Commun. math. Phys. 23, 275—284 (1971)

Boundary Conditions and Correlation Functions in the ν-Dimensional Ising Model at Low Temperature

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Received August 1, 1971

Abstract. The boundary condition dependence of the correlation functions in a phase transition region of the thermodynamic parameters is of great importance to understand the character and properties of the phase transition itself. In this paper we study the boundary condition dependence of certain correlation functions in the Ising model at low temperature.

§ 1. Motivation of the Paper. A Related Problem

The object of this note is the investigation of a problem, formulated in § 2, related to the old question of whether the value $\bar{m}^*(\beta)$ for the spontaneous magnetization in the Ising model, as computed by Onsager and Yang, coincides with the appropriate derivative of the free energy $f(\beta, h)$:

$$m^*(\beta) = \left. \frac{\partial f(\beta, h)}{\partial h} \right|_{h=0^+} . \tag{1.1}$$

As is well known Onsager's definition is essentially [1]:

$$\bar{m}^*(\beta) = \lim_{|x-y| \to \infty} \langle \sigma_x \sigma_y \rangle \tag{1.2}$$

where $\langle \sigma_x \sigma_y \rangle$ is the two-spin correlation function defined as a suitable thermodynamic limit of finite volume correlation functions.

The problem of showing the conjectured [1] identity of (1.1) and (1.2) can be formulated in the frame of a more general conjecture which we describe below [2].

Consider the sets of correlation functions $\langle \sigma_a \sigma_b \sigma_c \ldots \rangle$ $a, b, c, \ldots \in Z^\nu$ that can be obtained as a thermodynamic limit of finite volume correlation functions using all the possible boundary conditions.

More precisely the finite volume correlation functions are defined as follows: let Λ be a finite box and let $X = \{\sigma_1, \sigma_2, \ldots, \sigma_{N(\Lambda)}\}$ be a spin configuration (here $\sigma_i = \pm 1$ denotes the value of the spin in the ith lattice site of Λ); let $-J > 0$ and

$$H_0(X) = J \sum_{\substack{(ij) \\ i,j \in \Lambda}} \sigma_i \sigma_j - h \sum_{i \in \Lambda} \sigma_i \quad (ij) \text{ means nearest neighbour} \tag{1.3}$$

and define as "boundary condition" a function $\Sigma(X)$ such that

$$|\Sigma(X)| \leqq 2|J|\,\sigma(\Lambda) \tag{1.4}$$

where $\sigma(\Lambda)$ is the surface of Λ. We call $H(X)=H_0(X)+\Sigma(X)$ the hamiltonian for the Ising model with "boundary condition Σ"[1].

The finite volume correlation functions with boundary condition Σ are the family of the functions of $a, b, \ldots \in Z^\nu \cup \Lambda$ given by

$$\langle \sigma_a \sigma_b \sigma_c \sigma_d \ldots \rangle_{\Sigma,\Lambda} = \frac{\sum\limits_{\text{all}\,X} \sigma_a \sigma_b \ldots e^{-\beta H(X)}}{Z(\beta, h, \Sigma)} \tag{1.5}$$

where the partition function $Z(\beta, h, \Sigma)$ is

$$Z(\beta, h, \Sigma) = \sum_{\text{all}\,X} e^{-\beta H(X)}.$$

If for each Λ we are given a Σ_Λ we can consider the thermodynamic limit as $\Lambda \to \infty$ (i.e. as the sides of $\Lambda \to \infty$):

$$\beta f(\beta, h) = \lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \log Z(\beta, h, \Sigma) \tag{1.6}$$

which is well known to exist and to be Σ-independent.

In general, for fixed $a, b, c \ldots \in Z^\nu$, the limit as $\Lambda \to \infty$ of (1.5) does not exist. However passing to suitable subsequences the limit (1.5) can be assumed to exist for all $a, b, \ldots \in Z^\nu$.

Consider all the families of functions

$$\langle \sigma_a \rangle, \langle \sigma_a \sigma_b \rangle, \langle \sigma_a \sigma_b \sigma_c \rangle, \ldots \tag{1.7}$$

that can be obtained as limits of subsequences of (1.5) with all the possible choices of Σ_Λ.

Each set of functions in (1.7) will be called an equilibrium state corresponding to an external field h (see (1.3)) and a temperature β^{-1}. We shall say that the possible equilibrium states correspond to different boundary conditions.

One can ask when the set of equilibrium states consists of just one element i.e. when the limit as $\Lambda \to \infty$ of (1.5) exists and is Σ_Λ-independent. It has recently been possible to show that, if $h \neq 0$ and $J > 0$, the limit as $\Lambda \to \infty$ of (1.5) is indeed unique and Σ-independent [3]. However for

[1] The reason for limiting ourselves to (1.4) is twofold: first the "Cyclic boundary conditions" together with the boundary condition obtained by occupying all the sites outside Λ with spins having a prescribed value can be described by surface terms verifying (1.4); second it has been shown that, using boundary terms verifying (1.4) one can obtain in the thermodynamic limit all the possible equilibrium states (i.e. by allowing more general surface terms one does not get anything new) [8].

$h=0$ and β large enough it is known that there are at least two different equilibrium states. These two states have been investigated in detail and have equal even correlation functions and opposite odd correlation functions. Furthermore, if we denote them by μ^+ and μ^-, the following properties have been shown to hold for large β (i.e. low temperature) [4]:

$$\begin{aligned}
&\lim_{x-y\to\infty} \mu^+(\sigma_x\sigma_y) = m^2 = \lim_{x-y\to\infty} \mu^-(\sigma_x\sigma_y)\\
&\mu^+(\sigma_x) = m \qquad \mu^-(\sigma_x) = -m\\
&\mu^+(\sigma_x\sigma_y\sigma_z\ldots) = \lim_{h\to 0^+} \mu(\sigma_x\sigma_y\sigma_z\ldots)\\
&\mu^-(\sigma_x\sigma_y\sigma_z\ldots) = \lim_{h\to 0^-} \mu(\sigma_x\sigma_y\sigma_z\ldots)
\end{aligned} \tag{1.8}$$

where $\mu^\pm(\sigma_x\sigma_y\ldots)$ denotes the value of $\langle\sigma_x\sigma_y\ldots\rangle$ in the state $\mu^\pm$ and μ denotes the (unique) equilibrium state at the same β and in presence of a field $h \neq 0$.

Another important property of $\mu^\pm$ is that (for large β) [4]

$$\mu^+(\sigma_x) = -\mu^-(\sigma_x) = m = \lim_{h\to 0^+} \mu(\sigma_x) = \lim_{h\to 0^+} \frac{\partial f(\beta,h)}{\partial h}. \tag{1.9}$$

It is tempting and natural to conjecture that μ^+ and μ^- are essentially all the possible translationally invariant equilibrium states at $h=0$ and that they represent the two phases in which the system can be found. More precisely one can conjecture that if ϱ is another translationally invariant equilibrium state (corresponding to the same β and to $h=0$) then there exists α_ϱ such that

$$\varrho = \alpha_\varrho\mu^+ + (1-\alpha_\varrho)\mu^- \qquad 0 \leqq \alpha_\varrho \leqq 1 \tag{1.10}$$

where this means that $\varrho(\sigma_x\sigma_y\sigma_z\ldots) = \alpha_\varrho\mu^+(\sigma_x\sigma_y\ldots) + (1-\alpha_\varrho)\,\mu^-(\sigma_x\sigma_y\ldots)$.

It is easy to see that conjecture (1.10) together with the properties (1.8), (1.9) implies that the Onsager value (1.2) for the spontaneous magnetization coincides with (1.1) at least in the region of β where (1.8), (1.9) can be proven (i.e. low temperature).

Conjecture (1.10) implies several rather strong properties of other correlation functions. It is the object of this paper to study some of these properties and to prove that they in fact hold.

2. Formulation of the Problem

Consider a ν-dimensional Ising model ($\nu \geqq 2$) enclosed in a rectangular box Λ containing $N(\Lambda)$ lattice points. Consider the probability distribution induced at temperature β^{-1} on the spin configurations in Λ by the hamiltonian $H_0(X)$ and a boundary condition $\Sigma_\Lambda(X)$: the probability of a spin configuration X is:

$$P_{\Lambda,\Sigma}(X)=\frac{e^{-\beta H(X)}}{Z(\Lambda,\beta,\Sigma)}\,. \tag{2.1}$$

To each spin-configuration X in Λ we associate a set of lines constructed as follows: Consider the lattice bonds not lying on the boundary of Λ and having opposite spins at their extremes; draw a unit segment perpendicular to each of these bonds, through their center[2]. The set of lines thus obtained splits into several disjoint self-advoiding lines some of which are closed while the others begin and end on the boundary of Λ. Let $\Gamma=(\gamma_1,\gamma_2,\dots,\gamma_n)$ denote this set of lines. Clearly given Γ there are only two spin-configurations $X_1(\Gamma)$ and $X_2(\Gamma)$ having Γ as set of contours.

Let $\Sigma^1_\Lambda(\Gamma)$ and $\Sigma^2_\Lambda(\Gamma)$ be the surface terms associated with $X_1(\Gamma)$ and $X_2(\Gamma)$.

We can ask for the probability that given contours $\gamma_1,\gamma_2,\dots,\gamma_n$ are part of Γ: this probability is given by

$$\varrho_{\beta,\Sigma,\Lambda}(\gamma_1\dots\gamma_n)=\frac{\sum\limits_{\Gamma\supset(\gamma_1,\dots,\gamma_n)} e^{-2\beta J|\Gamma|}\left(e^{-\beta\Sigma^1_\Lambda(\Gamma)}+e^{-\beta\Sigma^2_\Lambda(\Gamma)}\right)}{Z(\beta,\Lambda,\Sigma)} \tag{2.2}$$

here $|\Gamma|=\sum\limits_{\gamma'\in\Gamma}|\gamma'|$ and $|\gamma'|=$ length of γ' (this follows from the fact that if $h=0$ $H(X)'=2J\Sigma|\gamma'|+\Sigma(\beta,X)+\text{const.}$).

Suppose that the contours in $\Gamma=\{\gamma_1,\gamma_2\dots\gamma_n\}$ are all *closed*, then it makes sense to consider

$$\lim_{\Lambda\to\infty}\bar{\varrho}_{\beta,\Sigma,\Lambda}(\Gamma) \tag{2.3}$$

where $\bar{\varrho}_{\beta,\Sigma,\Lambda}(\Gamma)$ denotes the translational average of $\bar{\varrho}_{\beta,\Sigma,\Lambda}(\Gamma)$ i.e.:

$$\bar{\varrho}_{\beta,\Sigma,\Lambda}(\Gamma)=\frac{1}{N(\Lambda)}\sum_{\Gamma+x\in\Lambda}\varrho_{\beta,\Sigma,\Lambda}(\Gamma+x) \tag{2.4}$$

here $\Gamma+x$ means the set of contours obtained by translating by x the contours in Γ.

The conjecture discussed in the preceding section implies that

$$\varrho_\beta(\Gamma)=\lim_{\Lambda\to\infty}\bar{\varrho}_{\beta,\Sigma,\Lambda}(\Gamma) \tag{2.5}$$

exists for all closed Γ and is Σ independent.

In this paper we prove that (2.5) is indeed true. The proof is based on the fact that the possible limits of the r.h.s. in (2.5) define possible tangent

[2] This construction is well known (see D. Ruelle: Statistical mechanics, Benjamin 1969, p. 117). For a detailed discussion of the ambiguities that arise when four lines meet in a corner see G. Gallavotti, A. Martin-Lof: preprint. "Surface tension in the 2-dimensional Ising model", to appear in Commun. math. Phys.

planes to a certain convex surface in an infinite dimensional Banach space. We show that these tangent planes are tangent at the same point 0 and that, at this point, the tangent plane is unique because its components verify an integral equation with a kernel which is an analytic contraction in a neighbourhood of 0.

A similar technique was developed in connection with the investigation of the thermodynamic properties at low density [6].

3. Geometric Interpretation of $\bar{\varrho}(\Gamma)$

Let B be the Banach space of all the translationally invariant "potentials" on the non-intersecting closed contours:

$$\begin{aligned} &\Phi(\Gamma) = \Phi^{(n)}(\gamma_1 \dots \gamma_n) \quad \text{if} \quad \Gamma = |\gamma_1 \dots \gamma_n\} \ \gamma_i = \text{closed} \\ &\|\Phi\| = \sum_{0 \in \Gamma} |\Phi(\Gamma)| < +\infty . \end{aligned} \tag{3.1}$$

Define

$$U_\Phi(\Gamma) = \sum_{\Gamma' \subset \Gamma} \Phi(\Gamma') \quad \text{(i.e.: } \Gamma' \text{ built with closed contours in } \Gamma). \tag{3.2}$$

and

$$P_{\Lambda,\Sigma}(\Phi) = \frac{1}{N(\Lambda)} \log \sum_{\Gamma \subset \Lambda} e^{-\beta|\Gamma|}(e^{-\beta\Sigma^{(1)}(\Gamma)} + e^{-\beta\Sigma^{(2)}(\Gamma)}) e^{-\beta U_\Phi(\Gamma)} .$$

We shall first show the following

Theorem. *Under the above conditions and if β is large enough* $(\forall \Phi \in B)$:

$$P(\Phi) = \lim_{\Lambda \to \infty} P_{\Lambda,\Sigma}(\Phi) \tag{3.3}$$

exists and is Σ-independent. (The limit $\Lambda \to \infty$ is taken over the net of increasing rectangles.)

Proof. Observe first that

$$\begin{aligned} 2e^{-2J\beta\sigma(\Lambda)} &\sum_{\Gamma \subset \Lambda} e^{-\beta|\Gamma|} e^{-U_\Phi(\Gamma)} \\ &\leqq \sum_{\Gamma \subset \Lambda} e^{-\beta|\Gamma|}(e^{-\beta\Sigma^{(1)}(\Gamma)} + e^{-\beta\Sigma^{(2)}(\Gamma)}) e^{-\beta U_\Phi(\Gamma)} \\ &\leqq 2e^{2J\beta\sigma(\Lambda)} \sum_{\Gamma \subset \Lambda} e^{-\beta|\Gamma|} e^{-\beta U_\Phi(\Gamma)} \end{aligned} \tag{3.4}$$

hence the limit (if it exists) is Σ independent.

Now, denoting $|\gamma| =$ length of γ and $|\Gamma| = \Sigma_i |\gamma_i|$ if $\Gamma = (\gamma_1, \gamma_2, \dots)$ and setting $2J = 1$:

$$\tilde{Z}_0(\beta, \Lambda) = \sum_{\Gamma \subset \Lambda} e^{-\beta|\Gamma|} e^{-\beta U_\Phi(\Gamma)} = \sum_{\Gamma' \cup \Gamma \subset \Lambda} e^{-\beta|\Gamma|} e^{-\beta|\Gamma'|} e^{-\beta U_\Phi(\Gamma)} \tag{3.5}$$

where $\tilde{Z}_0(\beta, \Lambda)$ is defined by the equation in (3.5) and the second sum runs over all the possible families of contours $\Gamma' \cup \Gamma$ with Γ' built with open contours and Γ built with closed ones and $\Gamma \cup \Gamma'$ is admissible (i.e. Γ and Γ' disjoint).

Clearly

$$\tilde{Z}_0(\beta, \Lambda) \geqq Z_0(\beta, \Lambda) = \sum_{\substack{\Gamma \text{ closed} \\ \Gamma \subset \Lambda}} e^{-\beta|\Gamma|} e^{-\beta U_\Phi(\Gamma)}, \tag{3.6}$$

$$\tilde{Z}_0(\beta, \Lambda) \leqq Z_0(\beta, \Lambda) \Bigg(\sum_{\substack{\Gamma' \subset \Lambda \\ \Gamma' \text{ open}}} e^{-\beta|\Gamma'|} \Bigg). \tag{3.7}$$

The sum in (3.7) can be estimated as follows: suppose there are K contours in Γ', i.e.: $\Gamma' = (\gamma'_1, \gamma'_2, \ldots, \gamma'_K)$ with lengths $n_1, n_2, \ldots, n_K$.

These K contours will start in $P_1, P_2, \ldots, P_K \in$ boundary of Λ. The points $(P_1, P_2, \ldots, P_K)$ can be fixed in $\binom{\sigma(\Lambda)}{K}$ different ways where $\sigma(\Lambda)$ is the length of the boundary of Λ.

For each of these ways there are at most $3^{n_1+n_2+\cdots+n_K}$ possible contours $\gamma'_1 \ldots \gamma'_K$ with respective lengths $n_1, n_2, \ldots, n_K$ and end points $(P_1, P_2, \ldots, P_K)$ hence

$$\begin{aligned} \sum_{\substack{\Gamma' \subset \Lambda \\ \Gamma' \text{ open}}} e^{-\beta|\Gamma'|} &\leqq \sum_{K=0}^{\sigma(\Lambda)} \binom{\sigma(\Lambda)}{K} \sum_{n_1 \ldots n_K} \prod_{i=1}^{k} (3e^{-\beta})^{n_i} \\ &\leqq \sum_K \binom{\sigma(\Lambda)}{K} \left(\frac{3e^{-\beta}}{1-3e^{-\beta}} \right)^K = \left(1 + \frac{3e^{-\beta}}{1-3e^{-\beta}} \right)^{\sigma(\Lambda)} \end{aligned} \tag{3.8}$$

hence if β is large enough (since $\sigma(\Lambda)/N(\Lambda) \to 0$):

$$\frac{1}{N(\Lambda)} \log \tilde{Z}_0(\beta, \Lambda) - \frac{1}{N(\Lambda)} \lg Z_0(\beta, \Lambda) \to 0. \tag{3.9}$$

It is now easy, using standard subadditivity techniques) to show the existence of the limit [5]

$$\lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \lg Z_0(\beta, \Lambda) = P(\Phi). \tag{3.10}$$

We are now in a position to discuss the geometric meaning of $\bar{\varrho}(\Gamma)$.

From (3.5) it is clear that:

$$\frac{1}{|\Gamma|} \bar{\varrho}_{\Phi, \Lambda, \Sigma}(\Gamma) = \frac{\partial P_{\Lambda, \Sigma}(\Phi)}{\partial \Phi(\Gamma)}$$

and therefore $\bar{\varrho}_{\Phi, \Lambda, \Sigma}$ are essentially the components of the tangent plane to the convex surface $P_{\Lambda, \Sigma}(\Phi)$ defined for $\Phi \in B$. Hence every limit point

$\varrho(\Gamma)$ defined as

$$\varrho(\Gamma) = \lim_{\Lambda \to \infty} \operatorname{point} \bar{\varrho}_{\Phi, \Lambda, \Sigma}(\Gamma)$$

defines a tangent plane to the Σ-independent surface

$$P(\Phi) = \lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \lg Z_0(\beta, \Phi).$$

In the next section we show that the tangent plane to $P(\Phi)$ in $\Phi = 0$ is unique if β is large enough.

§ 4. Uniqueness of $\varrho(\Gamma)$

We prove the following theorem:

Theorem. *The convex functional $P(\Phi)$ on B has a unique tangent plane at $\Phi = 0$ provided β is large enough.*

Proof. It is of course enough to show that one can construct a tangent plane whose components are continuous in Φ for $\|\Phi\|$ sufficiently small.

The key remark is that, as shown in the preceding section,

$$P(\Phi) = \lim_{\Lambda \to \infty} \frac{1}{N(\Lambda)} \log \sum_{X \subset \Lambda}^{*} e^{-\beta|X| - \beta U_{\Phi}(X)} \equiv \lim_{\Lambda \to \infty} P_{\Lambda}^{*}(\Phi)$$

where the sum runs over the sets of *closed* contours in Λ. Hence the functions

$$\varrho_{\Lambda}^{*}(Y) = \frac{\sum\limits_{X \supset Y}^{*} e^{-\beta|X| - \beta U_{\Phi}(X)}}{\sum\limits_{X}^{*} e^{-\beta|X| - \beta U_{\Phi}(X)}} \tag{4.0}$$

are such that their averages define the components of the tangent plane to the surface $P_{\Lambda}^{*}(\Phi)$ and therefore their limits define a tangent plane to the surface $P(\Phi)$. We shall now seek equations for ϱ_{Λ}^{*} and their limits as $\Lambda \to \infty$ and, using them, we shall show that $\lim_{\Lambda \to \infty} \varrho_{\Lambda}^{*}(X) = \varrho(X)$ exists for all X and defines a tangent plane to $P(\Phi)$ which depends continuously on Φ around $\Phi = 0$ if β is large enough.

Let us denote in this section with capital letters X, S, T, T' etc. sets of compatible closed contours (previously denoted by Γ, Γ' etc.).

Define [6]

$$U_{\Phi}^{1}(X) = \sum_{\gamma_1 \in S \subset X} \Phi(S) \tag{4.1}$$

where γ_1 is chosen arbitrarily in X (but with a fixed well defined criterion).

$$W_\Phi^1(X,Y)=\sum_{\gamma_1\in S\subset X}\Phi(S\cup Y)\,, \tag{4.2}$$

$$I_\Phi(X,Y)=\sum_{\substack{\gamma_1\in T\subset X\\ \emptyset\neq S\subset Y}}\Phi(T\cup S)=\sum_{\emptyset\neq S\subset Y}W_\Phi^1(T,S)\,, \tag{4.3}$$

$$\begin{aligned}K_\Phi(X\,|\,T)&=\sum_{n\geqq 1}\sum_{\substack{\{S_1\ldots S_n\}\\ U_iS_i=T}}\prod_{j=1}^{n}\left(e^{-W_\Phi(T,S_i)}-1\right) && T\neq\emptyset\,,\\ &=1 && T=\emptyset\,.\end{aligned} \tag{4.4}$$

Then consider a finite Λ and a family of contours $X=(\gamma_1\ldots\gamma_n)$ with the Γ's closed. We have (in close analogy with Ref. [7, 4]):

$$\begin{aligned}\varrho_\Lambda^*(X)&=Z^{-1}\sum_{X'\cap X=0}e^{-\beta(|X|+|X'|)}e^{-\beta U_\Phi(X)}\\ &=e^{-\beta U_\Phi^1(X)}e^{-\beta|\gamma_1|}Z^{-1}\sum_{X'\cap X=\emptyset}e^{-\beta(|X^{(1)}|+|X'|)}\\ &\quad\cdot e^{-\beta U_\Phi(X^{(1)}\cup X')}\sum_{T\subset X'}K(X,T)\\ &=e^{-\beta U_\Phi^1(X)}e^{-\beta|\gamma_1|}\sum_{T\cap X=\emptyset}K(X,T)\sum_{Y\cap(T\cup X)=\emptyset}e^{-\beta(|X^{(1)}|+|Y|+|T|)}\\ &\quad\cdot e^{-\beta U_\Phi(X^{(1)}Y\cup T)}\\ &=e^{-\beta U_\Phi^1(X)}e^{-\beta|\gamma_1|}\sum_{T\cap X=\emptyset}K(X,T)\sum_{\overline{X''\cap\gamma_1\neq\emptyset}}'(-1)^{N(X'')}\varrho_\Lambda^*(X^{(1)}\cup X''\cup T)\end{aligned} \tag{4.5}$$

where all the contours are contained inside Λ and $\overline{X''\cap\gamma_1\neq\emptyset}$ means that *all* the contours in X'' intersect γ_1. The prime means that $X''=\emptyset$ is allowed. Z is the normalization factor as in (4.0).

These equations can be regarded as integral equations of the form

$$\varrho_\Lambda^*=\chi_\Lambda\alpha_\Lambda+\chi_\Lambda K\varrho_\Lambda^* \tag{4.6}$$

where

$$\begin{aligned}\alpha(X)&=0 && \text{if number of contours in } X>1\\ &=e^{-\beta|\gamma_1|-\beta\Phi(\gamma_1)} && \text{if}\quad X=\gamma_1\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}(K\mathrm{f})(X)&=e^{-\beta|\gamma_1|}e^{-\beta U_\Phi^1(X)}\\ &\quad\cdot\sum_{T\cap X=\emptyset}K(X,T)\sum_{\overline{X''\cap\gamma_1\neq\emptyset}}^{*}(-1)^{N(X'')}\mathrm{f}(X^{(1)}\cup X''\cup T)\\ (\chi_\Lambda\mathrm{f})(X)&=\mathrm{f}(X)\quad\text{if}\quad X\subset\Lambda\\ &=0\qquad\text{if}\quad X\not\subset\Lambda\end{aligned} \tag{4.8}$$

where the $*$ means that if $X=\gamma_1$ (i.e. if X contains just one contour (γ_1) then the term $T=\emptyset$, $X''=\emptyset$ is missing (because it has been included in α_Λ).

If we consider the above equation as an equation over the space of the f's with the norm

$$\|f\| = \sup_X \frac{|f(X)|}{(e^{-\beta/2})^{|X|}} \tag{4.9}$$

we find

$$\|K\| \leqq e^{\beta\|\Phi\|} e^{-\left(\beta/2 - \sum_{n=1}^{\infty} 3^n e^{-\beta n/2}\right)} e^{(e^{\beta\|\Phi\|}-1)} = L(\beta, \Phi) \tag{4.10}$$

if β is large enough and $\|\Phi\|$ small enough.

Therefore we can write for all X

$$\varrho_\Lambda^*(X) = \sum_{K=0}^{\infty} [(\chi_\Lambda K)^K (\chi_\Lambda \alpha)](X) \qquad X \subset \Lambda \tag{4.11}$$

and

$$\|\varrho_\Lambda^*\| \leqq \|\alpha_\Lambda\| \frac{1}{1-\|K_\Lambda\|} < e^{-\beta/2} \frac{1}{1-L(\beta, \Phi)}$$

hence $|\varrho_\Lambda^*(X)| \leqq (\text{const}) \cdot \exp - \beta/2|X|$.

It is easy to see that, using the strong property (4.10), the

$$\lim_{\Lambda \to \infty} \varrho_\Lambda^*(X) = \varrho_\Phi(X) \tag{4.12}$$

exists and is a solution of the equation

$$\varrho_\Phi = \alpha + K_\Phi \varrho_\Phi . \tag{4.13}$$

The kernel K_Φ depends continuously on Φ (i.e. if $\Phi \to \Phi_0$ then $\|K_\Phi - K_{\Phi 0}\| \to 0$) hence $\varrho_\Phi(X)$ depends continuously on Φ for $\|\Phi\|$ sufficiently small and β fixed but large enough. Hence the tangent plane to $P(\Phi)$ is unique at $\Phi = 0$ if β is large enough.

The equation verified by the tangent plane components at $\Phi = 0$ are

$$\varrho(X) = e^{-\beta|\gamma_1|} \sum_{Y \cap \gamma_1 \neq \emptyset}' (-1)^{N(Y)} \varrho(X^{(1)} \cup Y), \tag{4.14}$$

i.e. are the equations found and studied by Minlos and Sinai [4].

5. Conclusion

We have considered the ν-dimensional Ising model and the equilibrium states for the system enclosed in a box Λ and subject to arbitrary boundary conditions. We have studied the probability that the boundary Γ' between the spins up and the spins down contains a given set Γ of closed disjoint contours. Calling $\varrho_\Lambda(\Gamma)$ this probability we have shown that the limit

$$\lim_{\Lambda \to \infty} \bar{\varrho}_\Lambda(\Gamma) = \varrho(\Gamma) \tag{5.1}$$

exists on the net of increasing rectangles and is independent on the boundary conditions used to compute

$$\bar{\varrho}_\Lambda(\Gamma) = \frac{1}{N(\Lambda)} \sum_{\substack{x \in Z \\ \Gamma + x \subset \Lambda}} \varrho_\Lambda(\Gamma + x)$$

translational average of $\varrho_\Lambda(\Gamma)$.

Two questions remain open:

1) whether

$$\lim_{\Lambda \to \infty} \varrho_\Lambda(\Gamma) = \varrho(\Gamma) \quad \Gamma \text{ built with closed contours,}$$

i.e. if (5.1) holds with $\bar{\varrho}_\Lambda$ replaced by ϱ_Λ.

2) whether the independence of $\varrho(\Gamma)$ from the boundary conditions implies that the genral translationally invariant equilibrium state for the Ising model can be expressed as a superposition of only two extremal states (observe that converse is true: i.e. (5.1) is a necessary condition in order that the equilibrium state for the Ising model be expressible as a linear combination of only two extremal states).

It is of some importance to observe that the solution of problem 1) above does not necessarily come before the solution of problem 2). In fact, as shown in [7, 8], a given translationally invariant equilibrium state of the infinite system can be obtained as the thermodynamic limit of a suitable boundary condition which is such that $\varrho_\Lambda(\Gamma)$ is already translationally invariant i.e. $\varrho_\Lambda(\Gamma) = \varrho_\Lambda(\Gamma + x)$ if Γ and $\Gamma + x \in \Lambda$.

Restricting ourselves to such boundary conditions (which are of course enough for a complete investigation of the translationally invariant equilibrium states) one can take the bar out of Eq. (5.1). Another interesting boundary condition in which $\varrho_\Lambda(\Gamma) = \varrho_\Lambda(\Gamma + x)$ is the periodic boundary condition in Λ.

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