Commun. math. Phys. 22, 295—300 (1971)

Almost Positive Perturbations of Positive Selfadjoint Operators

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Received April 30, 1971

Abstract. Let A be a positve selfadjoint operator and let B be a symmetric perturbation of A. We establish sufficient conditions for the essential selfadjointness of $A+B$ on domains where A is essentially selfadjoint. The results have application to the $\lambda\phi^4$ field theory in two space-time dimensions.

I. Introduction

Let A be a positive selfadjoint operator with domain $\mathscr{D}(A)$. We establish sufficient conditions for $A+B$ to be essentially selfadjoint on domains where A is essentially selfadjoint, in particular on $\mathscr{C}^\infty(A)=\bigcap_{n=0}^{\infty}\mathscr{D}(A^n)$. The methods used, both generalize and depend crucially upon, two fundamental theorems concerning regular perturbations. We begin, then, by stating these theorems, together with a few definitions. Proofs may be found in [1].

Definition 1.1. An operator A is relatively bounded with respect to an operator T (or T-bounded) if $\mathscr{D}(A)\supset\mathscr{D}(T)$ and if there are constants a and b such that

$$\|A\psi\|^2\leqq a^2\|\psi\|^2+b^2\|T\psi\|^2,\ \psi\in\mathscr{D}(T). \tag{1.1a}$$

The T-bound of A is defined as the greatest lower bound of all non-negative b for which (1.1a) holds.

Definition 1.2. An operator T has *strong control* over an operator A, if A is T-bounded with T-bound strictly less than 1.

Definition 1.3. An operator T has *weak control* over an operator A if A is T bounded and (1.1a) holds with $b=1$.

It is clear that T has weak control over A if it has strong control. It is less clear that A may be T bounded with T bound equal to 1, even though T does not have weak control over A.

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* Supported in part by the U.S. Air Force under Grant AFOSR 68-1453 MOD C.

Theorem 1.4. (Kato-Rellich). *Let T be essentially selfadjoint and let A be symmetric. If T has strong control over A, then $T+A$ is essentially selfadjoint and its closure $(T+A)^{\sim}$ is equal to $\tilde{T}+\tilde{A}$.*

Theorem 1.5. (Kato). *Let T be essentially selfadjoint and let A be symmetric. If T has weak control over A, then $T+A$ is essentially selfadjoint.*

We show that these theorems admit a simple generalization. Suppose that B is a symmetric perturbation of the positive selfadjoint operator A. Suppose that A does not have strong control over B, but that $A^p, p>1$, does. Then $A+A^p$ also has strong control over B and it follows from Theorem 1.4 that $A+A^p+B$ is selfadjoint. Now in certain instances, and this is the crucial point, $A+A^p+B$ has weak control over the operator $-A^p$. If this is the case, Theorem 1.5 implies that the restriction of $A+B$ to $\mathscr{D}(A^p)$ is essentially selfadjoint. Section II is an elaboration of this theme. Our principal result is Theorem 2.5 which is strikingly concordant with Theorem 8 of a singular perturbation theory developed by Glimm and Jaffe [2]. The note has been structured with this in mind.

In III, we apply Theorem 2.5 to establish the space cut-off dynamics of a scalar field with a quartic interaction in two space-time dimensions. We comment on what we feel are essential simplifications of the original solution of this problem by Glimm and Jaffe [2, 3]. We also present a modified version of the Cannon-Jaffe proof for the selfadjointness of the locally correct Lorentz generator for this model.

II. Generalization of Regular Perturbation Theory

Definition 2.1. If S is a closable operator with domain $\mathscr{C}$ and if its closure $\tilde{S}$ is equal to an operator T, then $\mathscr{C}$ is a *core* for T.

Lemma 2.2. *Let A and N be commuting selfadjoint operators. Suppose A, $N \geqq 0$ and that $A \geqq N$. Let B by a symmetric perturbation of A which is N^p bounded for some $p \geqq 1$ and let $\mathscr{C}$ be any core of A^p. Then for some $c>0$, and any d, the restriction of $A+B+cN^p+d$ to $\mathscr{C}$, $(A+B+cN^p+d)|_{\mathscr{C}}$, is essentially selfadjoint. In particular, $A+B+cN^p+d$ is essentially selfadjoint on $\mathscr{C}^{\infty}(A)$.*

Proof. Choose $c>1$ so that cN^p has strong control over B. By successive use of Theorems 1.4 or 1.5, it follows that the following operators, in succession, are essentially selfadjoint: $cA^p|_{\mathscr{C}}, (cA^p+A)|_{\mathscr{C}}, (cA^p+A+cN^p)|_{\mathscr{C}}$, $(A+cN^p)|_{\mathscr{C}}, (A+B+cN^p)|_{\mathscr{C}}$ and $(A+B+cN^p+d)|_{\mathscr{C}}$. $\mathscr{C}^{\infty}(A)$ is a core for A^p because $\mathscr{C}^{\infty}(A)=\mathscr{C}^{\infty}(A^p)$ and the analytic vectors of an operator are contained in its C^{∞} vectors. We make use here of a theorem by Nelson [4].

Lemma 2.3. *Let A, B, and N satisfy the conditions of Lemma 2.1 and let $\mathscr{C}'$ be some core for A^p. Then for some $c>0$ and any core $\mathscr{C}$ of A^p,*

$(A+B)|_{\mathscr{C}}$ is essentially selfadjoint if the following sesquilinear form inequality holds on $\mathscr{C}'\times\mathscr{C}'$,

$$c^2N^{2p}\leqq (A+B+cN^p+d)^2+e^2\,. \tag{2.3a}$$

Here, d and e are arbitrary constants.

Proof. Let c be determined as in the proof of the preceeding Lemma. It follows, that $A+B+cN^p+d$ is essentially selfadjoint on $\mathscr{C}'$. If inequality (2.2a) holds on $\mathscr{C}'\times\mathscr{C}'$, then by closure it extends to $\mathscr{D}(A^p)\times\mathscr{D}(A^p)$, as $\mathscr{D}(A^p)\subset\mathscr{D}((A+B+cN^p+d)^{\frown})$; and hence it holds on $\mathscr{C}\times\mathscr{C}$, where $\mathscr{C}$ is any core of A^p. The essential selfadjointness of $(A+B+d)|_{\mathscr{C}}$ follows immediately from Theorem 1.5, whence the essential selfadjointness of $(A+B)|_{\mathscr{C}}$.

Lemma 2.4. *Suppose A, B and N fulfill the conditions of Lemma 2.1. Then the conditions of Lemma 2.3 are satisfied if there are non-negative constants a, ε and b with $2a+\varepsilon<2$ such that the following sesquilinear form inequalities,*

$$aA+B+b\geqq 0 \tag{2.4a}$$

and

$$\varepsilon N^{p+1}+[N^{p/2},[N^{p/2},B]]+b\geqq 0 \tag{2.4b}$$

hold on $\mathscr{C}^\infty(A)\times\mathscr{C}^\infty(A)$.

Proof. Let c be determined as in Lemma 2.2. Each of the following sesquilinear forms on $\mathscr{C}^\infty(A)\times\mathscr{C}^\infty(A)$ are well defined, and each set of inequalities is implied by its successor:

$$c^2N^{2p}\leqq (A+B+b+cN^p)^2+bc\,, \tag{2.4c}$$

$$(A+B+b)^2+cN^p(A+B+b)+c(A+B+b)N^p+bc\geqq 0\,, \tag{2.4d}$$

$$\begin{aligned}2N^{p/2}(aA+B+b)N^{p/2}&+2(1-a)N^{p/2}AN^{p/2}\\&+[N^{p/2},[N^{p/2},B]]+b\geqq 0\,,\end{aligned} \tag{2.4e}$$

$$\begin{cases}aA+B+b\geqq 0\\ 2(1-a)N^{p+1}+[N^{p/2},[N^{p/2},B]]+b\geqq 0\end{cases} \tag{2.4f}$$

and

$$\begin{cases}aA+B+b\geqq 0\\ \varepsilon N^{p+1}+[N^{p/2},[N^{p/2},B]]+b\geqq 0\,.\end{cases} \tag{2.4g}$$

As an immediate consequence of these lemmas, we have the following theorem.

Theorem 2.5. *Let A and N be commuting selfadjoint operators. Suppose $A, N\geqq 0$ and that $A\geqq N$. Let B be a symmetric perturbation of A which is N^p bounded for some $p\geqq 1$ and let $\mathscr{C}$ be any core of A^p. Then $(A+B)|_{\mathscr{C}}$*

is essentially selfadjoint if there are non-negative constants a, ε and b with $2a+\varepsilon<2$ *such that the following sesquilinear form inequalities hold on* $\mathscr{C}^\infty(A)\times\mathscr{C}^\infty(A)$,

$$aA+B+b\geqq 0$$

and

$$\varepsilon N^{p+1}+[N^{p/2},[N^{p/2},B]]+b\geqq 0.$$

Remark 2.6. $A+B$, $A+\tilde{B}$ and $(A+B)|_{\mathscr{C}^\infty(A)}$ are essentially selfadjoint.

Remark 2.7. $\tilde{B}$ is nowhere assumed selfadjoint.

Finally, we conclude this section by commenting on conditions which insure that $A+\tilde{B}$ is closed. The following is contained, *mutatis mutandis*, in Section 1 of [2].

We assume the hypothesis of Theorem 2.5. $A+\tilde{B}$ is certainly closed if the quadratic estimate

$$A^2+B^2\leqq c(A+B)^2+c \tag{2.8}$$

holds on $\mathscr{C}^\infty(A)\times\mathscr{C}^\infty(A)$ for some $c>0$; and (2.8) holds if the following two inequalities are valid on $\mathscr{C}^\infty(A)\times\mathscr{C}^\infty(A)$,

$$\begin{gathered} aA+B+b\geqq 0 \quad \text{and} \\ \delta A^2+[A^{1/2},[A^{1/2},B]]+b\geqq 0. \end{gathered} \tag{2.9}$$

Here, a, b, and δ are non-negative constants such that $2a+\delta<1$.

III. Application to $\lambda\phi^4$

We apply Theorem 2.5 to establish the space cut-off dynamics for a scalar quantum field in two space-time dimensions with a quartic interaction [2, 3; see also 7–10]. The three principal estimates for this model are

$$\varepsilon H_0+H_I(g)+b\geqq 0 \quad [5, 6], \tag{3.1}$$

$$\varepsilon N^3+[N,[N,H_I(g)]]+b\geqq 0 \quad [3] \tag{3.2}$$

and

$$\varepsilon H_0^2+[H_0^{1/2},[H_0^{1/2},H_I(g)]]+b\geqq 0 \quad [3]. \tag{3.3}$$

Here, H_0 and $H_I(g)$ are the free and space cut-off Hamiltonians, N is the number of particles operator, $\varepsilon>0$ is arbitrary and b is a positive constant which depends on ε. If we set $A=H_0$, $N=N$, $B=H_I(g)$ and $p=2$; the essential selfadjointness of $H=H_0+H_I$ restricted to $\mathscr{C}^\infty(H_0)$ follows immediately from Theorem 2.5, the fact that $H_I(g)$ is relatively bounded by N^2 and estimates (3.1) and (3.2) alone. The selfadjointness of H follows from the additional estimate (3.3) and the remarks at the end of II. We feel our results shorten and simplify the original arguments used by Glimm and Jaffe, and we take note of the following particulars:

1) The essential selfadjointness or selfadjointness of H does not depend on the selfadjointness of $H_I(g)$.

2) One less estimate is required in actually establishing the cut-off dynamics. Indeed, estimates of the type (2.4a) and (2.4b) both hold in the ϕ^6 theory, provided the coupling constant is sufficiently small.

3) The essential selfadjointness of H on standard domains, such as $\mathscr{C}^\infty(H_0)$, follows *ab initio* and does not require a separate proof.

A variation of the above technique provides a modified version of the Cannon-Jaffe proof for the selfadjointness of the locally correct Lorentz generator, or Lorentzian, of the ϕ^4 model [11, 12]. The Lorentzian M is defined by $M = \alpha H_0 + T_0(g_0) + T_I(g_1)$, where H_0 is the free Hamiltonian, $\alpha > 0$, and $T_0(g_0)$ and $T_I(g_1)$ are space modulated free and interacting Hamiltonians, respectively. The reader is referred to [11] for a full explication of these objects. There, the following bound and estimate may be found:

$$\|T_0(g_0)(H_0+I)^{-1}\| \leqq \text{const.} \tag{3.4}$$

and

$$\varepsilon H_0^2 + H_0 T_0(g_0) + T_0(g_0) H_0 + b \geqq 0 \tag{3.5}$$

The estimate,

$$\varepsilon H_0^2 + H_0 T_I(g_1) + T_I(g_1) H_0 + b \geqq 0\,, \tag{3.6}$$

follows directly from (3.1) and (3.3).

Theorem 3.7 (Cannon-Jaffe). $M = \alpha H_0 + T_0(g_0) + T_I(g_1)$ *is selfadjoint on* $\mathscr{D}(H_0) \cap \mathscr{D}(T_I(g_1))$ *and is essentially selfadjoint on* $\mathscr{C}^\infty(H_0)$.

Proof. $T_0(g_0)$ is relatively bounded by H_0. This follows from bound (3.4). Choose $\alpha' > \alpha$ so that $\alpha' H_0$ has strong control over $T_0(g_0)$. It follows as a consequence of estimate (3.6) that the selfadjoint operator $\alpha' H_0 + T_I(g_1)$ also has strong control over $T_0(g_0)$. It follows from Theorem 1.4 that $\alpha' H_0 + T_I(g_1) + T_0(g_0)$ is selfadjoint with domain $\mathscr{D}(H_0) \cap \mathscr{D}(T_I(g_1))$ and essentially selfadjoint on $C^\infty(H_0)$. Finally, estimates (3.5) and (3.6) guarantee that $\alpha' H_0 + T_I(g_1) + T_0(g_0)$ has strong control over the operator $-(\alpha' - \alpha) H_0$, and it follows again from Theorem 1.4 that $M = \alpha H_0 + T_I(g_1) + T_0(g_0)$ is selfadjoint with domain $\mathscr{D}(H_0) \cap \mathscr{D}(T_I(g_1))$ and essentially selfadjoint on $\mathscr{C}^\infty(H_0)$.

Acknowledgements. I am grateful to Professor J. Sucher for his interest and encouragement. I would like to thank Dr. B. Simon and Professor A. Jaffe for valuable comments. Finally, I would like to express my utmost gratitude to Dr. G. Svetlichny for innumerable illuminating discussions on related areas of mathematical physics prior to this work.

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