field, relatively local with respect to $\psi(x)$ and $\tilde{\psi}(x)$.

$$\mathrm{Curl}\, W_\mu(x)=0 .$$

The charge operator Q corresponding to the conserved current $V^\mu(x)$ defined by Eq. (0.3) is non-trivial. However, Q commutes with $W_\mu(x)$. It follows from our theorem that for $W_\mu(x)$ there exists a local primitive field which however cannot be relatively local to $\psi(x)$ and $\tilde{\psi}(x)$.

2. Let $\psi^a(x)$ and $\psi^b(x)$ be two free independent Dirac fields with equal mass $m>0$.

$$W_\mu(x)=:\tilde{\psi}^a(x)\,\gamma^5\gamma_\mu\psi^a(x):\otimes \mathbb{1}^b + \partial_\mu\{\tilde{\psi}^a(x)\,\gamma^5\otimes\psi^b(x) - \psi^a(x)\,\gamma^{5^T}\otimes\tilde{\psi}^b(x)\}$$

is a hermitian local pseudovector field, relatively local with respect to

$$\psi^a_\alpha(x)\otimes\psi^b_\beta(x),\qquad \tilde\psi^a_\alpha(x)\otimes\psi^b_\beta(x),\qquad \psi^a_\alpha(x)\otimes\tilde\psi^b_\beta(x),\qquad \tilde\psi^a_\alpha(x)\otimes\tilde\psi^b_\beta(x)$$

and $:\tilde\psi^a_\alpha(x)\,\psi^a_\beta(x):\otimes\,\mathbb{1}^b$.

$$\operatorname{Curl} W_\mu(x)=0\,.$$

The charge operator corresponding to the conserved currents $V^\mu(x)$ (cf. Eq. (0.3)) is non-trivial and does not commute with $W_\mu(x)$. According to our theorem, for $W_\mu(x)$ there does not exist a local primitive field.

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References

1. Wightman, A. S.: Introduction to some aspects of the relativistic dynamics of quantized fields. In: Cargese Lectures in Theoretical Physics 1964. New York-London-Paris: Gordon and Breach 1967.
2. Federbush, P.: Phys. Rev. **121**, 1247 (1961); Progr. Theoret. Phys. **26**, 148 (1961).
3. Lehmann, H., Pohlmeyer, K.: Commun. math. Phys. **14**, 78 (1969).
4. Streater, R. F., Wightman, A. S.: PCT, spin and statistics and all that. New York: W. A. Benjamin, Inc. 1964.
5. Robinson, D. W.: Symmetries, broken symmetries, currents and charges. In: Symmetry principles and fundamental particles. Istanbul 1966, San Francisco and London: W. A. Freeman and Co. 1967.
6. Hepp, K.: Commun. math. Phys. **1**, 96 (1965).
7. Araki, H., Haag, R.: Commun. math. Phys. **4**, 77 (1967).
8. Ruelle, D.: Helv. Phys. Acta **35**, 147 (1962); – Araki, H., Hepp, K., Ruelle, D.: Helv. Phys. Acta **35**, 164 (1962).
9. Gelfand, I. M., Schilow, G. E.: Verallgemeinerte Funktionen (Distributionen) II. Berlin: VEB Deutscher Verlag der Wissenschaften 1962.

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