Thermodynamic States of Classical Systems

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Abstract. The existence of a unique thermodynamic state for dilute classical systems is proved for a class of regular multi-particle potentials. The method relies on integral equations for "modified" correlation functions.

Integral equations have been used successfully to prove the existence of equilibrium states of lattice systems defined by multi-particle potentials for both classical and quantum statistical mechanics [1–3], and of classical continuous systems with two-body potentials [4]. However, difficulty in deriving and solving integral equations for continuous systems with any additional multi-particle potentials has led to a belief that these linear methods do not generalize in a natural way to the case of multi-particle potentials in the continuum.

In this paper, we will demonstrate that for stable, multi-particle potentials satisfying a regularity condition, Definition 2.1, the Kirkwood-Salzburg equations for the correlation functions generalize to a set of integral equations having, for sufficiently high temperatures, a unique solution which is analytic in its parameters and to which the finite volume correlation functions of the Gibbs ensemble converge. For finite type potentials, a more transparent formulation of regularity is derived. We wish to emphasize that the success in deriving integral equations for multi-particle potentials appears to arise from an optimal use of the expected thermodynamic behavior of the solutions.

1. Integral Equations for Modified Correlation Functions

We shall study classical continuous systems of point particles in ν-dimensional Euclidean space, $\mathbb{E}^\nu$. We assume that the potential energy of the particles is specified by a sequence $\{\varphi^m\}_{m \geq 2}$ of m-body potentials; namely, for $n \geq 2$ and $x_1, \ldots, x_n \in \mathbb{E}^\nu$, then the potential energy of n

particles located at $x_1, \ldots, x_n$ is $U(x_1, \ldots, x_n)$, where

$$U(x_1, \ldots, x_n) = \sum_{m=2}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq n} \varphi^m(x_{i_1}, \ldots, x_{i_m}).$$

To simplify notation, we will generally use upper case letters to denote finite subsets of $\mathbb{E}^v$ and drop the integer superscripts on the potentials φ. In this notation,

$$U(X) = \sum_{T \subset X} \varphi(T).$$

Where it is desirable to indicate an n-element subset of $\mathbb{E}^v$, we will write X_n or $N(X) = n$, and we will abbreviate the integration symbol $\int_{\Lambda^n} dx_1 \ldots dx_n$ by $\int_{\Lambda^n} dX_n$.

Definition 1.1. The potential φ is stable [5] if there exists a positive constant B such that

$$U(X) \geqq - N(X) B$$

for all $X \subset \mathbb{E}^v$. We shall call φ ultra-stable if, for some $\mu, 0 < \mu < 1, B$ can be chosen so that

$$U(X \cup Y \cup z) - \mu(U(X \cup z) - U(X)) \geqq - N(X \cup Y) B$$

for all $X, Y, z \subset \mathbb{E}^v$ with $N(z) = 1$.

For $X \subset \mathbb{E}^v$ and any $x \in X$, define

$$W^x(X) = \sum_{S \subset X - x} \varphi(x \cup S) = U(X) - U(X - x)$$

and let $\omega(X)$ be the element of X which is the smallest element, in some lexicographic order, of the set $\{x \in X \mid W^x(X) \geqq W^y(X), \forall y \in X\}$. Then, for any $Y \subset \mathbb{E}^v$, define

$$W(X) = W^{\omega(X)}(X)$$

$$W(X, Y) = \sum_{S \subset X - \omega(X)} \varphi(S \cup Y \cup \omega(X)).$$

The finite volume correlation functions ϱ_Λ, for Λ a bounded, Lebesgue-measurable subset of $\mathbb{E}^v$, are given on $E = \bigcup_n (\mathbb{E}^v)^n$ by

$$\varrho_\Lambda(X) = Z_\Lambda^{-1} \chi_\Lambda(X) \sum_{n=0}^{\infty} \frac{z^{n+N(X)}}{n!} \int_{\Lambda^n} dY_n e^{-\beta U(X \cup Y_n)} \qquad (1)$$

where $\chi_A(X_n)$ for each $n \geq 1$ is the characteristic function of the region $A^n \subset E$, and Z_A is the grand canonical partition function:

$$Z_A = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{A^n} dY_n e^{-\beta U(Y_n)}.$$

Define modified correlation functions g_A^μ for $0 < \mu < 1$ by

$$g_A^\mu(X) = e^{\mu \beta W(X)} \varrho_A(X).$$

The modification factor in these functions will avoid the high energy divergence of the Kirkwood-Salzburg equations for multi-particle potentials.

Theorem 1.2. *For A a bounded Lebesgue-measurable subset of $\mathbb{E}^\nu$ and $\{\varphi^m\}_{m \geq 2}$ an ultra-stable potential, the finite volume modified correlation functions g_A^μ, $0 < \mu < 1$, satisfy the integral equations*

$$g_A^\mu(X) = \sum_{n=0}^{\infty} \int_{(\mathbb{E}^\nu)^n} \frac{dR_n}{n!} g_A^\mu(X \cup R_n - \omega(X)) K_A^\mu(X, R_n) + \chi_A(X) \alpha(X)$$

with the kernel K_A^μ defined by

$$K_A^\mu(X, R_n)$$
$$= (-1)^n z \chi_A(X) e^{\mu \beta W(X)} \sum_{W \subset R_n} (-1)^{N(W)} \exp\{-\beta U(X \cup W) + \beta U(X \cup W - \omega(X))$$
$$- \beta \mu W(X \cup R_n - \omega(X))\} \tag{2}$$

and

$$\alpha(X) = \begin{cases} 1, & N(X) = 0 \\ 0, & N(X) > 0 \end{cases}.$$

Proof [1]. The identity

$$z^{N(A)} e^{-\beta U(A)} = Z_A \sum_{m=0}^{\infty} \int_{A^m} \frac{u \cdot m}{m!} (-1)^m \varrho_A(A \cup V_m) \tag{3}$$

for $A \subset A$ follows from stability, since

$$\sum_{n=0}^{\infty} \int_{A^n} \frac{dV_n}{n!} (-1)^n \sum_{m=0}^{\infty} \int_{A^m} \frac{dS_m}{m!} e^{-\beta U(A \cup V_n \cup S_m)}$$

$$= \sum_{m=0}^{\infty} \int_{A^m} \frac{dR_m}{m!} e^{-\beta U(A \cup R_m)} \sum_{A \subset R_m} (-1)^{N(V)} = e^{-\beta U(A)}$$

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[1] The above equations were derived jointly with D. Robinson of Centre Universitaire, Marseille.

and the integrands are absolutely summable for stable potentials. Substitute (3) in (1) to obtain

$$
g_A^\mu(X) = Z_A^{-1} \chi_A(X) \, e^{\mu\beta W(X)} \sum_{n=0}^{\infty} \frac{z^{n+N(X)}}{n!} \int_{A^n} dS_n \, e^{-\beta U(X \cup S_n - \omega(X))}
$$

$$
\times \{ e^{-\beta U(X \cup S_n) + \beta U(X \cup S_n - \omega(X))} \}
$$

$$
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \chi_A(X) \, z \int_{A^n} \frac{dS_n}{n!} \int_{A^m} \frac{dV_m}{m!} \, g_A^\mu(X \cup S_n \cup V_m - \omega(X))
$$

$$
\times \{ (-1)^m e^{\mu\beta(W(X) - W(X \cup S_n \cup V_m - \omega(X)))} \, e^{-\beta U(X \cup S_n) + \beta U(X \cup S_n - \omega(X))} \}
$$

$$
= \sum_{n=0}^{\infty} \int_{A^n} \frac{dR_n}{n!} \, g_A^\mu(X \cup R_n - \omega(X)) \left\{ \chi_A(X) \, z \sum_{V \subset R_n} (-1)^{N(V)} e^{\mu\beta W(X)} \right.
$$

$$
\times \left. e^{-\beta U(X \cup (R_n - V)) + \beta U(X \cup (R_n - V) - \omega(X)) - \mu\beta W(X \cup R_n - \omega(X))} \right\}.
$$

Finally, observe that the R_n integration can be extended to $(\mathbb{E}^\nu)^n$, since the g_A^μ vanish outside A.

2. Regular Potentials

In order for the integral equations to have a unique solution in the limit $A \to \infty$, it will be necessary to restrict attention to a class of ultra-stable potentials satisfying a regularity requirement.

Definition 2.1. The potential φ is regular if the following properties are satisfied for some $\mu, 0 < \mu < 1$, and for β sufficiently small:

(a) Except for X in a set of Lebesgue-measure zero, if $X \subset \mathbb{E}^\nu$ and $W(X) < \infty$, then for every bounded, measurable region $A \subseteq \mathbb{E}^\nu, n > 0$, and non-degenerate covering $S_i, \dots, S_j$ of the set $Y = \{y_1, \dots, y_n\}$, $\left(\text{i.e., } \bigcup_{i=1}^{j} S_i = Y, \emptyset \neq S_i \neq S_k \text{ for } i \neq k \right)$ the function $e^{-\mu\beta W(X \cup Y_n - \omega(X))}$ $\times \prod_{i=1}^{j} |e^{-\beta W(X, S_i)} - 1|$ is integrable with respect to Y on A.

(b) Except for X in a set of Lebesgue-measure zero, if $X \subseteq \mathbb{E}^\nu$ and $W(X) = \infty$, then $W(X, R) > -\infty$ almost everywhere (with respect to R).

(c) if $A \to \infty$ in the sense of van Hove, then

$$
\lim_{A \to \infty} \underset{X \subset A}{\mathrm{ess\,sup}} \left\{ \sum_{n=1}^{\infty} \int \frac{dY_n}{n!} \sum_{j=1}^{\infty} \sum_{\substack{S_1, \dots, S_j \subset Y_n \\ \cup_i S_i = Y_n \\ \emptyset \neq S_i \neq S_k}} e^{-\mu\beta W(X \cup Y_n - \omega(X))} \right.
$$

$$
\left. \times \prod_{i=1}^{j} |e^{-\beta W(X, S_i)} - 1| \, e^{-\beta(1-\mu) W(X)} \right\} < \infty.
$$

We shall see that for an ultra-stable, regular potential, a unique infinite volume state can be defined for dilute systems. An equivalent formulation of regularity which gives a more transparent restriction on the potentials φ or the "interaction energies" W would obviously be desirable. Although we cannot derive this in general, we will give in Section 3 a sufficient criterion for regularity for a class of physically reasonable potentials.

Ultra-stability yields immediately.

Lemma 2.2. *For z sufficiently small, $g_\Lambda^\mu \in L^\infty(E)$, the Banach space of bounded, Lebesgue-measurable functions on E with sup norm $\| \ \|_\infty$.*

Lemma 2.3. *If φ is an ultra-stable, regular potential, then the kernel K_Λ^μ is a bounded operator on $L^\infty(E)$, with norm*

$$\|K_\Lambda^\mu\| \le \operatorname*{ess\ sup}_{X \subset \Lambda} \left\{ z \sum_{n=1}^\infty \int_{\Lambda^n} \frac{dY_n}{n!} \sum_{j=1}^\infty \sum_{\substack{S_1,\ldots,S_j \subset Y_n \\ \cup_i S_i = Y_n \\ \emptyset \neq S_i \neq S_k}} \prod_{i=1}^j |e^{-\beta W(X,S_i)} - 1| \right.$$

$$\left. \times \exp\left\{ -\beta(1-\mu)\, W(X) - \mu\beta\, W(X \cup Y_n - \omega(X)) \right\} \right\}.$$

Proof. Compute

$$K_\Lambda^\mu(X, R) \exp\left\{ \beta(1-\mu)\, W(X) + \beta\mu\, W(X \cup R - \omega(X)) \right\}$$

$$= z\chi_\Lambda(X)(-1)^{N(R)} \sum_{W \subset R} (-1)^{N(W)} \exp\left\{ -\beta \sum_{\substack{S \subset W \\ S \neq \emptyset}} W(X, S) \right\}$$

$$= z\chi_\Lambda(X)(-1)^{N(R)} \sum_{W \subset R} (-1)^{N(W)} \sum_{j=1}^\infty \sum_{\substack{S_1,\ldots,S_j \subset W \\ \emptyset \neq S_i \neq S_k}} \prod_{i=1}^j (e^{-\beta W(X,S_i)} - 1)$$

$$= z\chi_\Lambda(X) \sum_{j=1}^\infty \sum_{\substack{S_1,\ldots,S_j \subset R \\ \emptyset \neq S_i \neq S_k}} \sum_{V \subset R - \cup_i S_i} (-1)^{N(V)} \prod_{i=1}^j (e^{-\beta W(X,S_i)} - 1)$$

$$= z\chi_\Lambda(X) \sum_{j=1}^\infty \sum_{\substack{S_1,\ldots,S_j \subset R \\ \cup_i S_i = R \\ \emptyset \neq S_i \neq S_k}} \prod_{i=1}^j (e^{-\beta W(X,S_i)} - 1)$$

observing in the last equation that $\sum_{V \subset T} (-1)^{N(V)}$ vanishes unless $T = \emptyset$.

Lemma 2.4. *If φ is an ultra-stable, regular potential and β is sufficiently small, then for some $\mu < 1$, the K_Λ^μ converge to a bounded operator K_∞^μ on $L^\infty(E)$ as $\Lambda \to \infty$ in the following sense. For every sequence $\{\Lambda_i\}$*

which converges $\lim\limits_{i \to \infty} \Lambda_i = \infty$ *in the sense of van Hove, and for each* $f \in L^1(E)$,
$K_{\Lambda_i}^{\mu+} f \to K_\infty^{\mu+} f$ *strongly in* $L^1(E)$.

Here, and throughout, T^+, for T an operator on $L^\infty(E)$, signifies the adjoint of T restricted to the space $L^1(E)$ of absolutely Lebesgue-integrable functions with norm $\| \ \|_1$.

Proof. Define K_∞^μ pointwise by Eq. (2) with the characteristic function χ_Λ replaced by the identity function,

$$K_\Lambda^\mu(X, Y) = \chi_\Lambda(X) \, K_\infty^\mu(X, Y)$$

and let $\sigma(X)$ denote the Lebesgue measure of any set $S \subset E$.

If $\varepsilon > 0$ and $\sigma(C) < \infty$, there exists a bounded set $C' \subset C$ with $\sigma(C - C') < \varepsilon$. For i sufficiently large, $\Lambda_i \supset C'$, and then

$$K_{\Lambda_i}^{\mu+} f(s) = \chi_{\Lambda_i}(s) \, K_\infty^{\mu+} f(s) = K_\infty^{\mu+} f(s)$$

if $s \in C'$. Thus

$$\sigma(\{s \,|\, s \in C, \ K_{\Lambda_i}^{\mu+} f(s) - K_\infty^{\mu+} f(s) \neq 0\}) < \varepsilon.$$

That is, $K_{\Lambda_i}^{\mu+} f \to K_\infty^{\mu+} f$ in Lebesgue measure on every measurable set of finite measure.

Furthermore, $K_{\Lambda_i}^{\mu+} f \to K_\infty^{\mu+} f$ weakly. For if $\varepsilon > 0$, choose a bounded measurable region $D \subset E$ so that $\| f - f_D \|_1 < \varepsilon$, where f_D denotes the restriction of f to D. Then, for i sufficiently large, $\Lambda_i \supset D$, and for each $\tau \in L^\infty(E)$,

$$\left| \int_E f(\overline{X}) \, ((K_{\Lambda_i}^\mu - K_\infty^\mu)\, \tau)(X) \, dX \right| \leq \left| \int_D f(\overline{X}) \, ((K_{\Lambda_i}^\mu - K_\infty^\mu)\, \tau)(X) \, dX \right|$$
$$+ \varepsilon \| K_{\Lambda_i}^\mu - K_\infty^\mu \| \, \| \tau \|_\infty \leq \varepsilon (\| K_{\Lambda_i}^\mu \| + \| K_\infty^\mu \|) \, \| \tau \|_\infty.$$

The Lemma follows from the observation that strong convergence in L^1 is equivalent to weak convergence and convergence in measure on every measurable set of finite measure.

Definition 2.5. Let g_∞^μ be a solution of the integral equation

$$(1 - K_\infty^\mu) \, g_\infty^\mu = \alpha \,.$$

Call ϱ_∞, defined by

$$\varrho_\infty(X) = e^{-\beta \mu W(X)} g_\infty^\mu(X)$$

an infinite volume state.

When $\| K_\Lambda^\mu \| \leq C < 1$ for all Λ, K_∞^μ is a strict contraction on $L^\infty(E)$, and the integral equation in Definition 2.5 has a unique solution.

Theorem 2.6. *If φ is an ultra-stable, regular potential, then in a region of the $\beta - z$ plane defined by*

$$\lim_{\Lambda \to \infty} \operatorname*{ess\,sup}_{X \subset \Lambda} \left\{ \sum_{n=1}^{\infty} \int \frac{d\,Y_n}{n!} \sum_{j=1}^{\infty} \sum_{\substack{S_1,\ldots,S_j \subset Y_n \\ \cup_i S_i = Y_n \\ \emptyset \neq S_i \neq S_k}} \prod_{i=1}^{j} |e^{-\beta W(X,S_i)} - 1| \right.$$

$$\left. \times \exp\{-\beta(1-\mu)\,W(X) - \beta\mu\,W(X \cup Y_n - \omega(X))\} \right\}$$

$$< z^{-1}$$

there is a unique infinite volume state ϱ_∞. If $\Lambda \to \infty$ in the sense of van Hove, then g_Λ^μ converges weakly to g_∞^μ, and ϱ_Λ converges pointwise to ϱ_∞ almost everywhere. In particular, if $W(X)$ is bounded below, then ϱ_Λ converges to ϱ_∞ weakly.

Proof. It remains only to be proved that for any $f \in L^1(E)$, $\varepsilon > 0$, and sequence $\{\Lambda_i\}$ with $\Lambda_i \to \infty$,

$$|(f, (I - K_{\Lambda_i}^\mu)^{-1} \alpha \chi_{\Lambda_i}) - (f, (I - K_\infty^\mu)^{-1} \alpha)| < \varepsilon$$

if i is sufficiently large, where we have used (f, τ) to denote the integral over E of the product $f(X)\tau(X)$, $f \in L^1(E)$, $\tau \in L^\infty(E)$. Estimating $(I - K^+)^{-1} f$ by $\sum_{j=0}^{N} (K^+)^j f$ and $(K_\Lambda)^i - (K)^i$ by $\sum_{k=1}^{i} K_\Lambda^{i-k}(K_\Lambda - K)K^{k-1}$, it is clear that $(I - K_{\Lambda_i}^{\mu+})^{-1} f \to (I - K_\infty^{\mu+})^{-1} f$ strongly, by Lemma 2.4. Then:

$$|(f, (I - K_{\Lambda_i}^\mu)^{-1} \alpha \chi_{\Lambda_i}) - (f, (I - K_\infty^\mu)^{-1} \alpha)|$$

$$\leq |((I - K_{\Lambda_i}^{\mu+})^{-1} f - (I - K_\infty^{\mu+}) f, \alpha \chi_{\Lambda_i})|$$

$$+ |((I - K_\infty^{\mu+})^{-1} f, \alpha(1 - \chi_{\Lambda_i}))|$$

$$\leq \varepsilon/2 + |((I - K_\infty^{\mu+}) f, \alpha(1 - \chi_{\Lambda_i}))|$$

for i sufficiently large. But if $D \subset E$ is a bounded measurable region satisfying $\|(I - K_\infty^{\mu+})^{-1} f - ((I - K_\infty^{\mu+})^{-1} f)_D\|_1 < \varepsilon/2$, then

$$|((I - K_\infty^{\mu+})^{-1} f, \alpha(1 - \chi_{\Lambda_i}))| < \varepsilon/2$$

when $\Lambda_i \supset D$.

Corollary 2.7. *If φ is an ultra-stable, regular potential, then in the region of the $\beta - z$ plane defined in Theorem 2.6, $\varrho_\infty(X)$ is an analytic function of β and z. If φ is a translation-invariant potential, then ϱ_∞ is translation invariant.*

3. Potentials of Finite Type

For potentials which satisfy some physically reasonable restrictions of finiteness and continuity, a sufficient condition for regularity can be given which directly generalizes the usual definition of regularity [5] for two-body interactions.

Toward this end, define $d: E \to \mathbb{R}$ by $d(X) = \min_{\substack{x, y \in X \\ x \neq y}} |x - y|$. If $\{\delta_n\}_{n \geq 1}$ is a sequence of non-negative numbers, let $E_{\{\delta_n\}} = \{X \in E | d(X) > \delta_{N(X)}\} \subset E$, and write $I_X(Y)$ for $|e^{-\beta W(X, Y)} - 1|/N(Y)!$, $X, Y \in E$.

Definition 3.1. A potential $\{\varphi^n\}_{n \geq 2}$ is of finite type if
(a) for each n, φ^n is continuous on $(\mathbb{E}^\nu)^n \cap E_{\{0\}}$,
(b) $\varphi^2(x, y) \to \infty$ as $d(x \cup y) \to 0$, $x, y \in \mathbb{E}^\nu$,
(c) φ is bounded below on E,
(d) for each $X \in E$, there exists a sequence $\{t_X^n\}_{n \geq 1}$ of step functions in $L^1(E)$, with intervals of uniform volume $1/n$ and bounded below by I_X, such that $t_X^n \to I_X$ strongly and $\{t_X^n\}_{n \geq 1}$ is Cauchy uniformly in X.

Lemma 3.2. If φ is ultra-stable and of finite type, then $W(X)/N(X)$ is bounded below on E, and $W(X) \to \infty$ as $d(X) \to 0$.

Theorem 3.3. *A stable potential φ of finite type is regular if for β sufficiently small and $\Lambda \to \infty$ in the sense of van Hove,*

$$\lim_{\Lambda \to \infty} \sup_{X \subset \Lambda} \left\{ \sum_{n=1}^\infty \int dY_n I_X(Y_n) \right\} < \infty.$$

Proof. Let $f(n)$ be a positive decreasing function of the positive integers n, to be fixed later, and choose $\{\delta_n\}_{n \geq 1}$ to satisfy

$$e^{-\beta W(Y)} < f(n) \quad \text{whenever } N(Y) = n \text{ and } d(Y) < \delta_n.$$

It is clearly sufficient to prove:

$$\lim_{\Lambda \to \infty} \sup_{X \subset \Lambda} \left\{ \sum_{n=1}^\infty \int\limits_{E_{\{\delta_n\}}} \frac{dY_n}{n!} \sum_{j=1}^\infty \sum_{\substack{S_1, \ldots, S_j \subset Y_n \\ \cup_i S_i = Y_n \\ \emptyset \neq S_i \neq S_k}} \left(\prod_{i=1}^j |e^{-\beta W(X, S_i)} - 1| \right) \right\} < \infty \quad (5)$$

uniformly in $\{\delta_n\}$, and for each $\varepsilon > 0$ and $X \in E$,

$$\left\{ \sum_{n=1}^\infty \int\limits_{E - E_{\{\delta_n\}}} \frac{dY_n}{n!} \sum_{j=1}^\infty \sum_{\substack{S_1, \ldots, S_j \subset Y_n \\ \cup_i S_i = Y_n \\ \emptyset \neq S_i \neq S_k}} (f(n))^\mu \prod_{i=1}^j |e^{-\beta W(X, S_i)} - 1| \right\} < \varepsilon \quad (6)$$

for some $\{\delta_n\}_{n \geq 1}$.

Consider (6) first. Fix $X \in E$, let $\varepsilon > 0$, and assume φ is bounded below by $M < -1$. Then $W(X, S_i) > 2^{N(X)} M$ and $|e^{-\beta W(X, S_i)} - 1| < e^{2^{N(X)} \beta M}$.

By (4), Λ_0 can be chosen sufficiently large so that

$$\int_{E-\Lambda^n} \frac{dY_n}{n!} |e^{\beta W(X,Y_n)}-1| < \varepsilon \quad \text{and} \quad |e^{-\beta W(X,Y_n)}-1| < 1 \quad \text{if} \quad Y_n \cap \Lambda^n \neq Y_n$$

for all n and all $\Lambda \supset \Lambda_0$. Split the integration in (6) over the sets $(E - E_{\{\delta_n\}}) \cap \left(\bigcup_n \Lambda_0^n\right)$ and $(E - E_{\{\delta_n\}}) \cap \left(E - \bigcup_n \Lambda_0^n\right)$. Since the number of coverings of Y_n by j subsets is certainly less than $\binom{2^n}{j}$, the integration over $(E - E_{\{\delta_n\}}) \cap \left(\bigcup_n \Lambda_0^n\right)$ can be bounded by

$$\sum_{n=1}^{\infty} \left\{ \frac{V(\Lambda_0)^n}{n!} 2^{2n} e^{\beta M 2^{N(X)+n}} (f(n))^{\mu} \right\} \tag{7}$$

and the integration over $(E - E_{\{\delta_n\}}) \cap \left(E - \bigcup_n \Lambda_0^n\right)$ by ε times the above. Eq. (6) then is satisfied by choosing $f(n)$ so that (7) is less than ε.

To prove (5), fix Λ, let $P_n = \{[x_1, \dots, x_n]\}$ be a uniform partition of Λ^n with $V([x_1, \dots, x_n]) = \lambda$, for all n; here the intervals $[x_1, \dots, x_n]$ of the partition are labeled by the centers $(x_1, \dots, x_n)$ of the hypercubes $[x_1, \dots, x_n]$. Define

$$J_{\{x_1 \cup \dots \cup x_n\}}(X) = \sup_{S \in [x_1, \dots, x_n]} |e^{-\beta W(X,S)} - 1|$$

and choose λ sufficiently small so that

$$\left| \sum_{n=1}^{\infty} \left(\sum_{P_n} J_{\{x_1 \cup \dots \cup x_n\}}(X) \frac{\lambda}{n!} - \int_{\Lambda^n} \frac{dY_n}{n!} \{|e^{-\beta W(X,Y_n)} - 1|\} \right) \right| < \varepsilon$$

for all X. Then:

$$\sum_{n=1}^{\infty} \int_{E_{\{\delta_n\}} \cap \Lambda^n} \frac{dY_n}{n!} \sum_{j=1}^{\infty} \sum_{\substack{S_1, \dots, S_j \subset Y_n \\ \cup_i S_i = Y_n \\ \emptyset \neq S_i \neq S_k}} \prod_{i=1}^{j} |e^{-\beta W(X,S_i)} - 1|$$

$$\leqq \sum_{n=1}^{\infty} \sum_{P_n} \frac{\lambda}{n!} \sum_{j=1}^{\infty} \sum_{\substack{S_1, \dots, S_j \subset \{x_1 \cup \dots \cup x_n\} \\ \cup_i S_i = Y_n \\ \emptyset \neq S_i \neq S_k}} \left(\prod_{i=1}^{j} J_{\{S_i\}}(X) \lambda \right) \lambda^{-j}$$

$$\leqq \sum_{j=1}^{\infty} \frac{1}{j!} \left\{ \sum_{n=1}^{\infty} \frac{1}{n! \lambda} \sum_{P_n} J_{\{x_1 \cup \dots \cup x_n\}}(X) \lambda \right\}^j$$

$$\leqq \exp\left\{ \sum_{n=1}^{\infty} \frac{1}{\lambda} \int_{\Lambda^n} \frac{dY_n}{n!} \{|e^{-\beta W(X,Y_n)} - 1|\} + \frac{\varepsilon}{\lambda} \right\} < \infty$$

which completes the proof.

From the foregoing, it is easy to construct examples of regular multi-body potentials. The simplest such example is a translation-invariant potential with two and three body components only, satisfying:

i) $\varphi^2(0, x)$ is continuous, bounded below, and absolutely integrable outside a set of finite Lebesgue measure, and $\varphi^2(0, x) \to \infty$ as $|x| \to 0$.

ii) φ^3 is continuous, non-negative, and symmetric in its arguments, and $\varphi^3(0, x_1, x_2)$ vanishes outside a bounded region.

4. Integral Equation Techniques

The technique of using integral equations to study families of functions on a Banach space is most fruitful if either the Banach space is chosen to accomodate the special properties of the family, as in Ruelle's analysis of two-body potentials [5], or if, as above, the family is optimally modified to fit a fixed Banach space. In either case, one seeks *a priori* bounds for the correlation functions, and thus expects for thermodynamic inter-actions implicit cutoffs at high particle densities and high energy densities. It is for this reason that it is natural to study the modified correlation functions with their "modified" high energy behavior rather than the correlation functions themselves.

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