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On the Stability of the Taub Universe

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Abstract. An analysis of the stability of the Taub universe for arbitrary, initially small perturbations is carried out. It is found that the perturbations decrease during the expansion and increase during the contraction of the unperturbed space. In the process we obtain the general solution to a system of six coupled, linear, partial differential equations in six unknown functions of four variables.

I. Introduction

The solution to Einstein's field equations found by Taub [1] has aroused considerable interest on several counts. (a) It is a *non-flat* solution of the *empty-space* equations, having closed, homogeneous, space-like hypersurfaces which expand anisotropically. It can thus be interpreted as describing a universe containing nothing but gravitational radiation. (b) It is well behaved for all finite values of the coordinates, but becomes singular (in the sense that certain components of the Riemann tensor become infinite) at $t = \pm\infty$. The invariants $R_{ijkl}R^{ijkl}$ and $R_{ijkl}R^{ikjl}$, however, remain finite. (c) Newman, Unti, and Tamburino [2] have obtained a metric which extends the solution to values of the proper time outside the range covered by Taub's coordinates[1]. (d) The extended space-time is *maximal* [3] (i.e. not part of a still larger space-time), but it is *geodesically incomplete* [3] (i.e. there exist in it geodesics which cannot be extended to infinite values of their affine length). (e) In the part of the manifold outside the two singularities (NUT space) there exist closed time-like curves [4].

Some of these properties are more than mere mathematical curiosities. In particular the singularities and the expansion are features found in almost all cosmological solutions. With its anisotropy, Taub space thus appears to be the simplest generalization of the Friedman models – albeit without matter. In the early stages of the evolution of the universe, however, when the curvature is high, the presence or absence of matter

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[1] The proper time interval covered by Taub's coordinates is finite.

has little effect[2] on the evolution of the geometry, and Misner has used Taub-like metrics to discuss the nature of the singularity near the zero of time [5]. Finally, the investigations of Behr [6] imply that the presence of matter does not significantly alter the nature of the solution – in particular, the occurrence of singularities.

For a cosmological solution to be acceptable as a model for the real universe, however, it must be stable against small perturbations. It is easy to show that, when we introduce a small amount of pressureless matter, its density grows without limit as the singularity is approached. Since this particular perturbation does not remain small, one concludes that the Taub space is unstable [3]. We wish to point out, however, that this instability is intimately connected to the presence of a singularity in the future, and precisely the same behavior is found [7] when one perturbs the closed Friedman models which also have a future singularity. In fact, *all* perturbations of the closed Friedman models grow without limit as the singularity is approached [7]. Hence, the density argument is not by itself sufficient to rule out Taub space as a possible cosmological model, as long as one is willing to consider the closed Friedman models as such. The fact that, in the Taub case, the infinity destroys the smooth continuation into NUT should not concern us since NUT space, being acausal, is of no interest as a cosmological model.

In this paper we carry out a general perturbation analysis of the Taub metric. It is found that, just as in the positive curvature Robertson-Walker metrics, all perturbations decrease during expansion and increase during contraction. We therefore conclude that, as far as stability is concerned, the Taub metric is as good a description of a closed universe as are the positive curvature Robertson-Walker metrics.

But quite apart from strengthening the case for Taub space, the analysis carried out in this paper is interesting from a purely mathematical point of view, being a rigorous solution of a system of six coupled differential equations, linear and homogeneous with respect to six unknown functions of four variables, with coefficients depending on one variable (the time coordinate). Moreover, the steps followed in obtaining the solution provide, we believe, some new insight into the structure of Einstein's system of equations, which, in the final analysis, is what makes the solution of such a complicated problem possible.

II. The Exact Solution

The Taub-NUT metric can be written in the form:

$$ds^2 = l^2\{F(d\tau)^2 - A(\omega^1)^2 - A(\omega^2)^2 - B(\omega^3)^2\}, \tag{2.1}$$

[2] The energy of the field, being quadratic in the $g_{ij,k}$, dominates the energy of matter when the curvature is high enough (i.e. when the $g_{ij,k}$ are large).

where l is a constant length and F, A, and B are functions of the dimensionless time coordinate $\tau = \frac{ct}{l}$. The ω^α's are one-forms satisfying

$$d\omega^\alpha = \tfrac{1}{2}\varepsilon^\alpha{}_{\beta\gamma}\,\omega^\beta \wedge \omega^\gamma\,. \tag{2.2}$$

They can be expressed in terms of the Euler angles ψ, θ, φ as follows:

$$\left.\begin{aligned} \omega^1 &= -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\varphi\,,\\ \omega^2 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\varphi\,,\\ \text{and}\qquad \omega^3 &= d\psi + \cos\theta\, d\varphi\,. \end{aligned}\right\} \tag{2.3}$$

The field equations $R_{ij} = 0$ for this metric are

$$R_{00} = -\frac{\ddot{A}}{A} - \frac{1}{2}\frac{\ddot{B}}{B} + \frac{1}{2}\frac{\dot{A}^2}{A^2} + \frac{1}{4}\frac{\dot{B}^2}{B^2} + \frac{1}{2}\frac{\dot{A}\dot{F}}{AF} + \frac{1}{4}\frac{\dot{B}\dot{F}}{BF} = 0\,, \tag{2.4}$$

$$\frac{2F}{A}R_{11} = \frac{2F}{A}R_{22} = \frac{\ddot{A}}{A} - \frac{1}{2}\frac{\dot{A}\dot{F}}{AF} + \frac{1}{2}\frac{\dot{A}\dot{B}}{AB} - \frac{BF}{A^2} + \frac{2F}{A} = 0\,, \tag{2.5}$$

and

$$\frac{2F}{B}R_{33} = \frac{\ddot{B}}{B} - \frac{1}{2}\frac{\dot{B}\dot{F}}{BF} - \frac{1}{2}\frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{BF}{A^2} = 0\,, \tag{2.6}$$

(where a dot signifies differentiation with respect to τ) with the off-diagonal components being identically zero. There is one non-trivial Bianchi identity, namely

$$\begin{aligned}\left(R^i_0 - \frac{1}{2}R\delta^i_0\right)_{;i} &= \left(R^0_0 - \frac{1}{2}R\right)^{\cdot}\\ &\quad + \left(\frac{\dot{A}}{A} + \frac{1}{2}\frac{\dot{B}}{B}\right)R^0_0 - \frac{\dot{A}}{A}R^1_1 - \frac{1}{2}\frac{\dot{B}}{B}R^3_3 = 0\,;\end{aligned} \tag{2.7}$$

this implies that

$$2F\left(R^0_0 - \frac{1}{2}R\right) = \frac{1}{2}\frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{BF}{2A^2} + \frac{2F}{A} = 0 \tag{2.8}$$

is a first integral of (2.4)–(2.6).

As is evident from physical considerations, the field equations allow F to be specified arbitrarily (corresponding to the arbitrariness in the choice of the time coordinate). For any given F, Eq. (2.7) ensures the compatibility of the three Eqs. (2.4)–(2.6) in the two unknown functions A and B, while Eq. (2.8) implies a constraint on the constants of integration. If one chooses for F the general form $F = A^m B^n$, then the particular choice $m = 2$ makes (2.6) an equation for B alone, while $n = -1$ makes the difference (2.5)–(2.8) an equation for A alone. Once uncoupled, the equations can be integrated rather easily.

Taub's [1] choice was $F = A^2 B$ and he obtained

$$A = \frac{\cosh\tau}{2\{1+\cosh(\tau+a)\}}, \qquad B = \frac{1}{\cosh\tau} \tag{2.9}$$

which becomes singular ($B = 0$) at $\tau = \pm\infty$.

Misner [3, 4] chooses $F = 4/B$ and obtains

$$A = 1+\tau^2, \qquad B = \frac{4(1+2a\tau-\tau^2)}{1+\tau^2} \tag{2.10}$$

with $B = 0$ at $\tau = a \pm \sqrt{a^2+1}$ ($a = m/l$ in his notation).

Finally, if we choose $F = A^2 B^{-1}$ we find

$$A = \frac{1}{2(1+\cos\tau)}, \qquad B = \cos\tau + a\sin\tau \tag{2.11}$$

with $B = 0$ at $\cot\tau = -a$ (two values differing by π).

In the above expressions a stands for the only non-trivial constant of integration of (2.4)–(2.6). The other two are the origin and the unit of the time variable, which were chosen equal to 0 and 1, respectively. The last two solutions allow the metric to be continued analytically outside the "Taub" region, where $B < 0$ and τ becomes space-like, while ψ is now time-like. Since we will be using Misner's choice, we rewrite Eqs. (2.5), (2.6), and (2.8) for $F = 4/B$; we have

$$\frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} - \frac{4}{A^2} + \frac{8}{AB} = 0, \tag{2.12}$$

$$\frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{4}{A^2} = 0, \tag{2.13}$$

and

$$\frac{1}{2}\frac{\dot{A}^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} - \frac{2}{A^2} + \frac{8}{AB} = 0. \tag{2.14}$$

III. The Perturbation

We now write[3] $g'_{ij} = g_{ij} + \delta g_{ij}$ where g_{ij} are the known functions of time and δg_{ij} are small corrections depending on all four coordinates. It is well known that we can impose four arbitrary conditions on the δg_{ij}'s without restricting the generality of the perturbation. (What is more, there are still coordinate dependent perturbations which must be removed later – see Appendix C.) We choose

$$\delta g_{0\alpha} = 0 \tag{3.1}$$

for three of these conditions and as a fourth require that g'_{00} is the same function of $g'_{\alpha\beta}$ as g_{00} is of $g_{\alpha\beta}$. For example, if $F = 4/B$ we require that $F' = 4/B'$ which, to first order, becomes

$$\delta F = -\frac{4}{B^2}\,\delta B\,. \tag{3.1a}$$

In the following we specialize to Misner's choice of time coordinate $F = 4/B$ (even though everything can be done in the same way for arbitrary F), and parametrize the non-zero metric coefficients as follows:

$$g'_{ij} = g_{ij} + \delta g_{ij} = -\begin{pmatrix} -\dfrac{4}{B}\left(1-\dfrac{\beta}{B}\right) & 0 & 0 & 0 \\ 0 & A+\alpha+\gamma & \kappa & \lambda \\ 0 & \kappa & A+\alpha-\gamma & \mu \\ 0 & \lambda & \mu & B+\beta \end{pmatrix}, \tag{3.2}$$

where α, β, γ, κ, λ, and μ are six unknown functions of τ, ψ, θ, and φ. Here and in what follows A and B are the known functions of time given by Eq. (2.10), and satisfying Eqs. (2.12)–(2.14).

We also introduce a small amount of dust-like matter, with density ϱ, satisfying the conservation law

$$(\varrho u^i u^j)_{;j} = 0\,. \tag{3.3}$$

To first order this equation gives

$$\varrho A\sqrt{B} = \text{constant} \stackrel{\text{def}}{=} \left(\frac{8\pi G}{c^4}\right)^{-1} M \text{ (small of 1}^{\text{st}}\text{ order)}\,. \tag{3.3a}$$

The field equations[4] for the metric (3.2) with source $T_{ij} = \varrho u_i u_j$, where only terms linear in the δg_{ij} are retained, are given by

[3] Latin indices have the range 0, 1, 2, and 3; Greek 1, 2, and 3.

[4] The computations are most easily done using the methods of Cartan. We use the notation $R_{ij} = R^s_{isj}$ where $\frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l = d\omega^i_j + \omega^i_s \wedge \omega^s_j$. The ω^i_j's are found by solving $dg_{ij} = \omega_{ij} + \omega_{ji}$ and $d\omega^i = -\omega^i_s \wedge \omega^s$ for the metric (3.2) and the one-forms (2.3). The ω^i_j's are given in Appendix D.

$$\delta R_{00} = \frac{2M}{AB^{\frac{3}{2}}} \quad = -\left(\frac{\alpha}{A}\right)^{\cdot\cdot} - \left(\frac{\alpha}{A}\right)^{\cdot}\left(\frac{\dot{A}}{A} + \frac{1}{2}\frac{\dot{B}}{B}\right) - \frac{1}{2}\left(\frac{\beta}{B}\right)^{\cdot\cdot} - \left(\frac{\beta}{B}\right)^{\cdot}\left(\frac{\dot{B}}{B} + \frac{1}{2}\frac{\dot{A}}{A}\right) - 2\left[\frac{L_1^2 + L_2^2}{AB} + \frac{L_3^2}{B^2}\right]\frac{\beta}{B}, \tag{3.4}$$

$$\frac{\delta R_{11} - \delta R_{22}}{2} = 0 \quad = \frac{B}{8}\left\{\ddot{\gamma} + \dot{\gamma}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \gamma\left(\frac{\dot{A}^2}{A^2} + \frac{16}{B^2} - \frac{4}{A^2}\right)\right\} - \frac{L_3^2\gamma}{2B} - L_3\kappa\left(\frac{2}{B} - \frac{1}{2A}\right) + \frac{L_3(L_1\lambda - L_2\mu)}{2B} + \frac{L_1\mu + L_2\lambda}{B}, \tag{3.5}$$

$$\frac{\delta R_{11} + \delta R_{22}}{2} = \frac{M}{2\sqrt{B}} \quad = \frac{B}{8}\left\{\ddot{\alpha} + \dot{\alpha}\frac{\dot{B}}{B} + \alpha\frac{4}{A^2} + \dot{\beta}\frac{\dot{A}}{B} - \beta\frac{\dot{A}\dot{B} + 8}{B^2}\right\} - \frac{L_3^2\alpha}{2B} - \frac{(L_1^2 + L_2^2)\alpha}{2A} + \frac{(L_1^2 - L_2^2)\gamma + (L_1L_2 + L_2L_1)\kappa}{2A} + \frac{L_2\lambda - L_1\mu}{A} + \frac{L_3(L_1\lambda + L_2\mu)}{2B}, \tag{3.6}$$

$$\delta R_{33} = \frac{M\sqrt{B}}{2A} = \frac{B}{8}\left\{\ddot{\beta} + \dot{\beta}\frac{\dot{A}}{A} + \beta\frac{4}{A^2} + \dot{\alpha}\frac{\dot{B}}{A} - \alpha\left(\frac{\dot{A}\dot{B}}{A^2} + \frac{8B}{A^3}\right)\right\} + \frac{L_3^2\beta}{2B} - \frac{(L_1^2 + L_2^2)\beta}{2A} - \frac{L_3^2\alpha}{A} + \frac{L_3(L_1\lambda + L_2\mu)}{A} - \frac{B}{A^2}(L_2\lambda - L_1\mu), \tag{3.7}$$

$$\delta R_{01} = \quad 0 \quad = -L_1\left\{\left(\frac{\alpha - \gamma}{2A}\right)^{\cdot} + \frac{\dot{\beta}}{2B}\right\} + L_2\left(\frac{\kappa}{2A}\right)^{\cdot} + \frac{A}{2B}L_3\left(\frac{\lambda}{A}\right)^{\cdot} + \frac{A}{2B}\left(\frac{\mu}{A}\right)^{\cdot} - \frac{B}{2A}\left(\frac{\mu}{B}\right)^{\cdot}, \tag{3.8}$$

$$\delta R_{02} = \quad 0 \quad = -L_2\left\{\left(\frac{\alpha + \gamma}{2A}\right)^{\cdot} + \frac{\dot{\beta}}{2B}\right\} + L_1\left(\frac{\kappa}{2A}\right)^{\cdot} + \frac{A}{2B}L_3\left(\frac{\mu}{A}\right)^{\cdot} - \frac{A}{2B}\left(\frac{\lambda}{A}\right)^{\cdot} + \frac{B}{2A}\left(\frac{\lambda}{B}\right)^{\cdot}, \tag{3.9}$$

$$\delta R_{03} = \quad 0 \quad = -L_3\left\{\left(\frac{\alpha}{A}\right)^{\cdot} + \frac{\alpha}{2A}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + \frac{\beta}{2B}\frac{\dot{A}}{A}\right\} + \frac{B}{2A}\left(\frac{L_1\lambda + L_2\mu}{B}\right)^{\cdot}, \tag{3.10}$$

$$\delta R_{12} = \quad 0 \quad = \frac{B}{8}\left\{\ddot{\kappa} + \dot{\kappa}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \kappa\left(\frac{\dot{A}^2}{A^2} + \frac{16}{B^2} - \frac{4}{A^2}\right)\right\} - \frac{L_3^2\kappa}{2B} + L_3\gamma\left(\frac{2}{B} - \frac{1}{2A}\right) + \frac{L_3(L_1\mu + L_2\lambda)}{2B} - \frac{L_1\lambda - L_2\mu}{B}, \tag{3.11}$$

$$\delta R_{23} = \quad 0 \quad = L_2L_3\left(\frac{\beta}{2B} - \frac{\alpha + \gamma}{2A}\right) - L_1\left(\frac{\gamma}{A} + \frac{\beta}{2A} - \frac{B\alpha}{2A^2}\right) + \frac{B}{8}\left\{\ddot{\mu} + \mu\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot{A}^2}{A^2}\right)\right\} - \frac{L_1^2\mu}{2A} + \frac{L_1(L_2\lambda + L_3\kappa)}{2A} + \frac{L_3\lambda - L_2\kappa}{A}, \tag{3.12}$$

$$\delta R_{31} = \quad 0 \quad = L_1L_3\left(\frac{\beta}{2B} - \frac{\alpha - \gamma}{2A}\right) - L_2\left(\frac{\gamma}{A} - \frac{\beta}{2A} + \frac{B\alpha}{2A^2}\right) + \frac{B}{8}\left\{\ddot{\lambda} + \lambda\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot{A}^2}{A^2}\right)\right\} - \frac{L_2^2\lambda}{2A} + \frac{L_2(L_1\mu + L_3\kappa)}{2A} + \frac{L_1\kappa - L_3\mu}{A}. \tag{3.13}$$

The Bianchi identities take the form

$$\left.\begin{aligned}&\left(\frac{B}{4}\delta R_{00}-\frac{1}{2}\delta R\right)^{\cdot}-\frac{1}{A}(L_1\delta R_{01}+L_2\delta R_{02})-\frac{1}{B}L_3\delta R_{03}\\&+\frac{B}{4}\left(\frac{\dot A}{A}+\frac{1}{2}\frac{\dot B}{B}\right)\delta R_{00}+\frac{\dot A}{A}\frac{\delta R_{11}+\delta R_{22}}{2A}+\frac{1}{2}\frac{\dot B}{B}\frac{\delta R_{33}}{B}=0,\end{aligned}\right\}\tag{3.14}$$

$$\left.\begin{aligned}&\left(\frac{B}{4}\delta R_{01}\right)^{\cdot}-L_1\left\{\frac{\delta R_{11}+\delta R_{22}}{2A}+\frac{\delta R_{11}-\delta R_{22}}{2A}+\frac{1}{2}\delta R\right\}\\&-\frac{1}{A}L_2\delta R_{12}-\frac{1}{B}L_3\delta R_{13}+\frac{B}{4}\frac{\dot A}{A}\delta R_{01}+\frac{B-A}{BA}\delta R_{23}=0,\end{aligned}\right\}\tag{3.15}$$

$$\left.\begin{aligned}&\left(\frac{B}{4}\delta R_{02}\right)^{\cdot}-L_2\left\{\frac{\delta R_{11}+\delta R_{22}}{2A}-\frac{\delta R_{11}-\delta R_{22}}{2A}+\frac{1}{2}\delta R\right\}\\&-\frac{1}{A}L_1\delta R_{12}-\frac{1}{B}L_3\delta R_{23}+\frac{B}{4}\frac{\dot A}{A}\delta R_{02}+\frac{A-B}{AB}\delta R_{31}=0,\end{aligned}\right\}\tag{3.16}$$

$$\begin{aligned}&\left(\frac{B}{4}\delta R_{03}\right)^{\cdot}-\frac{1}{A}(L_1\delta R_{31}+L_2\delta R_{32})\\&\qquad-L_3\left\{\frac{\delta R_{33}}{B}+\frac{1}{2}\delta R\right\}+\frac{B}{4}\frac{\dot A}{A}\delta R_{03}=0.\end{aligned}\tag{3.17}$$

In the above equations, L_1, L_2, and L_3 are differential operators dual to the one-forms ω^α. They can be defined on an arbitrary function f by

$$\begin{aligned}df-\dot f\,d\tau&=\frac{\partial f}{\partial\psi}d\psi+\frac{\partial f}{\partial\varphi}d\varphi+\frac{\partial f}{\partial\theta}d\theta\\&\stackrel{\text{def}}{=}(L_1 f)\,\omega^1+(L_2 f)\,\omega^2+(L_3 f)\,\omega^3.\end{aligned}\tag{3.18}$$

As is to be expected from Eq. (2.2), they satisfy the angular momentum commutation rules: $[L_1, L_2]=-L_3$, etc. Hence, they are the angular momentum operators expressed in terms of the Euler angles (see Appendix A, and reference given therein, for their explicit expression and some of their properties).

IV. Reduction of the Equations

In this section we show that the solution of the system (3.4)–(3.13) is equivalent to the solution of:

(a) two pairs of partial differential equations of simple form, each pair involving only two unknown functions;

(b) two ordinary differential equations in two other functions;

(c) a quadrature.

Examining first Eq. (3.4) for δR_{00} and remembering that A and B are functions of time only, we see that α and β must be of the form $f_1(\tau) f_2(\psi, \theta, \varphi)$ with f_2 being an eigenfunction of the operators $L_1^2 + L_2^2$ and L_3. If we denote by S_m^l the eigenfunctions of L^2 and L_3 satisfying

$$\left.\begin{aligned} L^2 S_m^l &\stackrel{\text{def}}{=} (L_1^2 + L_2^2 + L_3^2) S_m^l = -l(l+1) S_m^l, \\ L_3 S_m^l &= im S_m^l, \\ \text{and} \quad (L_1^2 + L_2^2) S_m^l &= -L S_m^l, \quad \text{where} \quad L \stackrel{\text{def}}{=} l(l+1) - m^2, \end{aligned}\right\} \tag{4.1}$$

we can express α and β in terms of the complete set of functions S_m^l as follows (see Appendix A for a more complete description of the S_m^l's):

$$\left.\begin{aligned} \alpha &= A\left\{a_M(\tau) + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{l,m}(\tau) S_m^l(\psi, \theta, \varphi)\right\} \\ \text{and} \\ \beta &= B\left\{b_M(\tau) + \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{l,m}(\tau) S_m^l(\psi, \theta, \varphi)\right\} \end{aligned}\right\} \tag{4.2}$$

where we must demand that

$$(a_{l,m})^* = a_{l,-m} \quad \text{and} \quad (b_{l,m})^* = b_{l,-m} \tag{4.2a}$$

in order that α and β be real. In (4.2) we have taken out factors A and B from the time dependence of α and β and we have split up the space independent part of the expansion into two terms: $a_M(\tau)$ and $b_M(\tau)$ (which are proportional to M, the amount of matter introduced with the perturbation) and a_{00} and b_{00}; and these latter terms like all other $a_{l,m}$ and $b_{l,m}$ satisfy homogeneous equations. The equations satisfied by $a_M(\tau)$ and $b_M(\tau)$ are[5]

$$\frac{4M}{AB^{\frac{3}{2}}} = -\left\{2\ddot{a}_M + \dot{a}_M\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \ddot{b} + \dot{b}\left(2\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\right)\right\}, \tag{4.3}$$

$$\frac{4M}{AB^{\frac{3}{2}}} = \ddot{a}_M + \dot{a}_M\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + a_M\left(\frac{8}{A^2} - \frac{8}{AB}\right) + \dot{b}_M\frac{\dot{A}}{A} - \frac{8}{AB}b_M, \tag{4.4}$$

$$\frac{4M}{AB^{\frac{3}{2}}} = \ddot{b}_M + \dot{b}_M\left(2\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\right) + \dot{a}_M\frac{\dot{B}}{B} - \frac{8}{A^2}a_M. \tag{4.5}$$

Having taken care of the source terms in Eqs. (3.4)–(3.13) we find that, for each l, m, Eq. (3.4) for δR_{00} implies that $a_{l,m}$ and $b_{l,m}$ satisfy

$$2\ddot{a} + \dot{a}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \ddot{b} + \dot{b}\left(2\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\right) - 4b\left[\frac{L}{AB} + \frac{m^2}{B^2}\right] = 0. \tag{4.6}$$

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[5] See Eqs. (3.4), (3.6), and (3.7).

In this equation (and in what follows) we have suppressed the subscripts l, m from a and b and we have used the notation $L = l(l+1) - m^2$ (see (4.1)). Substituting expressions (4.2) for α and β in (3.10) for δR_{03}, we find that

$$\frac{L_1 \lambda + L_2 \mu}{B} = i \sum_{l=0}^{\infty} \sum_{m=-l}^{l} K_{l,m}(\tau) S_m^l, \tag{4.7}$$

where, for each l, m, $K_{l,m}$ satisfies the equation

$$\dot{K} = m \frac{A}{B} \left\{ 2\dot{a} + a\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B} \right) + b \frac{\dot{A}}{A} \right\}. \tag{4.8}$$

Turning next to Eq. (3.7) for δR_{33} and using (4.7) and (4.2), we find

$$\frac{L_2 \lambda - L_1 \mu}{A} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} R_{l,m}(\tau) S_m^l, \tag{4.9}$$

where, for each l, m, R is given by

$$R \stackrel{\text{def}}{=} G - mK \tag{4.10}$$

where

$$\begin{aligned} G \stackrel{\text{def}}{=} \frac{AB}{8} \bigg\{ & \ddot{b} + \dot{b}\left(2\frac{\dot{B}}{B} + \frac{\dot{A}}{A} \right) + 4b\left(\frac{L}{AB} - \frac{m^2}{B^2} \right) \\ & + \dot{a}\frac{\dot{B}}{B} + a\left(\frac{8m^2}{B^2} - \frac{8}{A^2} \right) \bigg\}. \end{aligned} \tag{4.11}$$

Finally, Eq. (3.6) for $(\delta R_{11} + \delta R_{22})/2$ now becomes

$$\frac{1}{A} \left\{ (L_1^2 - L_2^2)\, \gamma + (L_1 L_2 + L_2 L_1)\, \kappa \right\} = - \sum_{l=0}^{\infty} \sum_{m=-l}^{l} r_{l,m}(\tau) S_m^l, \tag{4.12}$$

where r is defined by

$$r \stackrel{\text{def}}{=} 2(F+G) - 3mK \tag{4.13}$$

where

$$\begin{aligned} F \stackrel{\text{def}}{=} \frac{AB}{8} \bigg\{ & \ddot{a} + \dot{a}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) + 4a\left(\frac{2}{A^2} + \frac{L-2}{AB} + \frac{m^2}{B^2} \right) \\ & + \dot{b}\frac{\dot{A}}{A} - \frac{8}{AB} b \bigg\}. \end{aligned} \tag{4.14}$$

It will be noted that all functions of time defined in (4.7)–(4.14) ultimately depend on a and b which so far satisfy only one Eq. (4.6). Like a and b they must be understood to carry indices l, m.

Having satisfied δR_{00}, $(\delta R_{11}+\delta R_{22})/2$, δR_{03}, and δR_{33} and remembering that

$$\delta R = \frac{B}{4}\,\delta R_{00} - \frac{\delta R_{11}+\delta R_{22}}{A} - \frac{\delta R_{33}}{B}, \tag{4.15}$$

so that $\delta R = 0$ also, we observe that the Bianchi identities (3.14) and (3.17) imply respectively

$$\begin{aligned} 0 &= L_1\,\delta R_{01} + L_2\,\delta R_{02} \\ &= \frac{1}{2} S^l_m \left\{ L\left(\dot a + \dot b + b\frac{\dot B}{B}\right) - \dot r - m\frac{A}{B}\left(\frac{B}{A}K\right)^{\cdot} + \frac{B}{A}\left(\frac{A}{B}R\right)^{\cdot} \right\}, \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} 0 &= L_1\,\delta R_{31} + L_2\,\delta R_{32} \\ &= \frac{im}{2} S^l_m \left\{ -r + \frac{B}{4m}\left[(BK)^{\cdot\cdot} + BK\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot A^2}{A^2} - \frac{8}{AB}\right)\right] \right. \\ &\qquad \left. + 3R + (a-b)\left(\frac{B}{A} + L\right)\right\}. \end{aligned} \tag{4.17}$$

Eqs. (4.16) and (4.17) suggest that we consider next the following combinations of derivatives of δR_{01}, δR_{02} and δR_{31}, δR_{32}:

$$L_2\,\delta R_{01} - L_1\,\delta R_{02} = 0 \tag{4.18}$$

and

$$L_2\,\delta R_{31} - L_1\,\delta R_{32} = 0\,. \tag{4.19}$$

Using the commutation rules (A.5)–(A.7) given in Appendix A, we obtain

$$\begin{aligned} &-L_3\left[\left(\frac{\alpha}{A}\right)^{\cdot} + \frac{\dot\beta}{B}\right] + \frac{d}{d\tau}\left[\frac{(L_2^2 - L_1^2)\,\kappa + (L_1L_2 + L_2L_1)\gamma}{A}\right] \\ &\qquad + \frac{A}{B}L_3\left(\frac{L_2\lambda - L_1\mu}{A}\right)^{\cdot} - \frac{B}{A}\left(\frac{L_1\lambda + L_2\mu}{B}\right)^{\cdot} = 0 \end{aligned} \tag{4.18a}$$

and

$$\left.\begin{aligned} &\left[L_3^2 + \frac{B}{A}(L_1^2 + L_2^2)\right]\left(\frac{\beta}{B} - \frac{\alpha}{A}\right) - (L_1^2 + L_2^2)\left(\frac{L_2\lambda - L_1\mu}{A}\right) \\ &+ \frac{B}{4}\left[(L_2\lambda - L_1\mu)^{\cdot\cdot} + (L_2\lambda - L_1\mu)\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot A^2}{A^2} - \frac{8}{AB}\right)\right] \\ &+ L_3\left[\frac{(L_2^2 - L_1^2)\,\kappa + (L_1L_2 + L_2L_1)\gamma}{A}\right] - 2L_3\left(\frac{L_1\lambda + L_2\mu}{A}\right) = 0\,. \end{aligned}\right\} \tag{4.19a}$$

We observe that in these equations λ and μ enter in combinations already "known" (see (4.7) and (4.9)) and thus we can express certain third order derivatives of κ and γ directly in terms of a and b as follows:

$$L_3\left\{\frac{(L_2^2-L_1^2)\,\kappa+(L_1L_2+L_2L_1)\gamma}{A}\right\}=-mkS_m^l\,, \tag{4.19b}$$

where

$$\left.\begin{aligned} mk &\stackrel{\text{def}}{=} (a-b)\left(m^2+\frac{B}{A}L\right)+LR+2m\frac{B}{A}K \\ &\quad+\frac{B}{4}\left[(AR)^{\cdot\cdot}+AR\left(\frac{6}{A^2}-\frac{1}{2}\frac{\dot{A}^2}{A^2}-\frac{8}{AB}\right)\right], \end{aligned}\right\} \tag{4.20}$$

and

$$\begin{aligned} &\frac{d}{d\tau}\left\{\frac{(L_2^2-L_1^2)\,\kappa+(L_1L_2+L_2L_1)\gamma}{A}\right\} \\ &\qquad=i\left\{m\left(\dot{a}+\dot{b}+b\frac{\dot{B}}{B}\right)-m\frac{A}{B}\dot{R}+\frac{B}{A}\dot{K}\right\}S_m^l\,. \end{aligned} \tag{4.18b}$$

For Eqs. (4.18b) and (4.19b) to be consistent with each other, the following integrability condition must be satisfied:

$$\left.\begin{aligned} m\dot{k} &\stackrel{\text{def}}{=} m\left\{m\left(\dot{a}+\dot{b}+b\frac{\dot{B}}{B}\right)-m\frac{A}{B}\dot{R}+\frac{B}{A}\dot{K}\right\} \\ &=\frac{d}{d\tau}\left\{(a-b)\left(m^2+\frac{B}{A}L\right)+2m\frac{B}{A}K+LR\right. \\ &\quad\left.+\frac{B}{4}\left[(AR)^{\cdot\cdot}+AR\left(\frac{6}{A^2}-\frac{1}{2}\frac{\dot{A}^2}{A^2}-\frac{8}{AB}\right)\right]\right\}. \end{aligned}\right\} \tag{4.21}$$

When a and b satisfy this equation (in Section VI we show that the terms in K drop out) we can write

$$\frac{1}{A}\left[(L_2^2-L_1^2)\,\kappa+(L_1L_2+L_2L_1)\gamma\right]=i\sum_{l=0}^{\infty}\sum_{m=-l}^{l}k_{l,m}S_m^l\,, \tag{4.22}$$

where k is given by (4.20) for $m \neq 0$ and both (4.18) and (4.19) are satisfied. Remembering that (4.16) and (4.17) are already satisfied, we find that we have succeeded in making δR_{01}, δR_{02}, δR_{31}, and δR_{32} vanish separately. The remaining two equations $(\delta R_{11}-\delta R_{22})/2$ and δR_{12} are then automatically satisfied by virtue of the two remaining Bianchi identities (3.15) and (3.16). All these statements will be verified in the next section after explicit expressions for γ, κ, λ, and μ have been given.

For particular values of m the integrability condition is satisfied as a consequence of some other equation. For example, when $m = 0$, $k =$ constant makes (4.21) trivial. These special cases will be discussed separately in Section VI. The following two equations will be useful in this connection:

$$m(k - r) = \frac{B}{4}\left[\ddot{X} + X\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot{A}^2}{A^2} - \frac{8(m+1)}{AB}\right)\right] + (L - m)\left[R + (a - b)\left(\frac{B}{A} - m\right)\right] \tag{4.23}$$

and

$$\dot{r} - \dot{k} = \frac{B}{A}\left(\frac{X}{B}\right)^{\cdot} + m\frac{A}{B}\left(\frac{X}{A}\right)^{\cdot} + (L - m)\left(\dot{a} + \dot{b} + b\frac{\dot{B}}{B}\right) \tag{4.24}$$

where

$$X \stackrel{\text{def}}{=} AR - BK\,. \tag{4.25}$$

Eq. (4.23) follows from (4.17) and (4.20), while (4.24) follows from (4.16) and the definition of $\dot{k}$ in (4.21). We note that if both (4.23) and (4.24) vanish, then the integrability condition is trivially satisfied.

We thus choose as our second equation which, together with (4.6), determines a and b, the integrability condition (4.21) (or any other equation which for particular values of l, m implies (4.21)). For each pair of eigenvalues l, m, we have now reduced the system of Eqs. (3.4)–(3.13) to the following:

(a) Two ordinary differential equations ((4.6) and (4.21) or its equivalent) determining a and b.

(b) Two pairs of partial differential equations, each pair involving only two functions ((4.7) and (4.9) for λ and μ, and (4.12) and (4.22) for κ and γ).

(c) A quadrature ((4.8) giving K when a and b are known).

V. Integration of the Space Derivatives

We now turn our attention to Eqs. (4.7) and (4.9), and (4.12) and (4.22) which determine the space dependence of λ, μ and γ, κ, respectively:

$$L_1\lambda + L_2\mu = iBKS^l_m\,, \tag{5.1}$$

$$L_2\lambda - L_1\mu = ARS^l_m\,, \tag{5.2}$$

$$\frac{1}{A}\left[(L_1^2 - L_2^2)\gamma + (L_1L_2 + L_2L_1)\kappa\right] = -rS^l_m\,, \tag{5.3}$$

and

$$\frac{1}{A}[(L_2^2 - L_1^2)\,\kappa + (L_1 L_2 + L_2 L_1)\,\gamma] = ik S_m^l\,. \tag{5.4}$$

Remembering that $L_\pm = L_1 \pm i L_2$ and hence

$$(L_\pm)^2 = L_1^2 - L_2^2 \pm i(L_1 L_2 + L_2 L_1)$$

we find that the above equations can be written in the following simple forms:

$$(5.1) \pm i(5.2) = L_\pm(\lambda \mp i\mu) = i(BK \pm AR)\,S_m^l \tag{5.5}$$

and

$$(5.3) \pm i(5.4) = (L_\pm)^2 \left(\frac{\gamma \mp i\kappa}{A}\right) = -(r \pm k)\,S_m^l\,. \tag{5.6}$$

Eqs. (5.5) and (5.6) imply that $\lambda \mp i\mu$ is proportional to $S_{m\mp 1}^l$ while $\gamma \mp i\kappa$ is proportional to $S_{m\mp 2}^l$. Making use of the phase convention (A.18) given in Appendix A we find that the following expressions satisfy (5.5) and (5.6):

$$\begin{aligned} \lambda \mp i\mu = {} & \mp i(-1)^m \frac{AR \pm BK}{\sqrt{(l \pm m)\,(l \mp m + 1)}}\, S_{m\mp 1}^l \\ & \mp i(-1)^l\, P_{\pm l}(\tau)\, S_{\pm l}^l + Q_\pm(\tau) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \frac{\gamma \mp i\kappa}{A} = {} & \frac{(r \pm k)\,S_{m\mp 2}^l}{\sqrt{(l \pm m)\,(l \mp m + 1)\,(l \pm m - 1)\,(l \mp m + 2)}} \\ & + \frac{f_{\pm l}(\tau)}{\sqrt{2l}}\, S_{\pm(l-1)}^l + g_{\pm l}(\tau)\, S_{\pm l}^l + h_\pm(\tau)\,. \end{aligned} \tag{5.8}$$

The first terms in the above expressions are not defined when $m = \pm l$ or $m = \pm(l-1)$. These are the particular values of m, however, for which the integrability condition (4.21) can be satisfied as a consequence of some other equation which eliminates these terms. This point will be discussed in detail in the next section. The functions of time P, Q, f, g, and h enter as "constants" of integration since $L_\pm(S_{\pm l}^l) = 0$ and $(L_\pm)^2 S_{\pm(l-1)}^l = 0$. Actually these functions are precisely the number needed to complete the expansion of λ, μ, γ, and κ in terms of the S_m^l's. (Note that the coefficients in these expansions which couple to a and b do not extend to $m = \pm l$; see Appendix B, Eqs. (B.6)–(B.9).) Ordinary differential equations determining these functions are obtained when we demand that λ, μ, γ, and κ as given by (5.7) and (5.8) satisfy δR_{01}, δR_{02}, δR_{31}, δR_{32}, $(\delta R_{11} - \delta R_{22})/2$, and δR_{12}. At the same time we will verify that the terms which depend on a and b (through R, K, r, k) do indeed cancel as was claimed at the end of Section IV. It is convenient to consider the combinations

$$\left.\begin{aligned}\delta R_{01}\pm i\delta R_{02}&=-L_{\pm}\left[\left(\frac{\alpha}{2A}\right)^{\cdot}+\frac{\dot\beta}{2B}\right]+L_{\mp}\left(\frac{\gamma\pm i\kappa}{2A}\right)^{\cdot}\\&+\frac{A}{2B}(L_3\mp i)\left(\frac{\lambda\pm i\mu}{A}\right)^{\cdot}\pm i\frac{\dot B}{2A}\left(\frac{\lambda\pm i\mu}{B}\right)^{\cdot},\end{aligned}\right\}\tag{5.9}$$

$$\left.\begin{aligned}\delta R_{31}\pm i\delta R_{32}&=L_{\pm}\left(L_3\mp i\frac{B}{A}\right)\left(\frac{\beta}{2B}-\frac{\alpha}{2A}\right)\pm iL_3\left(\frac{\lambda\pm i\mu}{A}\right)\\&+(L_3\mp i)\,L_{\mp}\left(\frac{\gamma\pm i\kappa}{2A}\right)\pm iL_{\pm}\left(\frac{L_2\lambda-L_1\mu}{2A}\right)\\&+\frac{B}{8}\left\{(\lambda\pm i\mu)^{\cdot\cdot}+(\lambda\pm i\mu)\left(\frac{6}{A^2}-\frac{1}{2}\frac{\dot A^2}{A^2}\right)\right\},\end{aligned}\right\}\tag{5.10}$$

and

$$\left.\begin{aligned}&\frac{1}{2}(\delta R_{11}-\delta R_{22})\pm i\delta R_{12}=L_{\pm}(L_3\mp i)\left(\frac{\lambda\pm i\mu}{2B}\right)\\&+\frac{AB}{8}\left\{\left(\frac{\gamma\pm i\kappa}{A}\right)^{\cdot\cdot}+\left(\frac{\dot A}{A}+\frac{\dot B}{B}\right)\left(\frac{\gamma\pm i\kappa}{A}\right)^{\cdot}+\left(\frac{16}{B^2}-\frac{8}{AB}\right)\left(\frac{\gamma\pm i\kappa}{A}\right)\right\}\\&\pm i\left(\frac{4A}{B}-1\right)L_3\left(\frac{\gamma\pm i\kappa}{2A}\right)-\frac{A}{B}L_3^2\left(\frac{\gamma\pm i\kappa}{2A}\right).\end{aligned}\right\}\tag{5.11}$$

In these expressions λ, μ, γ, and κ stand for the complete perturbations as in Eqs. (3.4)–(3.13), while in (5.7) and (5.8) they stand just for the l, m term in their expansions. We take this risk of slightly confusing the reader in order to avoid overburdening our equations with indices and summation signs.

Substituting (5.7), (5.8) and (4.2) in (5.9) and looking at the coefficient of the different harmonics, we find that the terms in a, b, R, K, r, and k cancel by virtue of (4.24) while the functions f, P, and Q must satisfy the equations[6]

$$\dot f_l-\frac{A}{B}(l+1)\left(\frac{P_l}{A}\right)^{\cdot}+\frac{B}{A}\left(\frac{P_l}{B}\right)^{\cdot}=0\tag{5.12}$$

and

$$\frac{A}{B}\left(\frac{Q}{A}\right)^{\cdot}-\frac{B}{A}\left(\frac{Q}{B}\right)^{\cdot}=0\,.\tag{5.13}$$

Doing the same with (5.10) we find that in this case it is (4.23) which makes the terms in a, b, R, K, r, and k cancel, while f, P, and Q must

[6] We drop the indices $\pm l$ since the two functions are complex conjugates of each other; see Eq. (B.3b).

satisfy the equations

$$\ddot{P}_l + P_l\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot{A}^2}{A^2} + \frac{8l}{AB}\right) + \frac{4(l+1)}{B} f_l = 0 \tag{5.14}$$

and

$$\ddot{Q} + Q\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot{A}^2}{A^2}\right) = 0\,. \tag{5.15}$$

Finally, we turn to Eq. (5.11). Here we must make use of the Bianchi identities (3.15) and (3.16) to prove that the terms in a, b, R, K, r, and k cancel. Since δR_{01}, δR_{02}, δR_{31}, and δR_{32} have been satisfied, these identities read simply

$$\tfrac{1}{2} L_1(\delta R_{11} - \delta R_{22}) + L_2 \delta R_{12} = 0 \tag{5.16a}$$

and

$$\tfrac{1}{2} L_2(\delta R_{11} - \delta R_{22}) - L_1 \delta R_{12} = 0\,; \tag{5.16b}$$

or

$$L_\pm \{\tfrac{1}{2}(\delta R_{11} - \delta R_{22}) \mp i \delta R_{12}\} = 0\,. \tag{5.17}$$

For the remaining terms to vanish, the functions P, f, g, and h must satisfy the equations

$$\ddot{f}_l + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{f}_l + 4\left[\frac{(l+1)^2}{B^2} - \frac{l+1}{AB}\right] f_l = -\frac{8l(l+1)}{AB^2} P_l\,, \tag{5.18}$$

$$\ddot{g}_l + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{g}_l + 4\left[\frac{(l+2)^2}{B^2} - \frac{l+2}{AB}\right] g_l = 0\,, \tag{5.19}$$

and

$$\ddot{h} + \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\dot{h} + 4\left[\frac{4}{B^2} - \frac{2}{AB}\right] h = 0\,. \tag{5.20}$$

It may be verified that the three Eqs. (5.12), (5.14), and (5.18) for the two functions f and P are consistent with each other, while (5.15) is a consequence of (5.13). In fact, the solution for Q may be found by inspection from (5.13). It is $Q = A - B$.

VI. The Time Dependence

In this section we consider the ordinary differential equations which determine the time dependence of the solution and show that, for each l, m, they allow the required number of constants to be specified corresponding to arbitrary initial perturbations. Two of these equations are (4.6) and (4.8) which we rewrite here:

$$2\ddot{a} + \dot{a}\left(2\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) + \ddot{b} + \dot{b}\left(2\frac{\dot{B}}{B} + \frac{\dot{A}}{A}\right) - 4b\left[\frac{L}{AB} + \frac{m^2}{B^2}\right] = 0 \tag{6.1}$$

and

$$\dot{K} = m\frac{A}{B}\left\{2\dot{a} + a\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + b\frac{\dot{A}}{A}\right\}. \tag{6.2}$$

The third equation depends on the value of m and of $l-m$[7]. If $l-m$ is different than 0 or 1 and $m \neq 0$ this equation is the integrability condition (4.21). Carrying out the indicated differentiation we find that the terms in K drop out[8] leaving

$$\left.\begin{aligned} &\dddot{R} + \left(3\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\ddot{R} + 4\left(\frac{4}{A^2} + \frac{L-6}{AB} + \frac{m^2}{B^2}\right)\dot{R} \\ &\quad + \frac{8G}{A^2}\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right) + \frac{4L}{A^2}\left\{\dot{a} - \dot{b} + (a-b)\left(\frac{\dot{B}}{B} - \frac{\dot{A}}{A}\right)\right\} \\ &\quad + \frac{8m^2}{AB}\left\{\dot{a} - \dot{b} + \frac{1}{2}(a+b)\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)\right\} = 0\,. \end{aligned}\right\} \tag{6.3}$$

Remembering that R contains $\ddot{b}$ we see that (6.3) is of the fifth order in b and, together with (6.1) and (6.2), eight constants are required to integrate the system. These eight constants correspond to the twelve initial values of α, β, γ, κ, λ, and μ and their first time derivatives subject to the four constraints $\delta R_i^0 - \frac{1}{2}\delta_i^0\,\delta R = 0$:

$$\begin{aligned} &R - r - 2mK + \frac{AB}{4}\left\{\dot{a}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right)\right. \\ &\quad \left. + 4a\left(\frac{1}{A^2} + \frac{L-2}{AB} + \frac{2m^2}{B^2}\right) + \dot{b}\frac{\dot{A}}{A} + \frac{4(L-2)}{AB}\,b\right\} = 0\,, \end{aligned} \tag{6.4a}$$

$$\dot{r} - L\left(\dot{a} + \dot{b} + b\frac{\dot{B}}{B}\right) + m\frac{A}{B}\left(\frac{B}{A}K\right)^{\cdot} - \frac{B}{A}\left(\frac{A}{B}R\right)^{\cdot} = 0, \tag{6.4b}$$

$$-\dot{k} + m\left(\dot{a} + \dot{b} + b\frac{\dot{B}}{B}\right) - m\frac{A}{B}\dot{R} + \frac{B}{A}\dot{K} = 0, \tag{6.4c}$$

and

$$\dot{K} - m\frac{A}{B}\left\{2\dot{a} + a\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) + b\frac{\dot{A}}{A}\right\} = 0\,. \tag{6.4d}$$

These constraints determine r, $\dot{r}$, $\dot{k}$, and $\dot{K}$ when a, $\dot{a}$, b, $\dot{b}$, R, $\dot{R}$, K, and k are given.

The equations being linear and homogeneous, only the seven ratios of these eight constants are meaningful. This is seen in the fact that neither

[7] We assume that $m > 0$. To obtain the corresponding equations for $m < 0$ we need only make the substitutions $m \to -m$, $K \to -K$, and $k \to -k$.

[8] $\dot{R}$ contains $\dot{K}$ which, by (6.2), is expressible in terms of a, b.

(6.1) nor (6.3) contain K. Hence

$$a=0, \quad b=0, \quad \text{and} \quad K=1 \tag{6.5}$$

is a particular solution of (6.1)–(6.3). By adding a multiple of this solution, we can always make $K(\tau_0)=1$. The magnitudes of the remaining seven initial value data are then fixed.

However, since Eqs. (6.1)–(6.3) include, as special cases, the coordinate dependent perturbations found in Appendix C, we can, by a change in the coordinate system, make three more (not *any* three) of the initial value data equal to anything we please. If we choose to make $a(\tau_0)$, $b(\tau_0)$, and $\dot{a}(\tau_0)$ equal to zero, then we must demand that the initial values of the functions u, p, and v determining the coordinate system satisfy the equations

$$Lv - \frac{\dot{A}}{A}u = -a(\tau_0), \tag{6.6}$$

$$2mp - \frac{\dot{B}}{B}u = -b(\tau_0), \tag{6.7}$$

and

$$\left(\frac{1}{2}\frac{\dot{A}^2}{A^2} - \frac{2}{A^2} + \frac{4L}{AB}\right)u - \frac{\dot{A}}{A}mp = -\dot{a}(\tau_0). \tag{6.8}$$

These equations follow from (C.29) and (C.30). In terms of the functions a, b, and K the four linearly independent solutions of (6.1)–(6.3) already known are given in the Table.

Table

a	b	K	Initial values of v	u	p	Description
0	0	1	–	–	–	Amplitude
1	0	$m\frac{A}{B}$	1	0	0	Coordinate Perturbations
$Lv-\frac{\dot{A}}{A}u$	$2mp-\frac{\dot{B}}{B}u$	$Lp+mv\left(\frac{LA}{B}-1\right)$	0	1	0	
$Lv-\frac{\dot{A}}{A}u$	$2mp-\frac{\dot{B}}{B}u$	$Lp+mv\left(\frac{LA}{B}-1\right)$	0	0	1	

In the last two lines, the functions u, p, and v satisfy equations (C.24) and (C.27) given in Appendix C. Since K does not enter in (6.3), the second solution in the table implies that a does not enter either. This can be verified directly, but the resulting equation is too long to be given here.

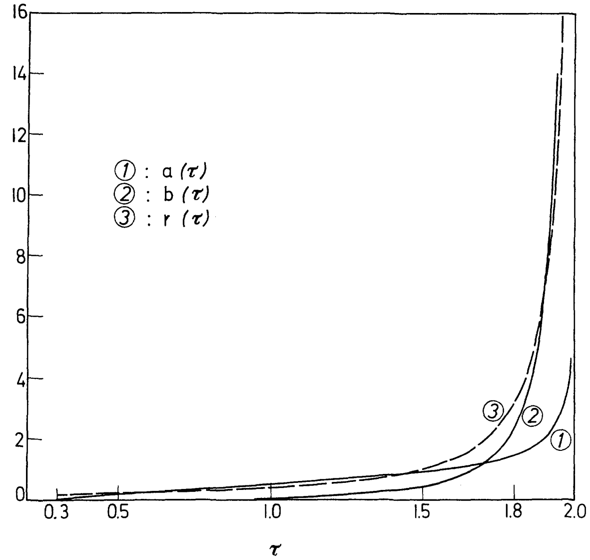

Fig. 1. The behavior of the functions a, b, and r for $m=0$, $l=2$. (Initial conditions: $\dot{a}=1$ and all other initial values equal to zero.) The same qualitative behavior is found for all other initial conditions (and also for other values of l) as long as $m=0$

As is to be expected from the $1/B$ terms in Eqs. (6.1)–(6.3), all perturbations grow without limit as the singularity $B=0$ is approached. The behavior of the solution for $m \neq 0$ is dramatically different from that when $m=0$. When $m=0$, the functions a, b, R, and r increase monotonically as the singularity is approached (see Fig. 1 – K and k are constants in this case). When $m \neq 0$, however, all six functions execute violent oscillations of increasing amplitude and frequency as they approach $B=0$; as can be seen from Figs. 2 and 3, the frequency depends on l and m. For a given l and m, the overall behavior is independent of the initial conditions, even though the amplitude depends strongly on which derivative of b is assumed non-zero initially (see Figs. 4 and 5). The oscillatory approach to the singularity is similar to what has been encountered in the Kasner [9] and the mix-master [5] universes.

We now turn our attention to special values of $l-m$ for $m \neq 0$. We will consider the case $m=0$ separately. When $l-m=0$, for our solution to be meaningful we must have $AR-BK=0$ *and* $r-k=0$ (see equations (5.7) and (5.8)), while at the same time the integrability condition must be satisfied. Now, when $l=m$, $L-m=l(l+1)-m^2-m=0$, and by (4.23) $AR-BK=0$ *implies* $r-k=0$. Since (4.24) is also equal to zero, the integrability condition is satisfied. Thus for the case $l=m$ we take in

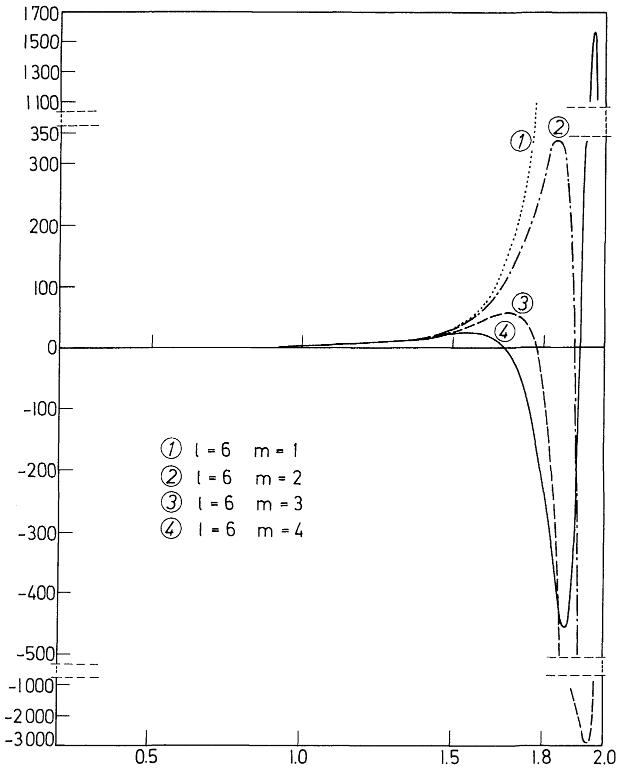

Fig. 2. The dependence of $r_{l,m}$ on m for fixed l. (Initial conditions: $\dot{a}=1$ and all other initial values equal to zero). The same qualitative behavior is true for the other functions also

place of (6.3) the equation

$$AR - BK = 0\,. \tag{6.9}$$

This is a second order equation so that the system (6.1), (6.2), and (6.9) admits five arbitrary constants. This is consistent with the fact that R, $\dot{R}$, and k are now given in terms of K, $\dot{K}$, and r and hence only a, $\dot{a}$, b, $\dot{b}$, and K are independent.

The other special case for $m \neq 0$ is $l - m = 1$, where we must have $r - k = 0$ in order that our solution be defined. To proceed in this case we rewrite Eqs. (4.23) and (4.24) making the substitution

$$\left.\begin{aligned} X &= AR - BK = (B + mA)\Pi \\ \text{where}\qquad \Pi &= \frac{AG}{B+mA} - K\,. \end{aligned}\right\} \tag{6.10}$$

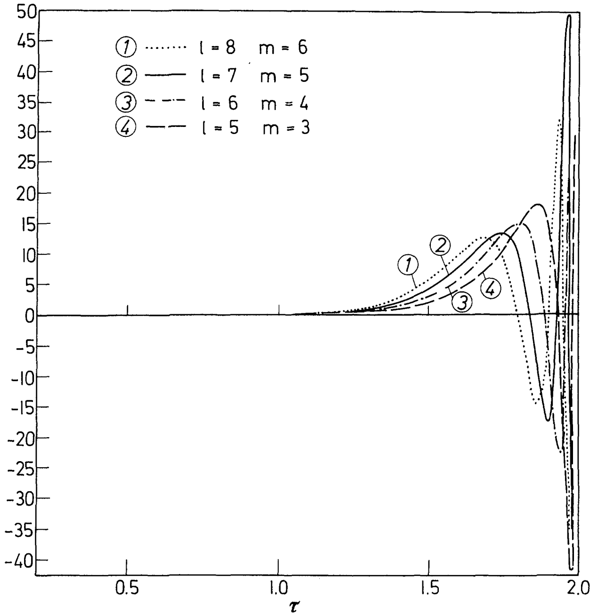

Fig. 3. The dependence of $b_{l,m}$ on m for fixed $l-m$. (Initial conditions: $\ddot{b}=1$ and all other initial values equal to zero.) The same qualitative behavior is true for the other functions also

They become

$$\left.\begin{aligned} m(k-r) = {} & \frac{B}{4(B+mA)}\frac{d}{d\tau}\left[(B+mA)^2\dot{\Pi}\right] \\ & -2m(m+1)\,\Pi+(L-m)\left[R+(a-b)\left(\frac{B}{A}-m\right)\right] \end{aligned}\right\} \tag{6.11}$$

and

$$\dot{r}-\dot{k}=\frac{(B+mA)^2}{AB}\dot{\Pi}+(L-m)\left(\dot{a}+\dot{b}+b\frac{\dot{B}}{B}\right). \tag{6.12}$$

If (6.12) is equal to zero, we can use it to eliminate the derivative of Π in (6.11). Since for $l-m=1$, $L-m=2(m+1)$ we find that (6.11) becomes in this case (the terms in K drop out)

$$\left.\begin{aligned} m(k-r)_{\left\{\substack{l-m=1\\ \dot{r}-\dot{k}=0}\right\}} = {} & \frac{B(m+1)}{2(B+mA)}\left\{4G-\left[AB\left(\dot{a}+\dot{b}+b\frac{\dot{B}}{B}\right)\right]^{\cdot}\right. \\ & \left.+\frac{4A}{B}\left(\frac{B}{A}+m\right)\left(\frac{B}{A}-m\right)(a-b)\right\}. \end{aligned}\right\} \tag{6.13}$$

Using the expression for G given in (4.11) and carrying out the differentiation, we find that the quantity in brackets in (6.13) is, apart from a non-zero factor, equal to the left-hand side of (6.1)! Hence, if a and b satisfy (6.1) *and* $\dot{r} - \dot{k} = 0$, then $r - k = 0$ also. Since both (4.23) and (4.24) vanish, we conclude that the integrability condition is satisfied. Eq. (6.12) being of third order in b, our system (6.1), (6.2), and (6.12) admits six arbitrary

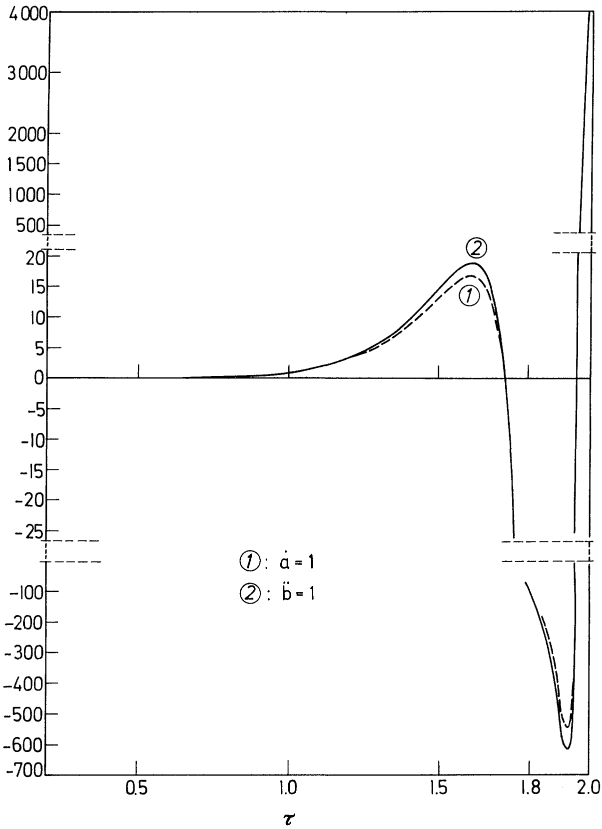

Fig. 4. The dependence of $r_{l,m}$ ($l = 5, m = 3$) on the initial conditions. Note that the positions of the peaks are the same for the two cases. The same is true for the other functions (as well as for other values of l and m)

constants. These are a, $\dot{a}$, b, $\dot{b}$, R, and K. Since in this case $\dot{r} = \dot{k}$, $\dot{R}$ cannot be given but follows from (6.4b) and (6.4c).

The case $m = 0$ needs special treatment in each of the above cases. We note that when $m = 0$ both K and k are imaginary constants (by 6.2, 4.21, and B.3a). In the general case ($l-m \neq 0$ or 1), the equation that replaces the integrability condition is $mk = 0$ (4.20) which becomes (since, by (4.10), $R = G$ when $m = 0$):

$$(AG)^{\cdot\cdot} + AG\left(\frac{6}{A^2} - \frac{1}{2}\frac{\dot{A}^2}{A^2} - \frac{8}{AB}\right) + \frac{4L}{B}\left[G + \frac{B}{A}(a-b)\right] = 0\,. \tag{6.14}$$

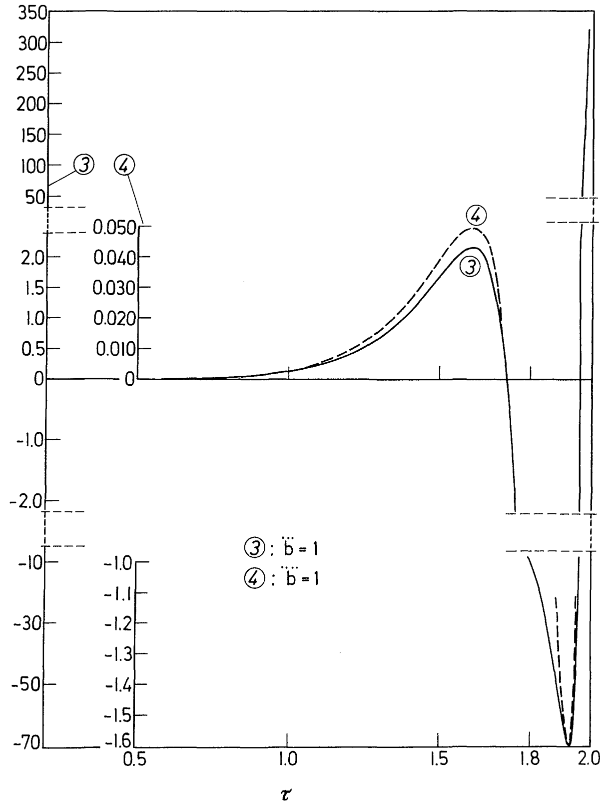

Fig. 5. The dependence of $r_{l,m}$ ($l = 5, m = 3$) on the initial conditions. Note that the positions of the peaks are the same as in Fig. 4, but the vertical scale changes drastically. The same is true for the other functions (as well as for other values of l and m)

This is a fourth order equation. Together with (6.1) and the constant values of K and k we still have eight constants.

When $m = 0$, the vanishing of (4.23) does not imply the vanishing of $r - k$, which is required for $l - m = 0$ or 1. Since, moreover, r is real while

k is imaginary, we must require, for both cases

$$r=0 \quad \text{and} \quad k=0\,. \tag{6.15}$$

We thus see that when $l=1$, $m=0$ we lose one constant of integration (the initial value of k). In view of the constraints (6.4), the remaining five constants are a, $\dot{a}$, b, $\dot{b}$, and K.

When $l=m=0$ we must, in addition, satisfy (6.9). Since R is real and K imaginary, we must set

$$R=0 \quad \text{and} \quad K=0\,. \tag{6.16}$$

In this case we lose one more constant (the value of K) and, since a, $\dot{a}$, b, and $\dot{b}$ are constrained by (6.4a), we can only assign three constants. This is consistent with the fact that a and b must now satisfy three equations: $r=0$, $R=0$, and (6.1). This system (with $l=m=0$) is exactly the same as (4.3)–(4.5) with the source term M equal to zero. This was to be expected since the separation of the space independent part of α and β into $a_M(b_M)$ and $a_{00}(b_{00})$ was artificial.

The functions P, Q, f, g, and h which do not couple to a and b satisfy second order equations and hence their time evolution is uniquely determined when the initial data P, $\dot{P}$, Q, $\dot{Q}$, f, and $\dot{f}$ satisfy the constraint Eqs. (5.12) and (5.13).

We note that the three coordinate dependent solutions found in Appendix C remain valid for all l, m. This completes the discussion of the time dependence of our equations for all values of l, m.

VII. Concluding Remarks

The reduction of the field equations given in Section IV implies that, for this particular metric, they can be divided into three sets of different importance:

(a) Four "main" equations δR_{00}, $(\delta R_{11}+\delta R_{22})$, δR_{33}, and δR_{03} which must be individually satisfied;

(b) Four "secondary" equations δR_{01}, δR_{02}, δR_{31}, and δR_{32} for which only two equations need be satisfied (the other two following from the Bianchi identities when the "main" equations are satisfied);

(c) Two "trivial" equations $(\delta R_{11}-\delta R_{22})$ and δR_{12} which are implied by the above equations and the Bianchi identities.

Even though one might expect a similar classification (and hence separation of the six unknown functions into three pairs) for other metrics with equally high symmetry for which the eigenfunctions of the operators L_1, L_2, and L_3 are known, the integration of the space derivatives need not be as trivial as in this case. One must, therefore, conclude that the solution of our problem is in large part due to the simple properties of the angular momentum operators.

Turning now to the physical significance of the results obtained in this paper, the only thing we can say with certainty is that the stability of the Taub space is exactly the same as that of the closed Friedman universe – the two differing only in the "dimensionality" of their singularity (Taub space collapses to a plane, Friedman to a point). We can also say that both are stable during expansion[9], since the initially small perturbations become even smaller and hence the first order approximation is justified.

In interpreting the instability during contraction[9] predicted by our linear analysis, we must bear in mind the limitations of linearization: the limitless growth of initially small perturbations does not exclude the possibility that they remain finite when higher order terms in the equations are retained.

Acknowledgements. I have profited from discussions with many people during different stages of this work. In particular, I would like to thank Professors A. Trautman and G. F. R. Ellis for their interest and help in clarifying some points. Above all, I am indebted to Professor S. Chandrasekhar for his continuous encouragement and advice and for several suggestions which proved essential in the solution of the problem. The research reported in this paper has in part been supported by the National Science Foundation under contracts GP-15973 and GP-28342 with the University of Chicago.

Appendix A

The operators L_α can be expressed in terms of the Euler angles ψ, θ, and φ as follows:

$$L_1 = -\sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \cos\psi\cot\theta\frac{\partial}{\partial\psi}, \tag{A.1}$$

$$L_2 = \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi} - \sin\psi\cot\theta\frac{\partial}{\partial\psi}, \tag{A.2}$$

and

$$L_3 = \frac{\partial}{\partial\psi}. \tag{A.3}$$

We define, as usual, $L_\pm = L_1 \pm iL_2$ in terms of which

$$L^2 = L_-L_+ + L_3^2 + iL_3 = L_+L_- + L_3^2 - iL_3\,. \tag{A.4}$$

[9] For the Taub case, "expansion" and "contraction" as used here refer to the sign of $\dot{B}$ when $B < A$.

The following commutation rules follow from $[L_1, L_2] = -L_3$ and its cyclic permutations and from the definition of $L_\pm$:

$$L_3(L_1L_2+L_2L_1)-(L_1L_2+L_2L_1)L_3 = 2(L_1^2-L_2^2)\,, \tag{A.5}$$

$$L_3(L_1^2-L_2^2)-(L_1^2-L_2^2)L_3 = -2(L_1L_2+L_2L_1)\,, \tag{A.6}$$

$$L_3(L_1^2+L_2^2)-(L_1^2+L_2^2)L_3 = 0\,, \tag{A.7}$$

$$[L_+, L_3] = -iL_+\,, \tag{A.8}$$

$$[L_-, L_3] = iL_-\,, \tag{A.9}$$

$$[L_+, L_-] = 2iL_3\,, \tag{A.10}$$

and

$$L_+L_- + L_-L_+ = 2(L_1^2+L_2^2)\,. \tag{A.11}$$

The eigenfunctions of L^2 and L_3 have the form

$$S^l_{mm'}(\psi, \theta, \varphi) = e^{im\psi} e^{im'\varphi} d^l_{mm'}(\theta)\,, \tag{A.12}$$

and are obviously eigenfunctions of $\frac{\partial}{\partial\varphi}$ also. In this paper we have suppressed the index m' on $S^l_{m,m'}$ since this eigenvalue does not enter in any of our equations. However, for our expansions (4.2) etc. to be completely general an additional sum over m' must be included. As follows from (4.1) and (A.12), the function $d^l_{m,m'}(\theta)$ satisfies

$$\left\{\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta} - \frac{m^2+m'^2-2mm'\cos\theta}{\sin^2\theta} + l(l+1)\right\} d^l_{mm'}(\theta) = 0\,; \tag{A.13}$$

$d^l_{m,m}(\theta)$ can be expressed in terms of powers of $\sin\theta$ and $\cos\theta$ as follows:

$$d^l_{mm'}(\theta) = \sqrt{\frac{(l+m)!\,(l-m)!}{(l+m')!\,(l-m')!}}\left(\cos\frac{\theta}{2}\right)^{m+m'}\left(\sin\frac{\theta}{2}\right)^{m-m'} P^{m-m',\,m+m'}_{l-m}(\cos\theta) \tag{A.14}$$

where

$$P^{\alpha,\beta}_n(x) = 2^{-n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k\,. \tag{A.15}$$

When either m or m' equal zero, $S^l_{0m'}$ or S^l_{m0} become proportional to spherical harmonics of the same indices.

The orthogonality of the $S^l_{m,m'}$ is expressed by

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{\pi}\int_0^{2\pi}(S^{l_1}_{m_1m_1'})^* S^{l_2}_{m_2m_2'}\,d\psi\sin\theta\,d\theta\,d\varphi = \frac{1}{2l_1+1}\delta_{l_1l_2}\delta_{m_1m_2}\delta_{m_1'm_2'}\,. \tag{A.16}$$

We choose the arbitrary phase in (A.16) so that

$$(S^l_{m,m'})^* = S^l_{-m,-m'}\,. \tag{A.17}$$

As is evident from (4.1), our operators are anti-hermitian and the Condon-Shortley phase convention relating $L_\pm S^l_m$ to $S^l_{m\pm 1}$ is inapplicable (it is impossible to satisfy (A.4)). The following phase convention is consistent with the L_α operators as given in this paper and (A.17):

$$L_\pm S^l_m = (-1)^m \sqrt{(l\mp m)(l\pm m+1)}\,S^l_{m\pm 1}\,. \tag{A.18}$$

Most of the expressions given in this Appendix can be found in reference [8], particularly in Chapters 2 and 4.

Appendix B

To make manifest the reality of the functions λ, μ, γ, and κ we must consider together the terms coming from S^l_m and S^l_{-m} in α and β. Adding to (5.7) and (5.8) the terms corresponding to $-m$ we find

$$\left.\begin{aligned}\lambda \mp i\mu = \mp i(-1)^m &\left[\frac{AR_{l,m}\pm BK_{l,m}}{\sqrt{(l\pm m)(l\mp m+1)}}\,S^l_{m\mp 1}\right.\\ &\left.+\frac{AR_{l,-m}\pm BK_{l,-m}}{\sqrt{(l\mp m)(l\pm m+1)}}\,S^l_{-m\mp 1}\right]\\ &\mp i(-1)^l P_{\pm l}S^l_{\pm l}+Q_\pm\end{aligned}\right\} \tag{B.1}$$

and

$$\left.\begin{aligned}\frac{\gamma\mp i\kappa}{A} = &\frac{(r_{l,m}\pm k_{l,m})S^l_{m\mp 2}}{\sqrt{(l\pm m)(l\mp m+1)(l\pm m-1)(l\mp m+2)}}\\ &+\frac{(r_{l,-m}\pm k_{l,-m})S^l_{-m\mp 2}}{\sqrt{(l\mp m)(l\pm m+1)(l\mp m-1)(l\pm m+2)}}\\ &+\frac{f_{\pm l}}{\sqrt{2l}}\,S^l_{\pm(l-1)}+g_{\pm l}S^l_{\pm l}+h_\pm\,.\end{aligned}\right\} \tag{B.2}$$

In these expressions we have written explicitly the indices l, $\pm m$. The reality conditions which follow from (4.2a) are

$$\left.\begin{aligned}(R_{l,m})^* &= R_{l,-m}, \qquad &(K_{l,m})^* &= -K_{l,-m}\,,\\ (r_{l,m})^* &= r_{l,-m}, \quad \text{and} \quad &(k_{l,m})^* &= -k_{l,-m}\,,\end{aligned}\right\} \tag{B.3a}$$

so that for $m=0$, K and k are pure imaginary, while R and r are real. Further we must demand of P, Q, f, g, and h that

$$\left.\begin{aligned}&P_l^* = P_{-l}, \quad Q_+^* = Q_-\,,\\ &f_l^* = f_{-l}, \quad g_l^* = g_{-l}, \quad \text{and} \quad h_+^* = h_-\,.\end{aligned}\right\} \tag{B.3b}$$

Using these conditions, we can replace all coefficients with $-m$ in terms of those with $+m$ and obtain

$$\frac{\alpha}{A} = a_M + a_{00} + \sum_{l=1}^{\infty} \left\{ a_{l,0} S_0^l + \sum_{m=1}^{l} \left[a_{l,m} S_m^l + (a_{l,m})^* S_{-m}^l \right] \right\}, \tag{B.4}$$

$$\frac{\beta}{B} = b_M + b_{00} + \sum_{l=1}^{\infty} \left\{ b_{l,0} S_0^l + \sum_{m=1}^{l} \left[b_{l,m} S_m^l + (b_{l,m})^* S_{-m}^l \right] \right\}, \tag{B.5}$$

$$\left.\begin{aligned} \lambda - i\mu = Q - i \sum_{l=1}^{\infty} \Bigg\{ &(-1)^l P_l S_l^l + \frac{AR_{l,0} + BK_{l,0}}{\sqrt{l(l+1)}} S_{-1}^l \\ &+ \sum_{m=1}^{l} (-1)^m \Bigg[\frac{AR_{l,m} + BK_{l,m}}{\sqrt{(l+m)(l-m+1)}} S_{m-1}^l \\ &+ \frac{AR_{l,m}^* - BK_{l,m}^*}{\sqrt{(l-m)(l+m+1)}} S_{-m-1}^l \Bigg] \Bigg\}, \end{aligned}\right\} \tag{B.6}$$

$$\left.\begin{aligned} \lambda + i\mu = Q^* + i \sum_{l=1}^{\infty} \Bigg\{ &(-1)^l P_l^* S_{-l}^l + \frac{AR_{l,0} - BK_{l,0}}{\sqrt{l(l+1)}} S_1^l \\ &+ \sum_{m=1}^{l} (-1)^m \Bigg[\frac{AR_{l,m} - BK_{l,m}}{\sqrt{(l-m)(l+m+1)}} S_{m+1}^l \\ &+ \frac{(AR_{l,m}^* + BK_{l,m}^*)}{\sqrt{(l+m)(l-m+1)}} S_{-m+1}^l \Bigg] \Bigg\}, \end{aligned}\right\} \tag{B.7}$$

$$\left.\begin{aligned} \frac{\gamma - i\kappa}{A} = h + \sum_{l=1}^{\infty} \Bigg\{ &g_l S_l^l + \frac{f_l}{\sqrt{2l}} S_{l-1}^l + \frac{r_{l,0} + k_{l,0}}{\sqrt{l(l+1)(l-1)(l+2)}} S_{-2}^l \\ &+ \sum_{m=1}^{l} \Bigg[\frac{(r_{l,m} + k_{l,m}) S_{m-2}^l}{\sqrt{(l+m)(l-m+1)(l+m-1)(l-m+2)}} \\ &+ \frac{(r_{l,m}^* - k_{l,m}^*) S_{-m-2}^l}{\sqrt{(l-m)(l+m+1)(l-m-1)(l+m+2)}} \Bigg] \Bigg\}, \end{aligned}\right\} \tag{B.8}$$

$$\left.\begin{aligned} \frac{\gamma + i\kappa}{A} = h^* + \sum_{l=1}^{\infty} \Bigg\{ &g_l^* S_{-l}^l + \frac{f_l^*}{\sqrt{2l}} S_{-l+1}^l \\ &+ \frac{r_{l,0} - k_{l,0}}{\sqrt{l(l+1)(l-1)(l+2)}} S_{+2}^l \\ &+ \sum_{m=1}^{l} \Bigg[\frac{(r_{l,m} - k_{l,m}) S_{m+2}^l}{\sqrt{(l-m)(l+m+1)(l-m-1)(l+m+2)}} \\ &+ \frac{(r_{l,m}^* + k_{l,m}^*) S_{-m+2}^l}{\sqrt{(l+m)(l-m+1)(l+m+1)(l-m+2)}} \Bigg] \Bigg\}. \end{aligned}\right\} \tag{B.9}$$

In these expressions m is strictly positive and a, b, R, K, r, and k are complex functions of time. Both the real and the imaginary parts of these functions satisfy the equations obtained in Section IV with $m>0$. (Similarly both the real and the imaginary parts of P, Q, f, g, and h satisfy the equations obtained in Section V.)

Noting that Eqs. (B.6) and (B.7) as well as (B.8) and (B.9) are complex conjugates of each other, we verify that λ, μ, γ, and κ (as well as α and β) are real as they should be.

Appendix C

As noted in Section III, the four gauge conditions (3.1) and (3.1a) do not completely specify the coordinate system. Before obtaining the form of the infinitesimal coordinate transformations still permitted by our gauge conditions, we will first derive general expressions for the perturbation to the metric tensor in tetrad form which is induced by an infinitesimal change of coordinates.

Let us define the matrices A_a^k and B_k^a which relate the tetrads to the coordinates by[10]

$$X_a = A_a^k \frac{\partial}{\partial x^k}, \qquad \omega^a = B_k^a \, dx^k ; \tag{C.1}$$

or

$$\frac{\partial}{\partial x^k} = B_k^a X_a, \qquad dx^k = A_a^k \omega^a , \tag{C.1a}$$

where we have used

$$A_a^k B_k^c = \delta_a^c . \tag{C.2}$$

In this Appendix letters from the beginning (middle) of the alphabet refer to tetrads (coordinates).

To obtain the change in $g_{ab}(x)$ under the infinitesimal coordinate transformation

$$x^i \to x^{i'} = x^i + \varepsilon \xi^i(x) , \tag{C.3}$$

with $\xi^i(x)$ four arbitrary functions, we start by demanding that the expression for the interval ds^2 remain unchanged, i.e.

$$g_{ab}(x) \, \omega^a(x) \, \omega^b(x) = g'_{ab}(x') \, \omega^a(x') \, \omega^b(x') . \tag{C.4}$$

Now, to the first order

$$g'_{ab}(x') = g'_{ab}(x) + \varepsilon \xi^r \frac{\partial g_{ab}(x)}{\partial x^r} , \tag{C.5}$$

---

[10] The X_a denote differential operators dual to the ω^a.

and

$$\begin{aligned}\omega^a(x') &= B^a_k(x')\,dx^{k'}\\ &= \left[B^a_k(x) + \varepsilon\xi^r \frac{\partial B^a_k}{\partial x^r}\right]\left(\delta^k_l + \varepsilon\frac{\partial \xi^k}{\partial x^l}\right)dx^l\\ &= \left[B^a_l(x) + \varepsilon\xi^r \frac{\partial B^a_l}{\partial x^r} + \varepsilon B^a_k \frac{\partial \xi^k}{\partial x^l}\right]A^l_b\,\omega^b\,, \qquad \text{(by C.1a)}\end{aligned}$$

so that

$$\omega^a(x') = [\delta^a_b + \varepsilon\Phi^a_b]\,\omega^b \tag{C.6}$$

where

$$\Phi^a_b \overset{\text{def}}{=} A^l_b \xi^r \frac{\partial B^a_l}{\partial x^r} + B^a_k X_b \xi^k\,. \tag{C.7}$$

Substituting (C.5) and (C.6) in (C.4), we obtain, to the first order in ε

$$g_{ab}(x) = g'_{ab}(x) + \varepsilon\xi^r \frac{\partial g_{ab}}{\partial x^r} + \varepsilon g_{ac}\Phi^c_b + \varepsilon g_{cb}\Phi^c_a\,. \tag{C.8}$$

Finally, making use of (C.7) we find

$$\left.\begin{aligned}\delta g_{ab}(x) &\overset{\text{def}}{=} g'_{ab}(x) - g_{ab}(x)\\ &= -\varepsilon\{g_{ac}B^c_k X_b \xi^k + g_{cb}B^c_k X_a \xi^k\}\\ &\quad - \varepsilon\xi^r\left\{\frac{\partial g_{ab}}{\partial x^r} + (g_{ac}A^l_b + g_{cb}A^l_a)\frac{\partial B^c_l}{\partial x^r}\right\}.\end{aligned}\right\} \tag{C.9}$$

Specializing now to the Taub metric (g_{ab} given by (2.1) with $F = 4/B$), we let 0, 1, 2, and 3 stand for τ, θ, φ, and ψ and rename the four functions ξ^i as follows:

$$\left.\begin{aligned}\varepsilon\xi^0 &= T(\tau,\theta,\varphi,\psi)\,,\\ \varepsilon\xi^1 &= \Theta(\tau,\theta,\varphi,\psi)\,,\\ \varepsilon\xi^2 &= \Phi(\tau,\theta,\varphi,\psi)\,,\\ \text{and}\qquad \varepsilon\xi^3 &= \Psi(\tau,\theta,\varphi,\psi)\,.\end{aligned}\right\} \tag{C.10}$$

The elements of the matrices A^k_a and B^a_k can be read off directly from Eqs. (A.1)–(A.3) and (2.3), respectively. Also, since $\omega^0 = d\tau$, $B^a_k = \delta^a_0$ and $A^k_a = \delta^k_0$. Substituting in (C.9) we find:

$$\delta g_{00} = -\frac{4}{B^2}\beta = -\frac{8}{B}\dot{T} + \frac{4}{B^2}T\dot{B}\,, \tag{C.11}$$

$$\delta g_{33} = -\beta = 2BL_3[\cos\theta\,\Phi + \Psi] + T\dot{B}\,, \tag{C.12}$$

$$\delta g_{03} = 0 = B[\cos\theta\,\Phi + \Psi]^{\cdot} - \frac{4}{B} L_3 T, \tag{C.13}$$

$$\delta g_{01} = 0 = A[-\sin\psi\,\Theta + \sin\theta\cos\psi\,\Phi]^{\cdot} - \frac{4}{B} L_1 T, \tag{C.14}$$

$$\delta g_{02} = 0 = A[\cos\psi\,\Theta + \sin\theta\sin\psi\,\Phi]^{\cdot} - \frac{4}{B} L_2 T, \tag{C.15}$$

$$\begin{aligned}\delta g_{11} &= -(\alpha+\gamma) = 2A\{-\sin\psi\, L_1\Theta + \sin\theta\cos\psi\, L_1\Phi\}\\ &\quad + T\dot{A} + 2A\Theta\cot\theta\cos^2\psi,\end{aligned} \tag{C.16}$$

$$\begin{aligned}\delta g_{22} &= -(\alpha-\gamma) = 2A\{\cos\psi\, L_2\Theta + \sin\theta\sin\psi\, L_2\Phi\}\\ &\quad + T\dot{A} + 2A\Theta\cot\theta\sin^2\psi,\end{aligned} \tag{C.17}$$

$$\begin{aligned}\delta g_{12} &= -\kappa = A\{\cos\psi L_1\Theta - \sin\psi\, L_2\Theta\\ &\quad + \sin\theta(\sin\psi\, L_1\Phi + \cos\psi\, L_2\Phi)\} + A\Theta\cot\theta\sin2\psi,\end{aligned} \tag{C.18}$$

$$\begin{aligned}\delta g_{23} &= -\mu = B[\cos\theta\, L_2\Phi + L_2\Psi]\\ &\quad + A[\cos\psi\, L_3\Theta + \sin\theta\sin\psi\, L_3\Phi] - B\Theta\sin\psi,\end{aligned} \tag{C.19}$$

$$\begin{aligned}\delta g_{31} &= -\lambda = B[\cos\theta\, L_1\Phi + L_1\Psi]\\ &\quad + A[-\sin\psi\, L_3\Theta + \sin\theta\cos\psi\, L_3\Phi] - B\Theta\cos\psi,\end{aligned} \tag{C.20}$$

where we made use of (3.2) to express δg_{ab} on terms of α, β, γ, κ, λ, and μ.
From (C.11) and (C.12) it follows that

$$L_3[\cos\theta\,\Phi + \Psi] = -\dot{T}. \tag{C.21}$$

Comparing with (C.13) we see that the functions T and $[\cos\theta\,\Phi + \Psi]$ can be expanded in the form

$$T = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{l,m}(\tau) S_m^l \tag{C.22}$$

and

$$[\cos\theta\,\Phi + \Psi] = i \sum_{l=0}^{\infty} \sum_{m=-l}^{l} p_{l,m}(\tau) S_m^l \tag{C.23}$$

where the functions $u_{l,m}$ and $p_{l,m}$ satisfy the equations

$$\dot{u}_{l,m} = m p_{l,m} \quad \text{and} \quad \dot{p}_{l,m} = \frac{4m}{B^2} u_{l,m}. \tag{C.24}$$

With T given by (C.22), Eqs. (C.14) and (C.15) can be solved for Θ and Φ. We find

$$\Theta = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} v_{l,m}(\tau)\,[\cos\psi\, L_2 - \sin\psi\, L_1]\, S_m^l \tag{C.25}$$

and

$$\sin\theta\,\Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} v_{l,m}(\tau)\,[\cos\psi\, L_1 + \sin\psi\, L_2]\, S_m^l \tag{C.26}$$

where the functions $v_{l,m}$ satisfy the equation

$$\dot{v}_{l,m} = \frac{4}{AB}\, u_{l,m}\,. \tag{C.27}$$

The obvious reality conditions for u, p, and v are

$$u_{l,m}^* = u_{l,-m}, \quad p_{l,m}^* = -p_{l,-m}, \quad \text{and} \quad v_{l,m}^* = v_{l,-m}\,. \tag{C.28}$$

We have now satisfied our gauge conditions (C.11)–(C.15). We next substitute the functions T, Θ, Φ, and Ψ as given above, in Eqs. (C.12) and (C.16)–(C.20) and carry out the indicated differentiations to obtain the following expressions for the perturbations to the metric tensor induced by such changes in the coordinate system as are allowed by our gauge conditions:

$$\alpha = A \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ \frac{\dot{A}}{A}\, u_{l,m} + v_{l,m}[L_1^2 + L_2^2] \right\} S_m^l\,, \tag{C.29}$$

$$\beta = B \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left\{ 2m p_{l,m} - \frac{\dot{B}}{B}\, u_{l,m} \right\} S_m^l\,, \tag{C.30}$$

$$\gamma + i\kappa = -A \sum_{l=0}^{\infty} \sum_{m=-l}^{l} v_{lm} L_+^2 S_m^l\,, \tag{C.31}$$

$$\gamma - i\kappa = -A \sum_{l=0}^{\infty} \sum_{m=-l}^{l} v_{l,m} L_-^2 S_m^l\,, \tag{C.32}$$

$$\lambda + i\mu = -i \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \{(B + mA) v_{l,m} + B p_{l,m}\}\, L_+ S_m^l\,, \tag{C.33}$$

$$\lambda - i\mu = i \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \{(B - mA) v_{l,m} - B p_{l,m}\}\, L_- S_m^l\,. \tag{C.34}$$

That these expressions are real follows from the reality conditions (C.28) and the fact that, with our phase conventions, $(L_+ S_m^l)^* = L_- S_{-m}^l$. Using Eqs. (C.24) and (C.27) for the derivatives of u, p, and v and the commutation rules given in Appendix A, it is a straightforward but extremely laborious procedure to verify that α, β, γ, κ, λ, and μ as given above satisfy Eqs. (3.4)–(3.13) as they should. We note that, for each l, m, the perturbations to the metric due to a change of coordinates form a three-parameter family – the three parameters being the initial values $u(\tau_0)$, $p(\tau_0)$, and $v(\tau_0)$ needed to integrate (C.24) and (C.27).

Appendix D

The ω^i_j's are, to the first order:

$$\omega^0_0 = -\frac{1}{2}\left\{\left(\frac{\dot{B}}{B} + \frac{\dot{\beta}}{B} - \frac{\beta}{B}\frac{\dot{B}}{B}\right)\omega^0 + \frac{L_\nu \beta}{B}\omega^\nu\right\}, \tag{D.1}$$

$$\begin{aligned}\omega^1_1 = \frac{1}{2}\Bigg\{&\left(\frac{\dot{A}}{A} + \frac{\dot{\alpha}+\dot{\gamma}}{A} - \frac{\alpha+\gamma}{A}\frac{\dot{A}}{A}\right)\omega^0 + \frac{L_\nu(\alpha+\gamma)}{A}\omega^\nu \\ &+\kappa\frac{2A-B}{A^2}\omega^3 - \frac{\lambda}{A}\omega^2\Bigg\},\end{aligned} \tag{D.2}$$

$$\begin{aligned}\omega^2_2 = \frac{1}{2}\Bigg\{&\left(\frac{\dot{A}}{A} + \frac{\dot{\alpha}-\dot{\gamma}}{A} - \frac{\alpha-\gamma}{A}\frac{\dot{A}}{A}\right)\omega^0 + \frac{L_\nu(\alpha-\gamma)}{A}\omega^\nu \\ &-\kappa\frac{2A-B}{A^2}\omega^3 + \frac{\mu}{A}\omega^1\Bigg\},\end{aligned} \tag{D.3}$$

$$\omega^3_3 = \frac{1}{2}\left\{\left(\frac{\dot{B}}{B} + \frac{\dot{\beta}}{B} - \frac{\beta}{B}\frac{\dot{B}}{B}\right)\omega^0 + \frac{L_\nu \beta}{B}\omega^\nu + \frac{\lambda}{A}\omega^2 - \frac{\mu}{A}\omega^1\right\}, \tag{D.4}$$

$$\begin{aligned}\omega^1_0 = &-\frac{2L_1\beta}{AB^2}\omega^0 + \frac{1}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{\alpha}+\dot{\gamma}}{A} - \frac{\alpha+\gamma}{A}\frac{\dot{A}}{A}\right)\omega^1 \\ &+\frac{1}{2}\left(\frac{\kappa}{A}\right)^{\cdot}\omega^2 + \frac{B}{2A}\left(\frac{\lambda}{B}\right)^{\cdot}\omega^3,\end{aligned} \tag{D.5}$$

$$\begin{aligned}\omega^2_0 = &-\frac{2L_2\beta}{AB^2}\omega^0 + \frac{1}{2}\left(\frac{\kappa}{A}\right)^{\cdot}\omega^1 \\ &+\frac{1}{2}\left(\frac{\dot{A}}{A} + \frac{\dot{\alpha}-\dot{\gamma}}{A} - \frac{\alpha-\gamma}{A}\frac{\dot{A}}{A}\right)\omega^2 + \frac{B}{2A}\left(\frac{\mu}{B}\right)^{\cdot}\omega^3,\end{aligned} \tag{D.6}$$

$$\begin{aligned}\omega^3_0 = &-\frac{2L_3\beta}{B^3}\omega^0 + \frac{A}{2B}\left(\frac{\lambda}{A}\right)^{\cdot}\omega^1 \\ &+\frac{A}{2B}\left(\frac{\mu}{A}\right)^{\cdot}\omega^2 + \frac{1}{2}\left(\frac{\dot{B}}{B} + \frac{\dot{\beta}}{B} - \frac{\beta}{B}\frac{\dot{B}}{B}\right)\omega^3,\end{aligned} \tag{D.7}$$

$$\left.\begin{aligned}\omega^1_2 = &\frac{1}{2}\left(\frac{\kappa}{A}\right)^{\cdot}\omega^0 + \frac{1}{2A}\left[L_2(\alpha+\gamma) - \lambda\right]\omega^1 \\ &+\frac{1}{A}\left[L_2\kappa - \mu - \frac{1}{2}L_1(\alpha-\gamma)\right]\omega^2 + \frac{1}{2A}\Bigg[L_3\kappa + L_2\lambda \\ &-L_1\mu + (2A-B)\left(1 - \frac{\alpha+\gamma}{A}\right) + 2\alpha - \beta\Bigg]\omega^3,\end{aligned}\right\} \tag{D.8}$$

$$\left.\begin{aligned}\omega_3^2 = {} & \frac{B}{2A}\left(\frac{\mu}{B}\right)^{\cdot}\omega^0 + \frac{1}{2A}\left[L_3\kappa - L_2\lambda + L_1\mu + B\left(1 - \frac{\alpha-\gamma}{A}\right)\right. \\ & \left. + \beta - 2\gamma\right]\omega^1 + \frac{1}{2A}\left[L_3(\alpha-\gamma) - \kappa\frac{2A-B}{A}\right]\omega^2 \\ & + \frac{1}{A}\left[L_3\mu - \lambda - \frac{1}{2}L_2\beta\right]\omega^3\,,\end{aligned}\right\} \tag{D.9}$$

$$\left.\begin{aligned}\omega_1^3 = {} & \frac{A}{2B}\left(\frac{\lambda}{A}\right)^{\cdot}\omega^0 + \frac{1}{B}\left[L_1\lambda - \kappa - \frac{1}{2}L_3(\alpha+\gamma)\right]\omega^1 \\ & + \frac{1}{2B}\left[B + 2\gamma - L_3\kappa + L_2\lambda + L_1\mu\right]\omega^2 \\ & + \frac{1}{2B}\left[L_1\beta - \mu\frac{B}{A}\right]\omega^3\,.\end{aligned}\right\} \tag{D.10}$$

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