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The b-Boundary of Tensor Bundles over a Space-Time

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Abstract. In [1] it was shown how to attach a boundary to any space-time. In the present paper a boundary is constructed for any bundle associated with the frame bundle of a space time. In such a way limits of tensor fields at boundary points of a space-time are defined. Using this we show that the Lorentz metric has always a unique continuous extension to the b-boundary of the space-time.

1. Introduction

In a recent paper [1], one of the present authors described a construction of a boundary $\dot{M}$ to any incomplete space-time M. The construction determines the point-set and topological structure of $\dot{M}$ uniquely. The question, if the notion of vectors, tensors, metric, ect., could be given a good sense at the points of $\dot{M}$, remained open. We shall show that there is a natural and very general way to do this. The notation of [1] will be used throughout.

Tensors can be considered as objects determined by their components in a linear frame and a transformation law of these components. Now, the construction of b-boundary yield a useful by-product – the boundary $\dot{L}(M)$ of the linear bundle. The idea is to regard the points of $\dot{L}(M)$ as generalized frames and to define tensors by means of their components therein. When the manifold is extendable beyond the boundary, this definition of boundary tensors coincide with the usual one. We have, moreover, a topology on the space of all such tensors: two "near" tensors must have "near" components in "near" frames. More generally, all associated bundles of $L(M)$ can be completed in this way.

Some simple applications to the tangent bundle of two-dimensional space-times is given in Section 3.

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In Section 4, the Lorentz metric is defined in the boundary points. The Lorentz metric on M can be regarded as a differentiable function $g: T(M) \to R$ such that its restriction $g|_{T_p(M)}$ to the tangent space $T_p(M)$ is a quadratic function of signature $(+, -, -, -)$ for every $p \in M$. Then, natural generalisation would be a continuous extension $\bar{g}$ of the function g to $\overline{T(M)}$, where $\overline{T(M)}$ is the completition of the tangent bundle $T(M)$ as defined in Section 2. $T(M)$ is dense in $\overline{T(M)}$, so if $\bar{g}$ exists, it is unique. We shall show that $\bar{g}$ always exists, and that its restriction $g|_{T_p}$ to T_p is a quadratic function of signature $(+, -, -, -)$ for every $p \in \bar{M}$ such that T_p is a linear space.

2. Associated Bundles

Let Z be a differentiable manifold, on which G acts differentiably to the left. Then, G acts as a topological group on $\overline{L(M)} \times Z$ to the right by the rule $(u, z) \cdot g = (u \cdot g, g^{-1} \cdot z)$. Denote $\overline{Z(M)}$ the set of all orbits of G in $\overline{L(M)} \times Z$. The map $\pi_1 : \overline{L(M)} \times Z \to \overline{Z(M)}$ sends every element of $\overline{L(M)} \times Z$ into its prbit. The requirement, that π_1 be continuous and open defines a topology on $\overline{Z(M)}$. The topological space $\overline{Z(M)}$ contains the Z associated bundle $Z(M)$ of $L(M)$ as a subspace; $\overline{Z(M)}$ will be called b-completition of $Z(M)$. The map $\pi_2 : \overline{L(M)} \times Z \to \overline{L(M)}$ sending every pair (u, z) into u is continuous and open [3]. Let $X \in \overline{Z(M)}$. $\pi_1^{-1}(X)$ is a class of elements of $\overline{L(M)} \times Z$ having the form $(u \cdot g, g^{-1} \cdot z)$, where $u \in \overline{L(M)}$ and $z \in Z$ are fixed, while g runs through G; therefore $\pi_2(\pi_1^{-1}(X)) = u \cdot G$ and $\pi(\pi_2(\pi_1^{-1}(X)))$ is exactly one element of $\bar{M}$. If we define $\pi_Z = \pi \circ \pi_2 \circ \pi_1^{-1}$, $\pi_Z : \overline{Z(M)} \to \bar{M}$ is a map, and π_Z must be continuous and open, because π, π_2, and π_1 are. π_Z is called projection of $\overline{Z(M)}$ onto $\bar{M}$. The restriction of π_Z to $Z(M)$ is the bundle projection of $Z(M)$.

Lemma 1. *It holds*

$$\pi_z(\pi_1(x)) = \pi(\pi_2(x)), \tag{1}$$

$$\pi^{-1}(\pi_z(x)) = \pi_2(\pi_1^{-1}(x)), \tag{2}$$

$$\pi_2^{-1}(\pi^{-1}(x)) = \pi_1^{-1}(\pi_z^{-1}(x)), \tag{3}$$

$$\pi_1(\pi_2^{-1}(x)) = \pi_z^{-1}(\pi(x)). \tag{4}$$

$$\begin{tikzcd}
\overline{L(M)} \times Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \overline{L(M)} \arrow[d, "\pi"] \\
\overline{Z(M)} \arrow[r, "\pi_z"'] & \bar{M}
\end{tikzcd}$$

Fig. 1

Proof. (1) follows directly from the definition of π_Z. (2) follows from the relation $\pi_2(\pi_1^{-1}(X)) = u \cdot G$. (3) follows from the definition of π_Z.

If $u \in \pi^{-1}(p)$, then $\pi_1(uxZ) \subset \pi_Z^{-1}(p)$, because of (1), $\pi_2(u \times Z) = u$, and $\pi(u) = p$. Let $X \in \pi_Z^{-1}(p)$ and $(v, z) \in \pi_1^{-1}(X)$. Then there is $g \in G$ such that $v = u \cdot g$ and this implies $(u, g \cdot z) \in \pi_1^{-1}(X)$. From this, we obtain even $\pi_1(u \times Z) = \pi_Z^{-1}(p)$, that is to say, (4).

Let $u \in \overline{L(M)}$ and G_u be the set of all $x \in G$ such that $u \cdot x = u$. G_u is a subgroup of G, called fix-group of u. If $u \in L(M)$, G_u contains only identity e. If $v = u \cdot g$, $g \in G$, then $G_v = g^{-1} \cdot G_u \cdot g$. If G_u is non-trivial, $\pi(u)$ is called a degenerate point of $\bar{M}$. G acts freely on the fibre of any non-degenerate point p, and there is a unique matrix $a(v, u) \in G$ such that $v = u \cdot a(v, u)$ for every two $u, v \in \pi^{-1}(p)$. The map $a : \pi^{-1}(p) \times \pi^{-1}(p) \to G$ is continuous, for $p \in M$ even differentiable.

The following two Lemmas show what the sets $\pi_z^{-1}(p)$ look like.

Lemma 2. *Let $p \in \bar{M}$, $u \in \pi^{-1}(p)$ and Z/G_u be the topological space, whose elements are orbits of the group G_u in Z and whose topology is determined by the requirement, that the map $\pi_u : Z \to Z/G_u$ sending every point of Z into its orbit be continuous and open.*

Then, $\pi_Z^{-1}(p)$ is homeomorph to Z/G_u.

Proof. Eq. (4) implies that the map $\varphi_u : Z \to \pi_z^{-1}(p)$ defined by $\varphi_u(z) = \pi_1(u, z)$ is onto. φ_u is continuous and open, because π_1 is. Let z_1, z_2 be two elements of Z satisfying $\varphi_u(z_1) = \varphi_u(z_2)$. Hence, there is $g \in G$ such that $(u \cdot g, g^{-1} \cdot z_1) = (u, z_2)$, i.e., $g \in G_u$ and $z_2 \in G_u \cdot z_1$. The map $\varphi'_u : Z/G_u \to \pi_z^{-1}(p)$ given by $\varphi'_u = \varphi_u \circ \pi_u^{-1}$ is, therefore, one-to-one, and furthermore continuous and open, so φ'_u is a homeomorphism, q.e.d.

Lemma 3. *Let p be a non-degenerate point of $\bar{M}$ and Z be a vector space, on which G acts linearly to the left. Then $\pi_z^{-1}(p)$ is a vector space isomorph to Z.*

Proof. The map φ_u is a homeomorphism of Z onto $\pi_z^{-1}(p)$; hence, φ_u defines a linear structure on $\pi_z^{-1}(p)$ by the rules:

If $\lambda \in R$, $X, Y \in \pi_z^{-1}(p)$, then

$$\lambda \cdot X = \varphi_u(\lambda \cdot \varphi_u^{-1}(X)), \qquad X + Y = \varphi_u(\varphi_u^{-1}(X) + \varphi_u^{-1}(Y)).$$

It is easily verified that all the corresponding axioms are satisfied. Moreover, the structure does not depend on the particular choice of $u \in \pi^{-1}(p)$: choose $v \in \pi^{-1}(p)$; then $\varphi_v(z) = \varphi_u(a^{-1}(u, v) \cdot z)$, so that $\varphi_v^{-1}(X) = a(u, v) \cdot \varphi_u^{-1}(X)$. This implies

$$\varphi_v(\lambda \cdot \varphi_v^{-1}(X)) = \varphi_u(a^{-1}(u, v) \cdot \lambda \cdot a(u, v) \cdot \varphi_u^{-1}(X)) = \varphi_u(\lambda \cdot \varphi_u^{-1}(X)),$$

$$\begin{aligned}\varphi_v(\varphi_v^{-1}(X) + \varphi_v^{-1}(Y)) &= \varphi_u(a^{-1}(u, v)(a(u, v) \cdot \varphi_u^{-1}(X) + a(u, v)\varphi_u^{-1}(Y)) \\ &= \varphi_u(\varphi_u^{-1}(X) + \varphi_u^{-1}(Y)),\end{aligned}$$

q.e.d.

At this stage, it can be already recognized, to what extent our main goal is reached. If $\overline{Z(M)}$ is a b-completition of a tensor bundle $Z(M)$, then $\pi_z^{-1}(p)$ is a tensor space if and only if p is non-degenerate. There is no natural way to define tensors in degenerate points.

An example of degenerate point is the cusp point of the ordinary conus: define the positive definite metric

$$ds^2 = dr^2 + r^2\, d\varphi^2$$

on a two-dimensional manifold M with coordinates $r, \varphi, 0 < r, 0 \leqq \varphi < \varphi_0$, φ_0 identified with 0. Then the b-boundary consists just of one point $r = 0$. The series of frames u_n in points defined by their components $r_n = 1/n$, $\varphi_n = 0$ with frame vectors $X_n = (1,0)$, $Y_n = (0, n)$ converges and defines a boundary point of $\overline{L(M)}$. Now, take the series v_n of frames in the same points obtained by parallel transfer of u_n along the line $r = r_n$. The length of the line in $L(M)$ is φ_0/n, whereas v_n is always rotated by the same angle of φ_0 relative to u_n. Thus, u_n and v_n define the same point of $\overline{L(M)}$, but $v_n = u_n \cdot g$, where g is non-trivial.

In order to vizualize, what a geometrical meaning the boundary points of $\overline{Z(M)}$ have, let us state the

Lemma 4. *Given a curve $C : (0, 1] \to M$ in M, $C((0, 1)) \subset M$, $p = C(1) \in \dot{M}$ and an arbitrary continuous field $u(t)$ of linear frames along C (i.e., $\pi(u(t)) = C(t)$) defining an end point $u \in \pi^{-1}(p)$. Let a curve $z : (0, 1) \to Z$ in Z have the property: there is $z \in Z$ such that, for every neighbourhood U of the set $G_u \cdot z$ in Z, a δ can be found satisfying: $\delta > 0$ and $z((1-\delta, 1)) \subset U$.*

Then the Z-field $X : (0, 1) \to Z(M)$ along C defined by

$$X(t) = \pi_1(u(t), z(t))$$

has a well defined limit $\lim_{t\to 1} X(t) = \pi_1(u, z)$ in $\overline{Z(M)}$.

Proof. Choose a neighbourhood V of X in $\overline{Z(M)}$. π_1 is continuous, so $\pi_1^{-1}(V)$ is a neighbourhood of $(u \times G_u \cdot z)$. Now, there is $\delta > 0$ such that the curve $(u(t), z(t))$ lies entirely in $\pi_1^{-1}(V)$ for $t > 1 - \delta$; then

$$X((1-\delta, 1)) \subset V,$$

q.e.d.

Lemma 4 implies in particular that a tensor field along a curve terminating in a non-degenerate boundary point has a limit, if its components in a parallelly propagated frame have a well-defined limits when approaching the boundary. And conversely, to every boundary tensor such a field can be constructed, taking, e.g., the components constant.

3. Accessibility

We choose now $Z = T = R^4$, so that every $z \in Z$ has four components z^i, $i = 0, \ldots, 3$, and define the action of G on Z as follows

$$z'^i = (a^{-1})^i_k z^k .$$

The corresponding Z associated bundle is the tangent bundle $T(M)$. The tangent space $T_p(\bar{M})$ to $\bar{M}$ in p is defined by $T_p(\bar{M}) = \pi_T^{-1}(p)$. The tangent vectors to $\bar{M}$ in p are defined by their components in linear frames. This definition is convenient, if we know what a linear frame is before we know what a tangent vector is, as in the case of the boundary $\dot{M}$. For points of M, another definition is usually used (cf. [2], p. 69) – a tangent vector is a class of smooth curves and a linear frame is an ordered four-tuple of linearly independent tangent vectors. This reversed approach emphasizes that property of tangent vectors which is basic for applications: every smooth curve through $p \in M$ has a tangent vector at p and, for any tangent vector X at p, there is at least one smooth curve having X as its tangent vector.

It is, therefore, of some interest to know what relation the smooth curves have to the tangent vectors at the boundary points as they were defined in the preceding Section. The following approach seems us to be natural. Let $C : (0, 1] \to \bar{M}$ be a map with properties:

1) $C((0, 1)) \subset M$, $p = C(1) \in \dot{M}$,

2) C is continuous on $(0, 1]$ and smooth on $(0, 1)$. We shall say that the curve C has a tangent vector $X \in T_p$ at p, if there is a continuous map $C' : (0, 1] \to \overline{T(M)}$ such that $C(t)$ is tangent to C for all $t \in (0, 1)$ and $C'(1) = X$.

The set $A_p \subset T_p$ consisting of all tangent vectors X at p such that at least one curve C exists having X as its tangent vector at p will be called accessibility of the point p.

The following simple Lemma enable us to illustrate the meaning of our construction.

Lemma 5. *Denote $\mathscr{M}$ the two-dimensional Minkowski space-time with the metric $ds^2 = dt^2 - dx^2$ in some coordinate system t, x, $L(\mathscr{M})$ its linear bundle with the projection π and $T(\mathscr{M})$ its tangent bundle with the projection π_T. Let M be a subspace of $\mathscr{M}$ such that the boundary of M in $\mathscr{M}$ is a piece-wise C^1-differentiable curve C. Then $\dot{M} = C$, $\dot{L}(M) = \pi^{-1}(C)$, $\dot{T}(M) = \pi_T^{-1}(C)$.*

By means of Lemma 5, the accessibility can easily be determined in the following cases as listed in the Table 1.

We observe that the point p of M, at which the boundary M is C^1, satisfies the following condition: Let $\dot{A}_p$ be the boundary of A_p in T_p;

then, for every neighbourhood U of $\dot{A}_p$ in $\overline{T(M)}$, there is a neighbourhood U_p of p in M such that $\dot{A}_q \subset U$ for any $q \in U_p \cap \dot{M}$.

These examples suggest that accessibility provides information about C^1-differentiability of M in regular, piece-wise differentiable cases. More detailed analysis goes beyond the aim of the present paper.

Table 1

M		$p \in \dot{M}$	A_p
$t-\lvert x\rvert<0$		$x(p)>0$	$t-x \leqq 0$
$t-\lvert x\rvert<0$		$x(p)<0$	$t+x \leqq 0$
$t-\lvert x\rvert<0$		$x(p)=0$	$t-\lvert x\rvert \leqq 0$
$t<-1$	$x=0$	$x(p) \neq 0$	$t \leqq \sin(1/x(p)) -$
$t-\sin(1/x)<0$	$x \neq 0$		$((x-x(p))/x^2(p))\cos(1/x(p))$
$t<-1$	$x=0$	$x(p)=0$	$t \leqq -1$
$t-\sin(1/x)<0$	$x \neq 0$		

In the first column, the condition is given that is satisfied by the points of M. In the second column, the condition is given that is satisfied by the coordinates of the point $p \in \dot{M}$, whose accessibility A_p is described in the third column by means of a relation between the coordinates of a point of A_p. A_p is regarded as a sub-set of $\mathscr{M}$, because T_p is isomorph to $\mathscr{M}$ for every $p \in M$.

4. Lorentz Metric

A Lorentz metric on M can be considered as a differentiable function $g: T(M) \to R$ such that the restriction $g|_{T_p}$ of g to T_p is a quadratic form of signature $(+, -, -, -)$ for every $p \in M$.

Definition. Lorentz metric on $\overline{M}$ is the continuous extension $\bar{g}$ of the function g to $\overline{T(M)}$.

Theorem. *Continuous extension $\bar{g}: \overline{T(M)} \to R$ of the function $g: T(M) \to R$ is in every point of $\dot{M}$ well-defined. If T_p is a linear space, then $\bar{g}|_{T_p}$ is a quadratic form of signature $(+, -, -, -)$.*

We show some Lemmas before approaching the proof of the Theorem.

Consider a positive definite Riemannian space V^n on which an involutive k-dimensional distribution S is given. Suppose that the set Q of all maximal integral manifolds is a differentiable manifold and that the projection π_Q mapping a point of V^n into the integral manifold passing through it is differentiable. Because V^n has a positive definite metric, S determines uniquely an orthogonal distribution $S^\perp$. Clearly $T_u(V^n) = S_u \oplus S_u^\perp$ holds and $(\pi_Q)_*$ induces a vector space isomorphismus $S_u^\perp \to T_{\pi(u)}(Q)$. Therefore a tangent vector $Z \in T_\alpha(Q)$ defines uniquely a vector field $\tilde{Z}$ along the integral manifold α, which is orthogonal to α at any point, by the condition $(\pi_Q)_* \tilde{Z} = Z$. Now we can formulate

Lemma 6. *If $g(\tilde{Z}, \tilde{Z})$ is constant along $\pi_Q^{-1}(\alpha)$ for any $Z \in T_\alpha(Q)$, then $g_Q(Z, Z) := g(\tilde{Z}, \tilde{Z})$ defines a positive definite metric on Q, called the quotient metric, which satisfies*

$$g_Q((\pi_Q)_* X, (\pi_Q)_* X) \leqq g(X, X), \qquad X \in T_u(V^n). \tag{5}$$

Furthermore

$$\begin{gathered} d_Q(\alpha, \beta) = d(x, \pi_Q^{-1}(\beta)) = d(\pi^{-1}(\alpha), y) = d(\pi_Q^{-1}(\alpha), \pi_Q^{-1}(\beta)), \\ x \in \pi_Q^{-1}(\alpha), \qquad y \in \pi_Q^{-1}(\beta) \end{gathered} \tag{6}$$

holds.

Remark. Eq. (6) means that the integral manifolds are equidistant and that the distance is given by the quotient metric.

Proof. From the definition of $\tilde{Z}$ corresponding to Z it is clear that g_Q is well defined. To prove (5) we decompose $X \in T_u(V^n)$ into its components in S_u and $S_u^\perp$. Then

$$g(X, X) \geqq g(X^\perp, X^\perp) = g_Q((\pi_Q)_* X, (\pi_Q)_* X)$$

implies (5). To prove (6) consider a curve $q(t)$ joining α and β. This curve defines a 1-parameter family of integral manifolds of S. The orthogonal curves have the same length as $q(t)$ and this together with (5) implies (6).

Let us now turn to the frame bundle.

In general, $L(M)$ consists of several components, which are isometric. Choose arbitrary one of them and denote it by $L_c(M)$. Let $O_c(M)$ be the set of all orthonormal frames in $L_c(M)$. $O_c(M)$ is a component of the set of all orthonormal frames in $L(M)$. Furthermore, it is a closed, ten-dimensional sub-manifold of $L_c(M)$. Denote by G_c the fix-group of $L_c(M)$ in $L(M)$. Elements of G_c act on $L_c(M)$ as diffeomorphisms, thus every set of the form $O_c(M) \cdot g$, $g \in G_c$, is a closed, ten-dimensional sub-manifold of $L_c(M)$, called Lorentz sub-manifold. Denote by $\mathscr{L}_c$ the fix-group of $O_c(M)$ in $L_c(M)$. $\mathscr{L}_c$ is a sub-group of the Lorentz group $\mathscr{L}$ including the component of identity of $\mathscr{L}$. The coset space $\mathscr{L}/\mathscr{L}_c$ gives the type of orientability of M [1].

Lemma 7. *The Lorentz sub-manifolds form a differentiable ten-parametric congruence of ten-dimensional equidistant closed sub-manifolds.*

$$\begin{tikzcd} L_c(M) \arrow[r, "\pi_Q"] & Q \\ \pi^{-1}(U) \arrow[u, "i_U"] \arrow[r, "\pi_U"] \arrow[d, "\eta"'] & Q_u \arrow[u, "\varphi_1"'] \arrow[d, "\varphi_2"] \\ G_c \arrow[r, "\pi_{\mathscr{L}}"'] & G_c/\mathscr{L}_c \end{tikzcd}$$

Fig. 2

Proof. Denote by Q the set of all Lorentz sub-manifolds of $L_c(M)$. The map $\pi_Q : L_c(M) \to Q$ sends every $u \in L_c(M)$ in the Lorentz sub-manifold $\alpha \in Q$ including u. The requirement, that π_Q be continuous and open, defines a topology on Q.

Choose $p \in M$. There is a neighbourhood U of p in M and a differentiable and open map $\eta : \pi^{-1}(U) \to G_c$ with the following properties:

The map $\psi : \pi^{-1}(U) \to U \times G_c$ defined by

$$\psi(u) = (\pi(u), \eta(u)) \tag{8}$$

is a diffeomorphism onto.

$$\eta(u \cdot g) = \eta(u) \cdot g \,. \tag{9}$$

$$\eta(\pi^{-1}(U) \cap O_c(M)) = \mathscr{L}_c \,. \tag{10}$$

The properties (8) and (9) follow immediately from the definition of a principal bundle (cf. [2], p. 294); since $O_c(M)$ includes the holonomy bundle, the "Reduction Theorem" ([4], p. 83) implies that there is a local orthonormal cross section in $L_c(M)$ from which we easily have (10).

The inclusion map $i_U : \pi^{-1}(U) \to L_c(M)$ is a differentiable open map, because $\pi^{-1}(U)$ is open.

Every $\alpha \in Q$ defines a non-empty set $\alpha_U = \pi^{-1}(U) \cap \pi_Q^{-1}(\alpha)$. If α and β are two distinct elements of Q, then the sets α_U and β_U are disjoint. Moreover, $\pi^{-1}(U) = \bigcup_{\alpha \in Q} \alpha_U$, because $L_c(M) = \bigcup_{\alpha \in Q} \pi_Q^{-1}(\alpha)$. Denote by Q_U the set of all α_U and $\pi_U : \pi^{-1}(U) \to Q_U$ the map sending every $u \in \pi^{-1}(U)$ in the set α_U, in which u lies. The requirement, that π_U be continuous and open, defines a topology on Q_U. It is easily seen that φ_1 defined by $\varphi_1 = \pi_Q \circ i_U \circ \pi_U^{-1}$ is a map, $\varphi_1 : Q_U \to Q$; φ_1 is one-to-one, continuous and open, so φ_1 is a homeomorphism.

$\mathscr{L}_c$ is a closed sub-group of G_c; the set $G_c/\mathscr{L}_c$ of all left cosets $\mathscr{L}_c \cdot g$ is, therefore, a differentiable manifold, and the projection map $\pi_{\mathscr{L}} : G_c \to G_c/\mathscr{L}_c$ is differentiable and open. From (9) and (10) follows that φ_2 defined by $\varphi_2 = \pi_{\mathscr{L}} \circ \eta \circ \pi_U^{-1}$ is a map, $\varphi_2 : Q_U \to G_c/\mathscr{L}_c$ is one-to-one, continuous and open, hence φ_2 is a homeomorphism.

Denote by $\varphi : Q \to G_c/\mathscr{L}_e$ the homeomorphism defined by $\varphi = \varphi_2 \circ \varphi_1^{-1}$. From the construction, it may easily be seen that

$$\varphi(\pi_Q(O_c(M) \cdot g)) = \pi_{\mathscr{L}}(g) \,.$$

Therefore, φ does not depend on U. The map φ defines a differentiable structure on Q by the requirement that φ be a diffeomorphism. Furthermore, we have

$$\pi_Q|_{\pi^{-1}(U)} = \varphi^{-1} \circ \pi_{\mathscr{L}} \circ \eta \,, \tag{11}$$

i.e., every point $u \in L_c(M)$ has a neighbourhood, on which π_Q is differentiable, because φ^{-1}, $\pi_{\mathscr{L}}$ and η are. This implies that π_Q is differentiable.

Now, it is clear that the Lorentz sub-manifolds form a differentiable ten-parametric congruence ($G_c/\mathscr{L}_c$ is ten-dimensional).

The group G_c acts on $G_c/\mathscr{L}_c$ to the right as a group of diffeomorphisms ([2], p. 230) by the rule $(\mathscr{L}_c \cdot x) \cdot g = \mathscr{L}_c \cdot xg$, so

$$\pi_{\mathscr{L}}(xg) = \pi_{\mathscr{L}}(x) \cdot g\,. \tag{12}$$

This action can be carried over to Q in the following way:

$$\alpha \cdot g = \varphi^{-1}(\varphi(\alpha) \cdot g)\,,$$

i.e.,

$$\varphi^{-1}(\varphi(\alpha)) \cdot g = \varphi^{-1}(\varphi(\alpha) \cdot g)\,. \tag{13}$$

With regard to the differentiability of φ, the action is differentiable. From (9), (11), (12) and (13) we have

$$\pi_Q(u \cdot g) = \pi_Q(u) \cdot g\,. \tag{14}$$

We will now show, that a quotient metric g_Q in the sense of Lemma 6 exists. Choose $\alpha \in Q$ and consider the corresponding Lorentz sub-manifold $\pi_Q^{-1}(\alpha)$. Take any E of the Lie algebra of G_c such that the vector field $\overset{*}{E}$ is orthogonal to $\pi_Q^{-1}(\alpha)$ at one point. Because the tangent space of $\pi_Q^{-1}(\alpha)$ is spanned at any point by B_i and $c^i_{\alpha k}\overset{*}{E}{}^k_i$ with $c^i_{\alpha k} = \text{const}$, $\overset{*}{E}$ is orthogonal to $\pi_Q^{-1}(\alpha)$ at any point. The field $\overset{*}{E}$ determines uniquely a 1-parameter sub-group $g(t)$ of G and (14) implies that $(\pi_Q)_* \overset{*}{E}_v$ is the same tangent vector in $T_\alpha(Q)$ for all points $v \in \pi_Q^{-1}(\alpha)$. Conversely every vector in $T_\alpha(Q)$ can be constructed in this way. As by the definition of the bundle metric $g(\overset{*}{E}, \overset{*}{E}) = \text{const}$, we can apply Lemma 6 and conclude that on Q a quotient metric exists with the properties (5) and (6). Hence the Lorentz submanifolds are equidistant.

Lemma 8. *Let $\{u_n\}$ and $\{v_n\}$ be two Cauchy sequences of points from $O_c(M)$, $u = \lim_{n\to\infty} u_n$, $v = \lim_{n\to\infty} v_n$, and $x \in G_c$ exists such that $v = u \cdot x$. Then $x \in \mathscr{L}_c$.*

Proof. Suppose that $x \notin \mathscr{L}_c$. $\{u_n \cdot x\}$ is a Cauchy sequence and $\lim_{n\to\infty} (u_n \cdot x) = v$ [1]. The points $u_n \cdot x$ lie in the Lorentz sub-manifold $O_c(M) \cdot x \neq O_c(M)$, which contradicts Lemma 7.

Proof of the Theorem:

Let us denote by $\overline{O_c(M)}$ the closure and by $\dot{O}_c(M)$ the boundary of $O_c(M)$ in $\overline{L_c(M)}$. The Theorem 4.2 in [1] implies that $\pi^{-1}(p) \cap \overline{O_c(M)} \neq \emptyset$ for every $p \in \bar{M}$. As G_c is a topological transformation group on $\overline{L_c(M)}$, we have then $u \cdot \mathscr{L}_c \subset \pi^{-1}(p) \cap \overline{O_c(M)}$ for every $u \in \pi^{-1}(p) \cap \overline{O_c(M)}$. But from the Lemma 8 it follows that, at the same time, $u \cdot \mathscr{L}_c \supset \pi^{-1}(p) \cap O_c(M)$. So, if $p = \pi(u)$,

$$u \cdot \mathscr{L}_c = \pi^{-1}(p) \cap \overline{O_c(M)} \tag{15}$$

for any $u \in \overline{O_c(M)}$.

Let $p \in \bar{M}$ and $X \in \pi_T^{-1}(p)$. The class $\pi_1^{-1}(X)$ includes at least one pair (u, z), $u \in \pi^{-1}(p) \cap \overline{O_c(M)}$. If (u', z') is another such pair, then, because of (15),

$$u' = u \cdot x$$

$$z' = x^{-1} \cdot y \cdot z\,,$$

where $x \in \mathscr{L}_c$ and $y \in G_u$. Suppose $y \notin \mathscr{L}_c$ and choose a sequence $\{u_n\}$ in $O_c(M)$ converging to u. The sequence $\{u_n \cdot y\}$ lies in the Lorentz submanifold $O_c(M) \cdot y \neq O_c(M)$, but it has the same limit as $\{u_n\}$ has. According to the proof of Lemma 6, this is impossible. Therefore, we have always $G_u \subset \mathscr{L}_c$ and

$$u' = u \cdot x_1\,,$$

$$z' = x_2 \cdot z\,,$$

where $x_1, x_2 \in \mathscr{L}_c$. It follows that every $X \in \pi_T^{-1}(p)$ is, in this way, mapped in a class $\mathscr{L}_c \cdot z$ of $T/\mathscr{L}_c$. Let us denote this map by $\mu : \pi_T^{-1}(p) \to T/\mathscr{L}_c$. Now, define the function $f : T \to R$ by $f(z) = (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2$; clearly $f(x \cdot z) = f(z)$ for any $x \in \mathscr{L}_c$; then, $\bar{g} = f \circ \mu$ is a well-defined map of $\pi_T^{-1}(p)$ in R. This construction can be performed in every point p of $\bar{M}$, so the function $\bar{g}$ is well-defined on $\overline{T(M)}$. Obviously, on $T(M)$, $\bar{g}$ coincides with the Lorentz metric g on M. If T_p is isomorph to T, then $\bar{g}$ is a quadratic form on T_p with just the right signature. It remains to show, that $\bar{g}$ is continuous.

Let $X \in \overline{T(M)}$ and $\pi_1(u, z) = X$. Choose a sequence $\{X_n\}$ in $T(M)$ converging to X, and a sequence $\{U_k\}$ of open neighbourhoods U_k of the point $(u, z) \in \overline{L(M)} \times T$ converging to (u, z). Then $\pi_1(U_k)$ is a neighbourhood of X in $\overline{T(M)}$ and there is N_k such that, for any $n > N_k$, $X_n \in \pi_1(U_k)$, and $\{N_k\}$ is a non-decreasing integer sequence. This means that $\pi_1^{-1}(X_n) \cap U_k \neq \emptyset$. Choose $(u_n, z_n) \in (\pi_1^{-1}(X_n) \cap U_k)$ for $N_k < n \leqq N_{k+1}$. The sequence (u_n, z_n) converges in $\overline{L(M)} \times T$ to (u, z) and $\pi_1(u_n, z_n) = X_n$. According to [1], there is a sequence $\{v_n\}$ with the following properties

$$\pi_1(v_n, z_n') = X_n, \quad \text{if} \quad z_n' = a(u_n, v_n) \cdot z_n\,,$$

$$v_n \in O_c(M)$$

$$\lim_{n\to\infty} v_n = u, \qquad \lim_{n\to\infty} a(u_n, v_n) = e\,.$$

But then $\lim_{n\to\infty} z'_n = z$ and we have $\lim_{n\to\infty} \bar{g}(X_n, X_n) = \bar{g}(X, X)$. This implies that, for any $\varepsilon > 0$, there is a neighbourhood U_X of the point $X \in \overline{T(M)}$ such that

$$|\bar{g}(Y, Y) - \bar{g}(X, X)| < \varepsilon \tag{16}$$

for every $Y \in (U_X \cap T(M))$. From this it easily follows that (16) holds for any $Y \in U_X$, because $T(M)$ is dense in $\overline{T(M)}$.

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