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Spin Waves and the BCS Model

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Abstract. We discuss the behaviour of the BCS model in the limit of infinitely many degrees of freedom. A new limiting procedure, based on spin waves, is proposed, by which the usual convergence difficulties can be overcome.

Introduction

This article is concerned with the behaviour of the Bardeen-Cooper-Schrieffer model [1] in the limit of infinitely many degrees of freedom. Since this problem has already been extensively studied by several authors [2–5], some explanation is needed for the publication of a new paper on this subject.

The method used by the above authors is, in essence, the following: for any finite number, say Ω, of degrees of freedom, the system is determined by a C^*-algebra $\mathfrak{A}_\Omega$ and a Hamiltonian H_Ω. The algebras $\mathfrak{A}_\Omega$ form an ascending series,

$$\mathfrak{A}_\Omega \subseteqq \mathfrak{A}_{\Omega'}$$

if $\Omega < \Omega'$, thus it is possible to define a new C^*-algebra $\mathfrak{A}_\infty$ by

$$\mathfrak{A}_\infty = \text{norm completion of } \bigcup_\Omega \mathfrak{A}_\Omega$$

$\mathfrak{A}_\infty$ is the smallest C^*-algebra containing all $\mathfrak{A}_\Omega$.

Now one constructs suitable representations π of $\mathfrak{A}_\infty$ – mostly the thermodynamic representations [6] which are readily obtained using the results of Thirring and Bogoliubov, Jr. [7] – and asks the following questions:

i) does $\pi(H_\Omega)$ converge, at least on a dense set?
ii) does $\pi(\exp iH_\Omega t)$ converge towards a unitary operator?
iii) does, for $S \in \pi(\mathfrak{A}_\infty)$

$$\pi(\exp iH_\Omega t)\, S\, \pi(\exp - iH_\Omega t)$$

converge and determine an automorphism of the algebra $\mathfrak{A}_\infty$? (This automorphism may, of course, be representation-dependent.)

It turns out that the situation is somewhat paradoxical. For temperature 0, $\pi(H_\Omega)$ converges weakly but not strongly on a dense set towards Bogoliubov's reduced Hamiltonian H_B. Due to this fact, $\pi(\exp iH_\Omega t)$ behaves completely irregularly and does not converge to a unitary operator. Nevertheless, H_B gives the correct time dependence in the sense that

$$\pi(\exp iH_\Omega t)\, S\pi(\exp - iH_\Omega t) \to \exp(iH_B t)\, S \exp(-iH_B t)$$

For finite temperatures, we cannot expect $\pi(H_\Omega)$ to converge in any sense, but, according to Ref. [8], one has to subtract suitable elements $K_\Omega \in (\mathfrak{A}_\infty)'$. Then one arrives at a similar situation: $\pi(H_\Omega) - K_\Omega$ is convergent, but only weakly, $\exp[it(\pi(H_\Omega) - K_\Omega)]$ behaves irregularly, but again $w\text{-}\lim(\pi(H_\Omega) - K_\Omega)$ gives the correct time dependence in the above-mentioned sense.

Thus the usual methods do not permit us to answer questions i) and ii) in a satisfying manner. One might argue that therefore the above described limiting procedure is not adequate to the problem. This idea is also confirmed by the occurence of certain pathologies discovered by Jelinek and Thirring [9] (see Appendix 1).

In this paper we propose another limiting procedure based on spin waves. While in the theory of ferromagnetism the usefulness of the concept of spin waves has been well known for a long time [10], we shall see that they also can be used with success in the BCS theory. Thus the words "spin wave" are not to be understood literally. We shall not consider the usual Bloch spin waves themselves, but shall rather construct a formal analogue of them which bears all the essential features.

As Bogoliubov [2] pointed out, the fundamental quantities in BCS theory are the quasi-particle creation and annihilation operators which are obtained from the ordinary creation and annihilation operators by means of a "Bogoliubov transformation". The BCS ground state (for the infinite system) is the quasi-particle vacuum, and the Hilbert space of the system is spanned by all n-quasi-particle states. For infinite temperatures, roughly speaking, the quasi-particle vacuum is replaced by some "thermal background" and the Hilbert space is spanned by all states differing from this background by a finite number of quasi-particles or holes.

Now we take the view that the above-mentioned states are not of physical relevance, but rather coherent linear combinations of them where each of these states enters with infinitesimal weight. We shall formulate this idea in a mathematically rigorous way, and shall see that our method enables us to give satisfactory answers to all three questions i)–iii).

Just one remark on the strategy of this paper. It is divided into two parts. Part 1 is concerned with generalities on the finite system, and the construction of spin wave operators and vectors (also for the finite case). In Part 2, the limit $\Omega \to \infty$ is performed by means of Trotter's theory of approximating sequences of Hilbert spaces. We shall see that the infinite BCS system occurs in two kinds of phases, normal-conducting ones and super-conducting ones. These phases are not in thermal equilibrium in the usual sense but only in a restricted sense. Nevertheless we obtain "all" thermal Green's functions. Furthermore, it will be possible to decide whether a given phase is stable or not, a problem which lies completely outside the scope of the usual approach.

1. The Finite System

A. Generalities

In the following, we shall deal with the strong coupling version of the BCS model [11] which is particularly simple. It is based on the assumption that an interaction only occurs between pairs formed by electrons placed around the Fermi surface. Thus it seems to be reasonable not to consider the original Hilbert space of the system, but to take into account pair excitations only. The Hilbert space $\mathscr{H}_\Omega$ obtained in this way is a tensor product of Ω two-dimensional Hilbert spaces,

$$\mathscr{H}_\Omega = \boldsymbol{C}^2 \otimes \boldsymbol{C}^2 \otimes \cdots \otimes \boldsymbol{C}^2 . \tag{1.1}$$

In $\mathscr{H}_\Omega$ an orthonormal basis is formed by the vectors

$$|x_1) \otimes |x_2) \otimes \cdots \otimes |x_\Omega) , \tag{1.2}$$

where

$$|x_p) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \forall p .$$

The first alternative states that the p^{th} pair state is occupied, while the second states that this is not the case. As can be seen easily, the number Ω of all pair states is proportional to the volume of the super-conductor.

Let us introduce the operators $\sigma_p^{(\alpha)} (\alpha = 1, 2, 3)$ as Pauli matrices acting on the p^{th} mode of $\mathscr{H}_\Omega$:

$$\sigma_p^{(\alpha)} = 1 \otimes 1 \otimes \cdots \otimes \sigma^{(\alpha)} \otimes \cdots \otimes 1 , \tag{1.3}$$

and let us define the operators $\sigma_p^{\pm}$ in a similar way. Then clearly σ_p^+ annihilates and σ_p^- creates the p^{th} pair. Furthermore, since

$$2\sigma_p^- \sigma_p^+ = 1 - \sigma_p^{(3)}, \tag{1.4}$$

the quantity $\varepsilon(1-\sigma_p^{(3)})$ where ε stands for the Fermi energy, turns out to be the operator of the kinetic energy of a pair.

The BCS Hamiltonian now takes a very simple form

$$H_\Omega = \varepsilon \sum_{p=1}^{\Omega} (1-\sigma_p^{(3)}) - \Omega^{-1} \sum_{p,q=1}^{\Omega} \sigma_p^- \sigma_q^+ . \tag{1.5}$$

Thereby we have chosen our units so that the "critical temperature" is $\frac{1}{2}$. A detailed discussion of this Hamiltonian may be found in the literature stated above. For the "quasi-spin method" just formulated, cf., the paper of Baumann, Eder, Sexl and Thirring [12].

B. Mixed States

If one wants to discuss thermodynamic properties of the system, one has to encounter mixed states. As it is well known (see, e.g., Ref. [8]), it is possible to use a Hilbert space formalism for their description. One simply takes as a new Hilbert space $\mathfrak{H}_\Omega$ the set of all operators acting on $\mathscr{H}_\Omega$ (or, in the general case, if the original Hilbert space is not finite-dimensional, the set of all Hilbert-Schmidt operators) and defines a scalar product by

$$(R, S) \rightsquigarrow \operatorname{tr}(R^* S) . \tag{1.6}$$

The map

$$R \rightsquigarrow SR \tag{1.7}$$

defines an operator $\bar{S}$ acting on $\mathfrak{H}_\Omega$. Similarly

$$R \rightsquigarrow RS \tag{1.8}$$

defines an operator $\tilde{S}$, and thus the commutator map

$$R \rightsquigarrow [S, R] \tag{1.9}$$

defines the operator $\hat{S} = \bar{S} - \tilde{S}$. The map

$$S \rightsquigarrow \bar{S} \tag{1.10}$$

is an isomorphism, while

$$S \rightsquigarrow \tilde{S} \tag{1.11}$$

is an anti-isomorphism.

Since

$$\operatorname{tr} \varrho S = \operatorname{tr} \varrho^{\frac{1}{2}} S \varrho^{\frac{1}{2}} , \tag{1.12}$$

ϱ being the density matrix, the thermal expectation value of the operator S equals in the new language the matrix element of $\bar{S}$ taken with $\varrho^{\frac{1}{2}}$, the latter quantity now being considered as an element of $\mathfrak{H}_\Omega$. In $\mathfrak{H}_\Omega$, the equation for the time evolution of an operator

$$i \frac{d}{dt} R = [H, R] \tag{1.13}$$

takes the form of a Schrödinger equation:

$$i\frac{d}{dt}|R\rangle = \hat{H}|R\rangle \tag{1.14}$$

where $|R\rangle$ shall indicate that the former operator R is now considered as vector in $\mathfrak{H}_\Omega$.

From usual textbooks, see, e.g. [13], one knows that $\mathfrak{H}_\Omega$ can be identified with

$$\mathscr{H}_\Omega \otimes \mathscr{H}_\Omega\,. \tag{1.15}$$

The operator $\overline{S}$ corresponds to

$$S \otimes 1\,, \tag{1.16}$$

while $\tilde{S}$ corresponds to

$$1 \otimes S^T\,. \tag{1.17}$$

Since $\mathscr{H}_\Omega$ is itself a tensor product, we obtain, by rearrangement of the modes, the following final form for $\mathfrak{H}_\Omega$:

$$\mathfrak{H}_\Omega = (C^2 \otimes C^2) \otimes (C^2 \otimes C^2) \otimes \cdots \otimes (C^2 \otimes C^2)\,. \tag{1.18}$$

Then

$$\overline{\sigma}_p^{(\alpha)} = (1 \otimes 1) \otimes \cdots \otimes \underbrace{(\sigma^{(\alpha)} \otimes 1)}_{p^{\text{th}}\,\text{mode}} \otimes \cdots \otimes (1 \otimes 1) \tag{1.19}$$

and

$$\tilde{\sigma}_p^{(\alpha)} = (1 \otimes 1) \otimes \cdots \otimes (1 \otimes \sigma^{(\alpha)T}) \otimes \cdots \otimes (1 \otimes 1)\,. \tag{1.20}$$

Now, of course,

$$\sigma^{(1)T} = \sigma^{(1)}, \qquad \sigma^{(2)T} = -\sigma^{(2)}, \qquad \sigma^{(3)T} = \sigma^{(3)} \tag{1.21}$$

and

$$\sigma^{+T} = \sigma^-, \qquad \sigma^{-T} = \sigma^+\,. \tag{1.22}$$

C. Bogoliubov Transformations

As already mentioned in the Introduction, the main tool in mathematical BCS theory is the Bogoliubov transformation. It takes, in terms of the pair annihilation and creation operators, a very simple form: it is a map

$$\sigma_p^{(\alpha)} \rightsquigarrow \tau_p^{(\alpha)},$$

where

$$\tau_p^{(\alpha)} = M^{\alpha\beta}\sigma_p^{(\beta)}, \tag{1.23}$$

M being a real orthogonal (3,3) matrix. This map clearly being an automorphism, for the "quasi-pair" operators $\tau_p^{(\alpha)}$, the same commutation relations are valid as for the $\sigma_p^{(\alpha)}$'s.

The matrix M depends on two angles Θ and Φ, and reads explicitly as

$$\begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\cos\Theta\,\sin\Phi & \cos\Theta\,\cos\Phi & -\sin\Theta \\ -\sin\Theta\,\sin\Phi & \sin\Theta\,\cos\Phi & \cos\Theta \end{pmatrix}. \tag{1.24}$$

In particular,

$$\begin{aligned} \sigma_p^{(3)} &= -\sin\Theta\,\tau_p^{(2)} + \cos\Theta\,\tau_p^{(3)} \\ 2e^{-i\Phi}\sigma_p^{+} &= \tau_p^{(1)} + i\cos\Theta\,\tau_p^{(2)} + i\sin\Theta\,\tau_p^{(3)} \\ 2e^{i\Phi}\sigma_p^{-} &= \tau_p^{(1)} - i\cos\Theta\,\tau_p^{(2)} - i\sin\Theta\,\tau_p^{(3)}. \end{aligned} \tag{1.25}$$

In the following Sections we shall only consider the τ's. It is obvious that it can be calculated with them in the same way as with the σ's. The vectors

$$|+) = \begin{pmatrix} \cos\Theta/2\,e^{-i\Phi/2} \\ i\sin\Theta/2\,e^{i\Phi/2} \end{pmatrix}$$

and

$$|-) = \begin{pmatrix} -i\sin\Theta/2\,e^{-i\Phi/2} \\ \cos\Theta/2\,e^{i\Phi/2} \end{pmatrix} \tag{1.26}$$

thereby replace the vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ since

$$\begin{aligned} &\tau^{+}|+) = \tau^{-}|-) = 0, \quad \tau^{(3)}|\pm) = \pm|\pm) \\ &\tau^{+}|-) = |+), \quad \tau^{-}|+) = |-). \end{aligned} \tag{1.27}$$

By means of these vectors we can define the quasi-particle vacuum $|\mathrm{vac}\rangle_\Omega$ in $\mathscr{H}_\Omega$:

$$|\mathrm{vac}\rangle_\Omega = |+)\otimes|+)\otimes\cdots\otimes|+) \tag{1.28}$$

and the n-quasi-particle states $|p_1, \ldots, p_n\rangle_\Omega$ by

$$|p_1, \ldots, p_n\rangle_\Omega = \tau_{p_1}^{-} \ldots \tau_{p_n}^{-}|\mathrm{vac}\rangle_\Omega \tag{1.29}$$

($1 \leqq p_i \leqq \Omega$, $p_i \neq p_k$ if $i \neq k$).

All these states together form an orthonormal basis for $\mathscr{H}_\Omega$, replacing the basis described by formula (1.2).

D. Spin Wave Operators

We shall now introduce spin wave operators along the lines of Dyson's paper [14]. The spin wave operators are quantities $\tau_\Omega^{(\alpha)}[\lambda]$ defined by

$$\tau_\Omega^{(\alpha)}[\lambda] = \Omega^{-\frac{1}{2}} \sum_{p=1}^{\Omega} e^{i\lambda p}\tau_p^{(\alpha)} \tag{1.30}$$

where λ is of the form $2\pi k/\Omega$, $k = 0, 1, \ldots, \Omega - 1$. (We shall denote the set of all these λ's by Λ_Ω.)

The commutation relations for the $\tau_\Omega^{(\alpha)}[\lambda]$'s are

$$[\tau_\Omega^{(\alpha)}[\lambda], \tau_\Omega^{(\beta)}[\mu]] = 2i\Omega^{-\frac{1}{2}}\varepsilon^{\alpha\beta\gamma}\tau_\Omega^{(\gamma)}[\lambda+\mu], \tag{1.31}$$

in particular

$$\begin{aligned}[][\tau_\Omega^{(3)}[\lambda], \tau_\Omega^{\pm}[\mu]] &= \pm 2\Omega^{-\frac{1}{2}}\tau_\Omega^{\pm}[\lambda+\mu] \\ [\tau_\Omega^{+}[\lambda], \tau_\Omega^{-}[\mu]] &= \Omega^{-\frac{1}{2}}\tau_\Omega^{(3)}[\lambda+\mu].\end{aligned} \tag{1.32}$$

(We make the convention that expressions like $\lambda+\mu$ or $-\lambda$ should always be understood modulo 2π.)

The index λ is not conserved under conjugation, but transformed into $-\lambda$:

$$\begin{aligned}(\tau_\Omega^{(\alpha)}[\lambda])^* &= \tau_\Omega^{(\alpha)}[-\lambda], \\ (\tau_\Omega^{\pm}[\lambda])^* &= \tau_\Omega^{\mp}[-\lambda].\end{aligned} \tag{1.33}$$

Formula (1.4) tells us that

$$\tau_\Omega^{(3)}[0] = \Omega^{\frac{1}{2}} - 2\Omega^{-\frac{1}{2}}N_\Omega \tag{1.34}$$

where N_Ω denotes the "quasi-pair number operator"

$$\sum_{p=1}^{\Omega} \tau_p^- \tau_p^+ = \sum_{\lambda\in\Lambda_\Omega} \tau_\Omega^-[-\lambda]\,\tau_\Omega^+[\lambda]. \tag{1.35}$$

E. Spin Wave Vectors in $\mathscr{H}_\Omega$

These vectors are obtained by application of the $\tau_\Omega^-[\lambda]$'s to $|\text{vac}\rangle_\Omega$:

$$|\lambda_1, \ldots, \lambda_n\rangle_\Omega = \tau_\Omega^-[\lambda_1] \ldots \tau_\Omega^-[\lambda_n]\,|\text{vac}\rangle_\Omega \tag{1.36}$$

(The λ_i need not be pair-wise different.)

They have more complicated properties than the $|p_1, \ldots, p_n\rangle_\Omega$'s. Firstly, one sees easily that the one-spin wave states are mutually orthogonal and also orthogonal to $|\text{vac}\rangle_\Omega$. However, if $n>1$, one finds only that

$$_\Omega\langle\lambda_1, \ldots, \lambda_n|\mu_1, \ldots, \mu_m\rangle_\Omega = 0 \tag{1.37}$$

if $n \neq m$. If $n=m$, then one can show that [15]:

$$_\Omega\langle\lambda_1, \ldots, \lambda_n|\mu_1, \ldots, \mu_n\rangle_\Omega = O(\Omega^{-1}) \tag{1.38}$$

if the two lists $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$ are essentially different, i.e., cannot be obtained from another one by permutations. (Note that if this were the case, then $|\lambda_1, \ldots, \lambda_n\rangle_\Omega = |\mu_1, \ldots, \mu_n\rangle_\Omega$ since alle $\tau_\Omega^-[\lambda]$'s commute.)

For the norm one finds that

$$\| |\lambda_1, \ldots, \lambda_n\rangle_\Omega \| = (\Pi\, a_\lambda!)^{\frac{1}{2}} + O(\Omega^{-\frac{1}{2}}). \tag{1.39}$$

Here we introduced the multiplicity function a_λ which denotes how often an index λ occurs in the list $(\lambda_1, \ldots, \lambda_n)$.

The action of the spin wave operators on these vectors is given by

$$\tau_\Omega^-[\mu]\,|\lambda_1, \ldots, \lambda_n\rangle_\Omega = |\mu, \lambda_1, \ldots, \lambda_n\rangle_\Omega \tag{1.40}$$

(this follows simply from the definition),

$$\begin{aligned} &\tau_\Omega^+[\mu]\,|\lambda_1, \ldots, \lambda_n\rangle_\Omega \\ &\quad = \delta_{\mu, -\lambda_1}|\hat{\lambda}_1, \lambda_2, \ldots, \lambda_n\rangle_\Omega \\ &\quad\quad + \cdots + \delta_{\mu, -\lambda_n}|\lambda_1, \ldots, \lambda_{n-1}, \hat{\lambda}_n\rangle_\Omega + O(\Omega^{-\frac{1}{2}}) \end{aligned} \tag{1.41}$$

($\hat{\lambda}_i$ means that the index λ_i is suppressed),

$$(\tau_\Omega^{(3)}[\mu] - \Omega^{\frac{1}{2}}\delta_{\mu,0})\,|\lambda_1, \ldots, \lambda_n\rangle_\Omega = O(\Omega^{-\frac{1}{2}}). \tag{1.42}$$

Thus, $\Omega^{-\frac{1}{2}}\tau_\Omega^{(3)}[\mu]$ is almost a c-number:

$$\Omega^{-\frac{1}{2}}\tau_\Omega^{(3)}[\mu]\,|\lambda_1, \ldots, \lambda_n\rangle_\Omega = \delta_{\mu,0}\,|\lambda_1, \ldots, \lambda_n\rangle_\Omega + O(\Omega^{-1}). \tag{1.43}$$

F. Spin Wave Vectors in $\mathfrak{H}_\Omega$. Definition and Simple Properties

The construction of the analogue of the above defined vectors in the spaces $\mathfrak{H}_\Omega$ is rather cumbersome.

To begin with, let us introduce an orthonormal basis for $C^2 \otimes C^2$ consisting of the four vectors

$$\begin{aligned} |\varrho^{(y)}) &= \sqrt{\frac{1+y}{2}}\,|+)\otimes|+) + \sqrt{\frac{1-y}{2}}\,|-)\otimes|-) \\ |\varrho^+) &= |+)\otimes|-) \\ |\varrho^-) &= |-)\otimes|+) \\ |\underline{\varrho}^{(y)}) &= \sqrt{\frac{1-y}{2}}\,|+)\otimes|+) - \sqrt{\frac{1+y}{2}}\,|-)\otimes|-)\,. \end{aligned} \tag{1.44}$$

By means of these vectors, an orthonormal basis for $\mathfrak{H}_\Omega$ is given by the vectors $|p_1, \ldots, p_l; q_1, \ldots, q_m; r_1, \ldots, r_n\rangle_\Omega$ defined as tensor products

$$|x_1)\otimes|x_2)\otimes\cdots\otimes|x_\Omega)$$

where

$$|x_i) = \begin{cases} |\varrho^+) & \text{if} \quad i \in \{p_1, \ldots, p_2\} \\ |\underline{\varrho}^{(y)}) & \text{if} \quad i \in \{q_1, \ldots, q_m\} \\ |\varrho^-) & \text{if} \quad i \in \{r_1, \ldots, r_n\} \\ |\varrho^{(y)}) & \text{otherwise}\,. \end{cases} \tag{1.45}$$

Physically, $|\varrho^{(y)})\otimes|\varrho^{(y)})\otimes\cdots\otimes|\varrho^{(y)})$ would correspond to the density matrix

$$\exp\left(\beta B\sum_{p=1}^{\Omega}\tau_p^{(3)}\right)\Big/\operatorname{tr}\exp\left(\beta B\sum_{p=1}^{\Omega}\tau_p^{(3)}\right) \tag{1.46}$$

the connection between y and β being given by

$$y=\tanh\beta B\,. \tag{1.47}$$

Now the spin wave vectors $|\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega$ are defined by the expression

$$\Omega^{-\frac{l+m+n}{2}}\sum_{\substack{p_1,\ldots,p_l\\ q_1,\ldots,q_m\\ r_1,\ldots,r_n\\ \text{(all different)}}} e^{i(\lambda_1p_1+\cdots+\lambda_lp_l+\mu_1q_1+\cdots+\nu_1r_1+\cdots)}|p_1,\ldots,p_l;q_1,\ldots,q_m;r_1,\ldots,r_n\rangle_\Omega\,. \tag{1.48}$$

Straightforward calculations show that for them similar properties hold as for the $|\lambda_1,\ldots,\lambda_n\rangle_\Omega$'s, namely

$$\begin{aligned}&{}_\Omega\langle\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\\ &\qquad\cdot|\lambda'_1,\ldots,\lambda'_{l'};\mu'_1,\ldots,\mu'_{m'};\nu'_1,\ldots,\nu'_{n'}\rangle_\Omega=0\end{aligned} \tag{1.49}$$

if $l\neq l'$, or $m\neq m'$, or $n\neq n'$. If $l=l'$, $m=m'$, and $n=n'$, one finds that

$$\begin{aligned}&{}_\Omega\langle\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\\ &\qquad\cdot|\lambda'_1,\ldots,\lambda'_l;\mu'_1,\ldots,\mu'_m;\nu'_1,\ldots,\nu'_n\rangle_\Omega=0(\Omega^{-1})\end{aligned} \tag{1.50}$$

if at least one of the three lists $(\lambda_1,\ldots,\lambda_l)$, $(\mu_1,\ldots,\mu_m)$, $(\nu_1,\ldots,\nu_n)$ is essentially different from its dashed counterpart.

Finally, for the norm, one obtains

$$\||\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega\|=(\Pi\, a_\lambda!\,b_\mu!\,c_\nu!)^{\frac{1}{2}}+0(\Omega^{-\frac{1}{2}}) \tag{1.51}$$

where a_λ, b_μ and c_ν are the multiplicity functions for the three lists.

G. The Effect of the Spin Wave Operators

The generalizations of formulae (1.40)–(1.43) are

$$\begin{aligned}&\bar{\tau}_\Omega^{\pm}[\varkappa]\,|\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega\\ &=\sqrt{\frac{1-y}{2}}\,|\lambda_1,\ldots,\lambda_l,\varkappa;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega\\ &\quad+\sqrt{\frac{1+y}{2}}\,(\delta_{\varkappa,-\nu_1}|\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\hat{\nu}_1,\nu_2,\ldots,\nu_n\rangle_\Omega+\cdots)\\ &\quad+0(\Omega^{-\frac{1}{2}})\end{aligned} \tag{1.52}$$

$$\begin{aligned}
&\bar{\tau}_\Omega^-[\varkappa]\,|\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega \\
&\quad=\sqrt{\frac{1+y}{2}}\,|\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n,\varkappa\rangle_\Omega \\
&\quad=+\sqrt{\frac{1-y}{2}}\,(\delta_{\varkappa,-\lambda_1}|\hat{\lambda}_1,\lambda_2,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega+\cdots) \\
&\qquad+0(\Omega^{-\frac{1}{2}})
\end{aligned} \tag{1.53}$$

$$\begin{aligned}
&(\bar{\tau}_\Omega^{(3)}[\varkappa]-y\Omega^{\frac{1}{2}}\delta_{\varkappa,0})\,|\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega \\
&\quad=\sqrt{1-y^2}\,|\lambda_1,\ldots,\lambda_l;\mu_1,\ldots,\mu_m,\varkappa;\nu_1,\ldots,\nu_n\rangle_\Omega \\
&\qquad+\sqrt{1-y^2}\,(\delta_{\varkappa,-\mu_1}|\lambda_1,\ldots,\lambda_l;\hat{\mu}_1,\mu_2,\ldots,\mu_m;\nu_1,\ldots,\nu_n\rangle_\Omega+\cdots) \\
&\qquad+0(\Omega^{-\frac{1}{2}})\,.
\end{aligned} \tag{1.54}$$

The corresponding formula for the $\tilde{\tau}_\Omega^{(\alpha)}[\varkappa]$'s would be rather complicated. However, observing that the behaviour of the σ's under transposition is very simple, it suffices to calculate the effect of the operators $(\tilde{\tau}_\Omega^{(\alpha)}[\varkappa])^T$ on the spin wave vectors. For simplicity, we shall write $\eta_\Omega^{(\alpha)}[\varkappa]$ instead of $(\tilde{\tau}_\Omega^{(\alpha)}[\varkappa])^T$.

Then, one only has to substitute

$$\begin{gathered}
\bar{\tau}_\Omega^+[\varkappa]\to\eta_\Omega^-[\varkappa]\,, \\
\bar{\tau}_\Omega^-[\varkappa]\to\eta_\Omega^+[\varkappa]\,, \\
y\to-y
\end{gathered} \tag{1.55}$$

in formulae (1.50) and (1.51), and

$$\bar{\tau}_\Omega^{(3)}[\varkappa]\to\eta_\Omega^{(3)}[\varkappa] \tag{1.56}$$

(y remaining unchanged), in formula (1.52) in order to get the new relations.

The connection between the $\tilde{\sigma}$'s and the η's is established by the formula

$$\tilde{\sigma}_\Omega^{(\alpha)}[\lambda]=L^{\alpha\beta}\eta_\Omega^{(\beta)}[\lambda]\,, \tag{1.57}$$

where the matrix L is related to the matrix M by

$$L^{\alpha\beta}=(-)^{\alpha-1}M^{\beta\alpha}\,. \tag{1.58}$$

Thus, in particular

$$\begin{aligned}
\tilde{\sigma}^{(3)}&=-\sin\Theta\,\eta^{(2)}+\cos\Theta\,\eta^{(3)} \\
2e^{i\Phi}\tilde{\sigma}^+&=\eta^{(1)}-i\cos\Theta\,\eta^{(2)}-i\sin\Theta\,\eta^{(3)} \\
2e^{-i\Phi}\tilde{\sigma}^-&=\eta^{(1)}+i\cos\Theta\,\eta^{(2)}+i\sin\Theta\,\eta^{(3)}
\end{aligned} \tag{1.59}$$

(we suppressed the index λ).

Finally, let us consider the operator

$$N_\Omega^{\mathrm{rel}} = -\tfrac{1}{2} \cdot \Omega^{\frac{1}{2}} (\bar{\tau}_\Omega^{(3)}[0] - \eta_\Omega^{(3)}[0]) . \tag{1.60}$$

This operator can be interpreted as "relative quasi-pair number operator" since

$$\begin{aligned} N_\Omega^{\mathrm{rel}} &|\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; \nu_1, \ldots, \nu_n\rangle_\Omega \\ &= (n-l)\,|\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; \nu_1, \ldots, \nu_n\rangle_\Omega . \end{aligned} \tag{1.61}$$

2. Transition to Infinitely many Degrees of Freedom

A. Trotter's Theory

Formulae (1.38) and (1.39) can be interpreted in such a way that, for $\Omega \to \infty$, the spin wave vectors $|\lambda_1, \ldots, \lambda_n\rangle_\Omega$ approach an orthogonal system, the norm of each of these vectors tending to $(\Pi a_\lambda !)^{\frac{1}{2}}$. A more sophisticated formulation of this idea is obtained by the use of Trotter's theory of "approximating sequences of Hilbert spaces" [16].

We say that a sequence of Hilbert spaces $\mathscr{H}_\Omega$ (which, for the moment, need not be the spaces considered in Part 1) is approximating a Hilbert space $\mathscr{H}_\infty$ if there exists a sequence of linear maps Φ_Ω: $\mathscr{H}_\infty \to \mathscr{H}_\Omega$ such that, for any vector $|x\rangle_\infty \in \mathscr{H}_\infty$ we have

$$\|\Phi_\Omega |x\rangle_\infty\|_\Omega \to \||x\rangle_\infty\|_\infty \tag{2.1}$$

and, if, in addition:

$$\sup_\Omega \|\Phi_\Omega\| < \infty . \tag{2.2}$$

[In order to avoid misunderstandings, we write $\|\cdots\|_\Omega$ for the norm in $\mathscr{H}_\Omega$, $\|\cdots\|_\infty$ for the norm in $\mathscr{H}_\infty$. The norm of Φ_Ω is defined by

$$\sup_{|x\rangle_\infty} \|\Phi_\Omega |x\rangle_\infty\|_\Omega \; \||x\rangle_\infty\|_\infty^{-1} .]$$

Now the usual definition of strong convergence of a sequence of vectors can be generalized as follows: we shall say that a sequence of vectors $|x\rangle_\Omega \in \mathscr{H}_\Omega$ is (strongly) convergent towards $|x\rangle_\infty \in \mathscr{H}_\infty$ – in which case we shall write $|x\rangle_\Omega \to |x\rangle_\infty$ – if

$$\||x\rangle_\Omega - \Phi_\Omega |x\rangle_\infty\|_\Omega \to 0 . \tag{2.3}$$

A sequence of bounded operators S_Ω in $\mathscr{H}_\Omega$ is said to be convergent towards a not necessarily bounded operator S_∞ in $\mathscr{H}_\infty$ – $S_\Omega \to S_\infty$ – if, for any $|x\rangle_\infty$ in the domain of S_∞ we have

$$S_\Omega \Phi_\Omega |x\rangle_\infty \to S_\infty |x\rangle_\infty . \tag{2.4}$$

The main result of Trotter's theory is: suppose that the sequence of self-adjoint operators $\boldsymbol{S}_\Omega$ converges towards the essentially self-adjoint operator $\boldsymbol{S}_\infty$. Then, for any bounded continuous function f,

$$f(\boldsymbol{S}_\Omega) \to f(\boldsymbol{S}_\infty^*)\,. \tag{2.5}$$

Thus, in particular

$$\exp(i\boldsymbol{S}_\Omega t) \to \exp(i\boldsymbol{S}_\infty^* t)$$

for real t.

B. Application to the Spaces $\mathscr{H}_\Omega$ and $\mathfrak{H}_\Omega$

It is almost obvious how Trotter's theory applies to the sequences $\mathscr{H}_\Omega$ and $\mathfrak{H}_\Omega$. Let us first construct the limiting space $\mathscr{H}_\infty$ of the sequence $\mathscr{H}_\Omega$. (In order to avoid immaterial complication, Ω shall run only through the powers of two.)

In $\mathscr{H}_\infty$ an orthogonal basis will be given by a vector

$$|\mathrm{vac}\rangle_\infty$$

and vectors

$$|\lambda_1, \ldots, \lambda_n\rangle_\infty \tag{2.6}$$

where the λ_i belong to the set $\Lambda = \cup \Lambda_\Omega$ which consists of all numbers of the form $2\pi \cdot k \cdot 2^{-j}$. Thereby two vectors

$$|\lambda_1, \ldots, \lambda_n\rangle_\infty \quad \text{and} \quad |\mu_1, \ldots, \mu_n\rangle_\infty$$

are considered to be equal if the two lists $(\lambda_1, \ldots, \lambda_n)$ and $(\mu_1, \ldots, \mu_n)$ are not essentially different.

The norm of $|\mathrm{vac}\rangle_\infty$ is 1, the norm of $|\lambda_1, \ldots, \lambda_n\rangle_\infty$ is $(\Pi a_\lambda!)^{\frac{1}{2}}$. The maps Φ_Ω are determined by

$$\Phi_\Omega |\lambda_1, \ldots, \lambda_n\rangle_\infty = \begin{cases} |\lambda_1, \ldots, \lambda_n\rangle_\Omega & \text{if this vector is defined in } \mathscr{H}_\Omega, \text{ i.e., all } \lambda_i \text{ belong to } \Lambda_\Omega, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.7}$$

The validity of relations (2.1) and (2.2) follows readily from formulae (1.38) and (1.39) and the relation

$$\|\Phi_\Omega\| = 1\,\forall\,\Omega \tag{2.8}$$

which is derived in Appendix 2.

The above considerations can be generalized easily to the case of the sequence $\mathfrak{H}_\Omega$. It will be approximating a space $\mathfrak{H}^{(y)}$ in which an orthonormal basis is given by vectors

$$(\Pi a_\lambda!\, b_\mu!\, c_\nu!)^{-\frac{1}{2}} |\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; \nu_1, \ldots, \nu_n\rangle_\infty \tag{2.9}$$

where again we disregard the order of the indices within the three lists $(\lambda_1, \ldots)$, $(\mu_1, \ldots)$ and $(\nu_1, \ldots)$.

The maps Φ_Ω are to be replaced by the maps $\Phi_\Omega^{(y)}$ determined by

$$\Phi_\Omega^{(y)} |\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; \nu_1, \ldots, \nu_n\rangle_\infty = \begin{cases} |\lambda_1, \ldots, \lambda_l; \mu_1, \ldots, \mu_m; \nu_1, \ldots, \nu_n\rangle_\Omega & \text{if this vector is defined in } \mathfrak{H}_\Phi, \\ 0 & \text{otherwise}. \end{cases} \tag{2.10}$$

As before it can be shown (cf., [15]) that conditions (2.1) and (2.2) are fulfilled.

C. The Limit of the Spin Wave Operators

It is readily seen that the sequences of operators $\tau_\Omega^\pm[\mu]$ are convergent in the sense of Trotter, their limits $\tau_\infty^\pm[\mu]$ being defined on the set of all finite linear combinations of $|\lambda_1, \ldots, \lambda_n\rangle_\infty$'s. (Unless otherwise stated, we shall always understand that the domain of operators acting on $\mathscr{H}_\infty$ is this set.) Explicitly, the $\tau_\infty^\pm[\mu]$ are given by

$$\tau_\infty^-[\mu] |\lambda_1, \ldots, \lambda_n\rangle_\infty = |\mu, \lambda_1, \ldots, \lambda_n\rangle_\infty$$

and

$$\tau_\infty^+[\mu] |\lambda_1, \ldots, \lambda_n\rangle_\infty = \delta_{\mu, -\lambda_1} |\hat{\lambda}_1, \lambda_2, \ldots, \lambda_n\rangle_\infty + \cdots \tag{2.11}$$

Furthermore

$$\tau_\Omega^{(3)}[\mu] - \Omega^{\frac{1}{2}} \delta_{\mu,0} \to 0\,. \tag{2.12}$$

With some elaboration, one also can show that

$$A_\Omega^{(1)} \ldots A_\Omega^{(n)} \to A_\infty^{(1)} \ldots A_\infty^{(n)} \tag{2.13}$$

where $A_\Omega^{(i)}$ stands for any of the above operators $\tau_\Omega^\pm[\mu]$ or $(\tau_\Omega^{(3)}[\mu] - \Omega^{\frac{1}{2}} \delta_{\mu,0})$, and

$$A_\infty^{(i)} = \lim A_\Omega^{(i)}$$

(cf. [15]).

Thus the $\tau_\infty^\pm[\mu]$ obey boson commutation relations:

$$[\tau_\infty^+[\mu], \tau_\infty^-[\lambda]] = \delta_{\lambda, -\mu}\,. \tag{2.14}$$

The Hermitian combinations

$$1/\sqrt{2}(\tau_\infty^+[\mu] + \tau_\infty^-[-\mu])$$

and

$$i/\sqrt{2}(\tau_\infty^+[\mu] - \tau_\infty^-[-\mu]) \tag{2.15}$$

are essentially self-adjoint (this follows from standard arguments, see, e.g., Putnam [17], Chapter 4) and may be considered as some sort of coordinate and momentum operators of the spin wave with frequency μ.

Finally, we remark that the sequence N_Ω converges towards N_∞ given by

$$N_\infty |\lambda_1, \ldots, \lambda_n\rangle_\infty = n |\lambda_1, \ldots, \lambda_n\rangle_\infty \tag{2.16}$$

hence

$$N_\infty = \sum_{\lambda \in \Lambda} \tau_\infty^-[-\lambda]\, \tau_\infty^+[\lambda]. \tag{2.17}$$

Analogous results are true for the operators $\bar{\tau}_\Omega^{(\alpha)}[\varkappa]$ and $\eta_\Omega^{(\alpha)}[\varkappa]$. We find that

$$\bar{\tau}_\Omega^\pm[\varkappa] \to \bar{\tau}_{\infty,y}^\pm[\varkappa]. \tag{2.18}$$

The operator on the right-hand side being defined on the set of all finite linear combinations of $|\lambda_1, \ldots, \lambda_l;\ \mu_1, \ldots, \mu_m;\ \nu_1, \ldots, \nu_n\rangle_\infty$'s. (Again we shall always understand that operators acting on $\mathfrak{H}_\infty^{(y)}$ are defined on this domain.) The effect of the $\bar{\tau}_{\infty,y}^\pm[\varkappa]$ on the $|\lambda_1, \ldots, \lambda_l;\ \mu_1, \ldots, \mu_m;$ $\nu_1, \ldots, \nu_n\rangle_\infty$'s is easily deduced from formulae (1.50) and (1.51).

Also

$$\bar{\tau}_\Omega^{(3)}[\varkappa] \to \bar{\tau}_{\infty,y}^{(3)}[\varkappa] \tag{2.19}$$

if $\varkappa \neq 0$. In contrast to the sequence $\tau_\Omega^{(3)}[\varkappa]$, the limits $\bar{\tau}_{\infty,y}^{(3)}[\varkappa]$ are not zero but can equally be easily deduced from formula (1.52). This formula also tells us that the sequence $(\bar{\tau}_\Omega^{(3)}[0] - y\Omega^{\frac{1}{2}})$ is convergent:

$$(\bar{\tau}_\Omega^{(3)}[0] - y\Omega^{\frac{1}{2}}) \to \chi_{\infty,y}. \tag{2.20}$$

One obtains thus the commutation relations

$$[y^{-\frac{1}{2}}\bar{\tau}_{\infty,y}^+[\varkappa],\ y^{-\frac{1}{2}}\bar{\tau}_{\infty,y}^-[\varkappa']] = \delta_{\varkappa,-\varkappa'} \tag{2.21}$$

and

$$[\bar{\tau}_{\infty,y}^\pm[\varkappa],\ \bar{\tau}_{\infty,y}^{(3)}[\varkappa']] = [\bar{\tau}_{\infty,y}^\pm[\varkappa],\ \chi_{\infty,y}] = 0 \quad (\varkappa' \neq 0). \tag{2.22}$$

Again it can be shown that the Hermitian combinations of $\bar{\tau}_{\infty,y}^\pm[\varkappa]$'s are essentially self-adjoint, as well as the $\bar{\tau}_{\infty,y}^{(3)}[\varkappa]$'s and $\chi_{\infty,y}$ are. The algebra generated by all these operators is of type III. Similar statements are true for the η's.

Concerning the sequence N_Ω^{rel}, it is plain that its limit $N_{\infty,y}^{\mathrm{rel}}$ exists and equals

$$y^{-1} \sum_{\varkappa \in \Lambda} (\bar{\tau}_{\infty,y}^-[-\varkappa]\,\bar{\tau}_{\infty,y}^+[\varkappa] - \eta_{\infty,y}^-[-\varkappa]\,\eta_{\infty,y}^+[\varkappa]). \tag{2.23}$$

D. The Ground State of the BCS Model

The first problem we want to discuss in our spin wave formalism is that of the ground state of the BCS model in case of infinitely many degrees of freedom. We shall show that, if the angles Θ and Φ are properly chosen, the Hamiltonians H_Ω converge towards an essentially self-adjoint operator H_∞, and consequently,

$$\exp(iH_\Omega t) \to \exp(iH_\infty^* t) \tag{2.24}$$

so that the well-known difficulties of the usual treatment (cf., the Introduction) do not occur in our new method.

In terms of the spin wave operators, H_Ω can be written as

$$-\varepsilon\Omega^{\frac{1}{2}}\sigma_\Omega^{(3)}[0] - \sigma_\Omega^-[0]\,\sigma_\Omega^+[0] \tag{2.25}$$

(here, and in the following, we shall always skip irrelevant c-numbers). Thus after performing a Bogoliubov transformation, we arrive at the expression

$$H_\Omega = \mathrm{I}_\Omega + \mathrm{II}_\Omega + \mathrm{III}_\Omega + \mathrm{IV}_\Omega \tag{2.26}$$

where

$$\mathrm{I}_\Omega = -\varepsilon\cos\Theta\,\Omega^{\frac{1}{2}}\tau_\Omega^{(3)}[0] - \tfrac{1}{4}\sin^2\Theta\,(\tau_\Omega^{(3)}[0])^2, \tag{2.27}$$

$$\mathrm{II}_\Omega = -\tfrac{1}{4}(\tau_\Omega^{(1)}[0])^2 - \tfrac{1}{4}\cos^2\Theta\,(\tau_\Omega^{(2)}[0])^2, \tag{2.28}$$

$$\mathrm{III}_\Omega = \tfrac{1}{2}\Omega^{-\frac{1}{2}}\cos\Theta\,\tau_\Omega^{(3)}[0] - \tfrac{1}{2}\Omega^{-\frac{1}{2}}\sin\Theta\,\tau_\Omega^{(2)}[0], \tag{2.29}$$

$$\begin{aligned}\mathrm{IV}_\Omega = {} & \varepsilon\Omega^{\frac{1}{2}}\sin\Theta\,\tau_\Omega^{(2)}[0] \\ & -\tfrac{1}{4}\sin\Theta\cos\Theta\,(\tau_\Omega^{(2)}[0]\,\tau_\Omega^{(3)}[0] + \tau_\Omega^{(3)}[0]\,\tau_\Omega^{(2)}[0]).\end{aligned} \tag{2.30}$$

Now always

$$\mathrm{III}_\Omega \to \tfrac{1}{2}\cos\Theta, \tag{2.31}$$

$$\mathrm{II}_\Omega \to -\tfrac{1}{4}(\tau_\infty^{(1)}[0])^2 - \tfrac{1}{4}\cos^2\Theta\,(\tau_\infty^{(2)}[0])^2, \tag{2.32}$$

whereas

$$\mathrm{I}_\Omega \to (2\varepsilon\cos\Theta + \sin^2\Theta)\,N_\infty. \tag{2.33}$$

The crucial quantity is IV_Ω. We obtain convergence only if either $\sin\Theta = 0$ or

$$2\varepsilon = \cos\Theta, \tag{2.34}$$

the latter condition will be referred to as "gap equation".

If $\sin\Theta = 0$, then $\lim H_\Omega$ exists but is not semi-bounded from below. Thus we do not obtain a description of the ground state of the infinite system. We are rather dealing with the "normal conducting phase" which is unstable at low temperatures. We shall discuss this question in more detail in Section G.

If, however, the gap equation holds, then H_Ω converges towards

$$\tfrac{1}{4}\sin^2\Theta\,(\tau_\infty^{(2)}[0])^2 + \sum_{\lambda\neq 0}\tau_\infty^-[-\lambda]\,\tau_\infty^+[\lambda] \tag{2.35}$$

("superconducting phase").

This result can be interpreted in the following way: in the infinite case the spin waves behave like a system of independently moving bosons, indexed by the set Λ. The operators $\tau_\infty^+[\lambda]$ and $\tau_\infty^-[-\lambda]$ annihilate or create, respectively, the λ^{th} boson. Thereby, all but one "particles" behave like harmonic oscillators with frequency one, while the remaining one behaves like a free particle.

There is an essential difference between Bogoliubov's reduced Hamiltonian and the Hamiltonian H_∞ which we have obtained above. Bogoliubov's Hamiltonian has a pure point spectrum, whereas H_∞ contains also a continuum in its spectrum. This continuum contribution is caused by the spin wave with frequency 0. We shall discuss later on what this means physically.

E. The Limit of the Sequence $\hat{H}_\Omega(y<1)$

By virtue of formula (1.57), we can decompose $\hat{H}_\Omega$ in an analogous manner as before H_Ω:

$$\hat{H}_\Omega = \mathrm{I}'_\Omega + \mathrm{II}'_\Omega + \mathrm{III}'_\Omega + \mathrm{IV}'_\Omega . \tag{2.36}$$

The connection between I_Ω and I'_Ω (and similarly between the other terms) is given by

$$\mathrm{I}'_\Omega = \bar{\mathrm{I}}_\Omega - \mathrm{I}''_\Omega \tag{2.37}$$

[cf., Eq. (1.7)], I''_Ω being obtained from $\bar{\mathrm{I}}_\Omega$ by substituting

$$\bar{\tau}_\Omega^{(\alpha)}[\lambda] \to \eta_\Omega^{(\alpha)}[\lambda] . \tag{2.38}$$

Now

$$\mathrm{I}'_\Omega \to (2\varepsilon \cos\Theta + y \sin^2\Theta)\, N_{\infty,y}^{\mathrm{rel}} \tag{2.39}$$

$$\begin{aligned} \mathrm{II}'_\Omega \to & -\tfrac{1}{4}[(\bar{\tau}_{\infty,y}^{(1)}[0])^2 - (\eta_{\infty,y}^{(1)}[0])^2] \\ & -\tfrac{1}{4}\cos^2\Theta\,[(\bar{\tau}_{\infty,y}^{(2)}[0])^2 - (\eta_{\infty,y}^{(2)}[0])^2] . \end{aligned} \tag{2.40}$$

$$\mathrm{III}'_\Omega \to 0 . \tag{2.41}$$

As before, the crucial quantity is IV'_Ω. In order to get convergence, we must require that either $\sin\Theta = 0$ ("normal-conducting phase") or

$$2\varepsilon = y\cos\Theta \tag{2.42}$$

("super-conducting phase"), the latter equation will be referred to as "temperature dependent gap equation". The reason for this name will become clear in the next Section.

If (2.42) is fulfilled, IV'_Ω converges to

$$-\tfrac{1}{2}\sin\Theta\cos\Theta\,\chi_{\infty,y}(\bar{\tau}_{\infty,y}^{(2)}[0] - \eta_{\infty,y}^{(2)}[0]) \tag{2.43}$$

and thus $\hat{H}_\Omega$ converges to

$$\hat{H}_{\infty,y}^{sc(0)} + \hat{H}_{\infty,y}^{sc(1)} \tag{2.44}$$

where $\hat{H}^{sc(0)}_{\infty,y}$ equals

$$\begin{aligned}\tfrac{1}{4}\sin^2\Theta\,&[(\bar{\tau}^{(2)}_{\infty,y}[0])^2-(\eta^{(2)}_{\infty,y}[0])^2]\\&-\tfrac{1}{2}\sin\Theta\cos\Theta\,[\bar{\tau}^{(2)}_{\infty,y}[0]-\eta^{(2)}_{\infty,y}[0]]\,\chi_{\infty,y}\end{aligned}\tag{2.45}$$

and $\hat{H}^{sc(1)}_{\infty,y}$ equals

$$\sum_{\lambda\neq 0}(\bar{\tau}^{-}_{\infty,y}[-\lambda]\,\bar{\tau}^{+}_{\infty,y}[\lambda]-\eta^{-}_{\infty,y}[-\lambda]\,\eta^{+}_{\infty,y}[\lambda])\,.\tag{2.46}$$

Therefore the following equations of motion are valid:

$$\begin{aligned}&\frac{d^2}{dt^2}\bar{\tau}^{(\alpha)}_{\infty,y}[\lambda]=-y\bar{\tau}^{(\alpha)}_{\infty,y}[\lambda]\qquad(\alpha=1,2)\\&\frac{d}{dt}\bar{\tau}^{(3)}_{\infty,y}[\lambda]=0\end{aligned}\tag{2.47}$$

if $\lambda\neq 0$, and

$$\begin{aligned}&\frac{d^2}{dt^2}\bar{\tau}^{(\alpha)}_{\infty,y}[0]=0\qquad(\alpha=1,2)\\&\frac{d}{dt}\chi_{\infty,y}=0\,.\end{aligned}\tag{2.48}$$

In the normal-conducting case we find similarly

$$\hat{H}_\Omega\rightarrow\hat{H}^{nc(0)}_{\infty,y}+\hat{H}^{nc(1)}_{\infty,y}\,,\tag{2.49}$$

where $\hat{H}^{nc(0)}_{\infty,y}$ equals

$$(2\varepsilon y^{-1}-1)\,[\bar{\tau}^{-}_{\infty,y}[0]\,\bar{\tau}^{+}_{\infty,y}[0]-\eta^{-}_{\infty,y}[0]\,\eta^{+}_{\infty,y}[0]]\tag{2.50}$$

and

$$\hat{H}^{nc(1)}_{\infty,y}=2\varepsilon y^{-1}\sum_{\lambda\neq 0}(\bar{\tau}^{-}_{\infty,y}[-\lambda]\,\bar{\tau}^{+}_{\infty,y}[\lambda]-\eta^{-}_{\infty,y}[-\lambda]\,\eta^{+}_{\infty,y}[\lambda])\,.\tag{2.51}$$

The first group of equations of motion are obtained from Eq. (2.47) by changing y into 2ε, while the second group is now

$$\begin{aligned}&\frac{d^2}{dt^2}\bar{\tau}^{(\alpha)}_{\infty,y}[0]=(y-2\varepsilon)\,\bar{\tau}^{(\alpha)}_{\infty,y}[0]\qquad(\alpha=1,2)\\&\frac{d}{dt}\chi_{\infty,y}=0\,.\end{aligned}\tag{2.52}$$

F. Thermodynamic Properties

We are now going to discuss the question whether the operators $\bar{\tau}^{(\alpha)}_{\infty,y}[\lambda]$ and $\chi_{\infty,y}$ are describing an infinite system in thermal equilibrium. This would be the case if the KMS condition

$$\int f(t-i\beta)\,\langle BA_t\rangle\,dt=\int f(t)\,\langle A_tB\rangle\,dt\tag{2.53}$$

(where the function f is the Fourier transform of a class $\mathscr{D}$ test function) were true. Thereby, A and B denote polynomials in the above-mentioned operators,

$$A_t = e^{i\hat{H}_\infty t} A e^{-i\hat{H}_\infty t} \tag{2.54}$$

[$\hat{H}_\infty$ stands either for the normal-conducting Hamiltonian (2.49) or for the super-conducting one (2.44)], and $\langle \ldots \rangle$ means the matrix element with the spin wave with $l = m = n = 0$.

Now it is clear that in general the KMS condition must be violated since the $\bar{\tau}^{(\alpha)}_{\infty,y}[0]$ ($\alpha = 1, 2$) behave in the super-conducting case like a free particle, whereas we meet a runaway situation if in the normal conducting case $2\varepsilon < y$. However, this "non-thermal" behaviour of the spin wave operators referring to the 0 mode is not too surprising. In fact, the Hamiltonian of the finite system is invariant under all permutations of the indices p, as well as the operators $\sigma^{(\alpha)}_\Omega[0]$ are. Comparing this situation with ordinary many-body systems, one sees that the $\sigma^{(\alpha)}_\Omega[0]$ are just the analogues of the centre-of-mass variables, and thus have to be disregarded in questions of thermal equilibrium. We shall discuss this point in detail in the next Section.

It is readily verified that the KMS condition is true if A and B do not contain spin wave operators referring to the 0 mode. (See also Verbeure and Verboven [18].) Hence we can say that all $\lambda \neq 0$ modes are in thermal equilibrium at a certain temperature which is uniquely determined by (2.53).

To prove this, let us first assume that

$$A = y^{-\frac{1}{2}} \bar{\sigma}^{-}_{\infty,y}[-\lambda], \qquad B = y^{-\frac{1}{2}} \bar{\sigma}^{+}_{\infty,y}[\lambda]$$

and compare the matrix elements

$$\langle A^n B^n \rangle \tag{2.55}$$

with the expression

$$(1 - e^{-\beta}) \operatorname{tr}[e^{-\beta a^* a} (a^*)^n a^n] \tag{2.56}$$

where the a and a^* are the ordinary annihilation and creation operators from quantum mechanics. Once this has been done, the general result is easily deduced and it is thus found that the KMS condition in the above stated restricted form is true if

$$y = \tanh \beta\varepsilon \tag{2.57}$$

in the normal-conducting case, and

$$y = \tanh \beta y/2 \tag{2.58}$$

in the super-conducting case. These relations are in agreement with Thirring's results [7].

Finally, it remains to establish the connection between the thermal Green's functions for the finite system

$$\langle \sigma_{p_1}^{(\alpha_1)} \ldots \sigma_{p_n}^{(\alpha_n)} \rangle_{\beta,\Omega} = \operatorname{tr}(e^{-\beta H_\Omega} \sigma_{p_1}^{(\alpha_1)} \ldots \sigma_{p_n}^{(\alpha_n)}) / \operatorname{tr} e^{-\beta H_\Omega} \tag{2.59}$$

and our Green's functions

$$\langle \bar{\sigma}_{\infty,y}^{(\alpha_1)}[\lambda_1] \ldots \bar{\sigma}_{\infty,y}^{(\alpha_n)}[\lambda_n] \rangle .$$

A simple combinatorial argument shows that

$$\begin{aligned} &\lim_{\Omega\to\infty} \sum_{p_1,\ldots,p_n=1}^{\Omega} e^{i(\lambda_1 p_1 + \cdots)} \langle \sigma_{p_1}^{(\alpha_1)} \ldots \sigma_{p_n}^{(\alpha_n)} \rangle_{\beta,\Omega} \\ &= \lim_{\Omega\to\infty} \sum_{p_1,\ldots,p_n=1}^{\Omega} e^{i(\lambda_1 p_1 + \cdots)} \lim_{\Omega\to\infty} \langle \sigma_{p_1}^{(\alpha_1)} \ldots \sigma_{p_n}^{(\alpha_n)} \rangle_{\beta,\Omega} \end{aligned} \tag{2.60}$$

where all λ_i have to be different from 0. Utilizing the standard results on

$$\lim_{\Omega\to\infty} \langle \sigma_{p_1}^{(\alpha_1)} \ldots \sigma_{p_n}^{(\alpha_n)} \rangle_{\beta,\Omega}$$

[7], one finds

$$\lim_{\Omega\to\infty} \langle \sigma_{\Omega}^{(\alpha_1)}[\lambda_1] \ldots \sigma_{\Omega}^{(\alpha_n)}[\lambda_n] \rangle_{\beta,\Omega} = \langle \bar{\sigma}_{\infty,y}^{(\alpha_1)}[\lambda_1] \ldots \bar{\sigma}_{\infty,y}^{(\alpha_n)}[\lambda_n] \rangle \tag{2.61}$$

if β is smaller than the inverse critical temperature β_0 determined by

$$2\varepsilon = \tanh \beta_0 \varepsilon . \tag{2.62}$$

(The matrix element on the right-hand side of formula (2.61) refers to the normal-conducting phase.)

If $\beta > \beta_0$,

$$\lim_{\Omega\to\infty} \langle \sigma_{\Omega}^{(\alpha_1)}[\lambda_1] \ldots \sigma_{\Omega}^{(\alpha_n)}[\lambda_n] \rangle_{\beta,\Omega} = \int \frac{d\Phi}{2\pi} \langle \bar{\sigma}_{\infty,y}^{(\alpha_1)}[\lambda_1] \ldots \bar{\sigma}_{\infty,y}^{(\alpha_n)}[\lambda_n] \rangle \tag{2.63}$$

the matrix element in the integrand referring to the superconducting phase. Since the gap equation only fixes Θ, the angle Φ remains free and we have to average over all possibilities.

It can also be shown [15] that any of the expressions

$$\lim_{\Omega\to\infty} \langle \sigma_{p_1}^{(\alpha_1)} \ldots \sigma_{p_n}^{(\alpha_n)} \rangle_{\beta,\Omega}$$

can be constructed from the

$$\langle \bar{\sigma}_{\infty,y}^{(\alpha_1)}[\lambda_1] \ldots \bar{\sigma}_{\infty,y}^{(\alpha_n)}[\lambda_n] \rangle$$

or

$$\int \frac{d\Phi}{2\pi} \langle \bar{\sigma}_{\infty,y}^{(\alpha_1)}[\lambda_1] \ldots \bar{\sigma}_{\infty,y}^{(\alpha_n)}[\lambda_n] \rangle .$$

respectively, ($\lambda_i \neq 0$). Therefore the thermodynamic information yielded by the $\lambda \neq 0$ modes is as complete as the information which one obtains in the usual treatment.

G. Stability of Phases

As a last point we have to discuss the physical meaning of $\hat{H}^{nc(0)}_{\infty,y}$ and $\hat{H}^{sc(0)}_{\infty,y}$. We shall see that there is a close connection between these quantities and the stability of phases.

To begin with, let us consider the normal conducting phase at high temperatures, i.e., if $y<2\varepsilon$. Then the equations of motion (2.52) tell us that not only

$$\langle \bar{\tau}^{(1)}_{\infty,y}[0]\rangle = \langle \bar{\tau}^{(2)}_{\infty,y}[0]\rangle = \langle \chi_{\infty,y}\rangle = 0 \tag{2.64}$$

for all times, but that also the fluctuations

$$\langle (\bar{\tau}^{(1)}_{\infty,y}[0])^2\rangle \qquad \text{etc.,} \tag{2.65}$$

remain bounded if time increases.

This is no longer true when the corresponding frequency becomes imaginary, i.e., if $y>2\varepsilon$. Then the fluctuations become larger and larger. If the system were large but finite, this would mean that the vector

$$s_\Omega = \Omega^{-1} \sum_{p=1}^{\Omega} \boldsymbol{\sigma}_p = \Omega^{-\frac{1}{2}} \boldsymbol{\sigma}_\Omega[0] \tag{2.66}$$

is displaced more and more until it arrives at a new position. Since a phase is completely characterized by the limit of the expectation value of s_Ω, we conclude that the normal conducting phase has the tendency to go over in another phase, i.e., is unstable.

The temperature β_0^{-1} where the normal-conducting phase changes from being stable to unstable is given by

$$2\varepsilon = y\,. \tag{2.67}$$

and, inserting (2.57), we find the equation

$$2\varepsilon = \tanh \beta_0 \varepsilon$$

which coincides with Eq. (2.62). For the super-conducting phase the situation is different. Since

$$\frac{d}{dt}\bar{\tau}^{(\alpha)}_{\infty,y}[0] = i[H^{sc(0)}_{\infty,y}, \bar{\tau}^{(\alpha)}_{\infty,y}[0]] \tag{2.68}$$

where $H^{sc(0)}_{\infty,y}$ stands for the expression

$$\tfrac{1}{4}\sin^2\Theta\,(\bar{\tau}^{(2)}_{\infty,y}[0])^2 - \tfrac{1}{2}\sin\Theta\cos\Theta\,\chi_{\infty,y}\,\bar{\tau}^{(2)}_{\infty,y}[0]\,, \tag{2.69}$$

we see that a displacement of the system in the τ^1 direction does not cost energy. This corresponds to the fact that the Hamiltonian of the finite system is invariant under rotations around the 3 axis. Thus we expect that any super-conducting phase characterized by y and Φ (Θ is fixed by the gap equation) has the tendency to mix up with all other phases characterized by the same value of y but other angles Φ'. (This is the analogue of the usual spreading of a wave packet in quantum mechanics.)

The situation may be illustrated by the following figure:

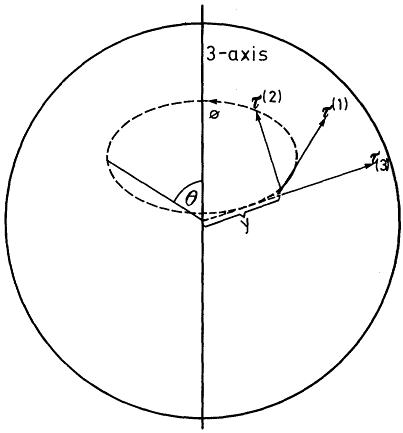

Fig. 1

However, $\langle \chi^2_{\infty,y} \rangle$ remains bounded in time and thus there are no further possibilities of phase transitions.

Another point of view is the following: if we consider the BCS model as a ferromagnet (with long-range interaction), then the parameter y means the average spontaneous magnetization since

$$y^2 = \lim_{\Omega\to\infty} \left\langle \Omega^{-2} \left(\sum_{p=1}^{\Omega} \bar{\sigma}_p \right)^2 \right\rangle . \tag{2.70}$$

The connection between Θ and y is given by the gap equation and is shown in the next figure:

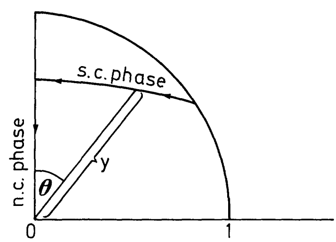

Fig. 2

whereas the relation between y and β is graphically:

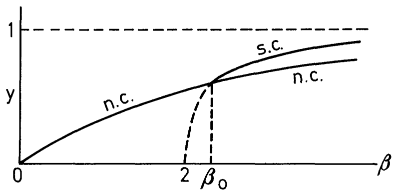

Fig. 3

The two curves are given by Eqs. (2.57) and (2.58). However, in the super-conducting case, one must be careful; while the equation

$$y = \tanh \beta y/2$$

has a non-trivial solution for all $\beta \geqq 2$, the gap equation

$$2\varepsilon = y \cos \Theta$$

can only be fulfilled if $y \geqq 2\varepsilon$, i.e., if $\beta \geqq \beta_0$.

If now $\beta > \beta_0$, then the "ferromagnet" may occur in two phases, and clearly the phase with higher order is preferred, i.e., the phase with larger y is expected to be stable, and indeed the lower y corresponds to the normal conducting phase which we have seen to be unstable in this region.

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Appendices

1. Some Pathologies

In the conventional treatment, one meets the following strange situation: let us decompose H_Ω into two parts,

$$H_\Omega = H_\Omega^{\text{kin}} + H_\Omega^{\text{int}}, \tag{A.1}$$

$$H_\Omega^{\text{kin}} = \varepsilon \sum_{p=1}^{\Omega} \sigma_p^{(3)}, \tag{A.2}$$

$$H_\Omega^{\text{int}} = -\Omega^{-1} \sum_{p,q=1}^{\Omega} \sigma_p^- \sigma_q^+ . \tag{A.3}$$

Let π be a thermodynamic representation of the algebra of the σ's at a certain temperature T, above the critical temperature. Then

$$\pi(H_\Omega^{\text{int}}) \rightharpoonup 0 . \tag{A.4}$$

that means, that for high temperatures the system cannot be distinguished from the free system characterized by H_Ω^{kin} only. Since the free system does not exhibit a phase transition if it is cooled down, we arrive at the paradox that in the infinite case one and the same Hamiltonian describes two systems which thermodynamically behave completely differently.

If, in particular, $\varepsilon = 0$ and $T \geqq \frac{1}{2}$, then all thermodynamic representations coincide and are the "chaotic" hyperfinite type II_1 factor representation – a very strange situation since Hugenholtz' theorem [19] predicts a type III factor representation for finite temperatures. There is no time evolution, i.e., $\pi(\sigma_p^{(\alpha)})$ is always constant in time. A careful analysis shows that in this case the concept of temperature loses its meaning – or one could equally say that the system is in thermal equilibrium with any system of temperature $\geqq \frac{1}{2}$ [20].

2. *Derivation of the Relation* $\|\Phi_\Omega\| = 1$

Since ${}_\Omega\langle \lambda_1, \ldots, \lambda_n | \mu_1, \ldots, \mu_m \rangle_\Omega = 0$ if $n \neq m$, it suffices to show that, for fixed n, the inequality

$$\Big\| \sum_{(\lambda_1, \ldots, \lambda_n)} C_{(\lambda_1, \ldots, \lambda_n)} (\Pi\, a_\lambda!)^{-\frac{1}{2}} |\lambda_1, \ldots, \lambda_n\rangle_\Omega \Big\|_\Omega^2 \leqq \sum_{(\lambda_1, \ldots, \lambda_n)} |C_{(\lambda_1, \ldots, \lambda_n)}|^2 \qquad \text{(A.5)}$$

is true. Thereby the C's are arbitrary coefficients and the summation is taken over all essentially different lists.

But this inequality is an immediate consequence of Dyson's theorem [14] which says that the operator

$$\sum_{(\lambda_1, \ldots, \lambda_n)} (\Pi\, a_\lambda!)^{-1} |\lambda_1, \ldots, \lambda_n\rangle_\Omega\, {}_\Omega\langle \lambda_1, \ldots, \lambda_n| \qquad \text{(A.6)}$$

acts as the unit operator in the subspace of $\mathscr{H}_\Omega$ spanned by all $|\lambda_1, \ldots, \lambda_n\rangle_\Omega$'s.

Thus, if M denotes a matrix with elements

$$M_{(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n)} = (\Pi\, a_\lambda!\, b_\mu!)^{-\frac{1}{2}}\, {}_\Omega\langle \lambda_1, \ldots, \lambda_n | \mu_1, \ldots, \mu_n\rangle_\Omega \qquad \text{(A.7)}$$

we have

$$M = M^* = M^2. \qquad \text{(A.8)}$$

Consequently

$$\sum_{(\lambda_1, \ldots), (\mu_1, \ldots)} C^*_{(\lambda_1, \ldots, \lambda_n)} M_{(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n)} C_{(\mu_1, \ldots, \mu_n)} \leqq \sum_{(\lambda_1, \ldots)} |C_{(\lambda_1, \ldots, \lambda_n)}|^2. \qquad \text{(A.9)}$$

This proves the assertion since the expressions on the left-hand side of formulae (A.5) and (A.9) coincide.

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