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The Quadratic Lagrangians in General Relativity

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Abstract. The solutions of the General Relativity equations with quadratic Lagrangians $R_{iklm}R^{iklm}$, $R_{ik}R^{ik}$, R^2 are studied. It is shown that nontrivial Euclidian (at $r \to \infty$) solution of the theory equations does not exist when $T \neq 0$ (T is a trace of the energy-momentum tensor of matter). The Schwarzschild solution is not an external part of a total solution when $T \neq 0$. Under condition $T = R = 0$ Lagrangians $R_{iklm}R^{iklm}$, $R_{ik}R^{ik}$ lead to the identical field equations, so there exist the only quadratic Lagrangian and the only field equations. This equation has a solution with an external part being a standard Schwarzschild solution for the statical spherically symmetric case.

1. Introduction

It is known that the standard external Schwarzschild solution satisfies the equations of the quadratic Lagrangians theory. A conclusion is likely to be made that with respect to its experimental consequences the gravitation theory with quadratic Lagrangians

$$L_1 = R_{iklm}R^{iklm}, \qquad L_2 = R_{ik}R^{ik}, \qquad L_3 = R^2 \tag{1}$$

is equivalent to the usual formulation of General Relativity. However bearing in mind a real distribution of a matter energy-momentum tensor it is not evident that a nontrivial Schwarzschild solution will be the external one for the total spherically symmetric gravitational field described with Lagrangians (1).

It is known that Lagrangian of the Einstein theory $L_0 = R$ is not invariant with respect to a change of the units measuring the interval. The necessity of the such invariance seems to be natural for a zero mass field. So the question arises on the possibility of using the another particular Lagrangian of course while the main theory principles being conserved. The quadratic Lagrangians satisfy all the necessary invariance requirements.

The properties of Lagrangians (1) and ideas of the necessity for the conformal invariance of a gravitation theory were discussed in papers [1–41].

In the present paper I show that under condition $T \neq 0$ the nontrivial Schwarzschild solution is not a solution of the quadratic Lagrangians theory equations. Under condition $T=0$ the equations have a solution with the external part being the Schwarzschild solution.

2. The Field Equations

The equations of the quadratic Lagrangians theory satisfy the condition [4, 5, 7, 11]

$$\delta(\sqrt{-g}\, L_1) = \delta(\sqrt{-g}\,(4L_2 - L_3)). \tag{2}$$

So the linear combination $L = \frac{1}{4}(AL_2 + BL_3)$ is the most general quadratic Lagrangian of the theory. Let us write the action integral in the form $J = \int (L_m - L)\sqrt{-g}\, d^4\Omega$, where L_m is Lagrangian describing the matter. When obtaining the field equations it is convenient to use the following formulas

$$\delta R^n{}_{ikl} = \nabla_k \delta \Gamma^n_{il} - \nabla_l\, \delta \Gamma^n_{ik}, \tag{3}$$

$$\delta \Gamma^{\,i}_{kl} = -1/2 g^{im}(\nabla_k \delta g_{ml} + \nabla_l\, \delta g_{mk} - \nabla_m \delta g_{kl}). \tag{4}$$

The field equations have a form

$$\begin{aligned} T^k_i = {} & A(R^{kn}R_{in} - 1/4\delta^k_i R_{mn}R^{mn} + \nabla_n(1/2\nabla^n R^k_i - \nabla_i R^{kn} + 1/4\delta^k_i \nabla^n R)) \\ & + B(R(R^k_i - 1/4\delta^k_i R) + (\nabla_n \nabla^n \delta^k_i - \nabla_i \nabla^k)\, R), \end{aligned} \tag{5}$$

where T^k_i is the symmetrical energy-momentum tensor of the matter, $T^k_i = 2/\sqrt{-g}\;\delta(\sqrt{-g}\, L_m)/\delta g_{ik}$. The Eqs.(5) correspond to: Lagrangian R^2 when $A=0$, $B=1$, Lagrangian $R_{ik}R^{ik}$ when $A=1$, $B=0$, Lagrangian $R_{iklm}R^{iklm}$ when $A=4$, $B=1$. The trace of Eqs. (5) has a form

$$T = (A + 3B)\nabla_i \nabla^i R. \tag{6}$$

3. Statical Spherically Symmetric Solution. $T \neq 0$

Let us take the expression for the metrics in the standard form

$$dS^2 = U(r)\, dt^2 - dr^2/V(r) - r^2\, d\theta^2 - r^2 \sin^2\theta d\varphi^2. \tag{7}$$

Eq. (6) can be written down in the form

$$r^{-2}\frac{\partial}{\partial r}\left(\sqrt{UV}\, r^2 \frac{\partial}{\partial r} R\right) = -T\sqrt{U/V}/(A+3B), \tag{8}$$

from which it follows that

$$\frac{\partial}{\partial r} R = -((A+3B)\sqrt{UV}\, r^2)^{-1} \int_0^r T\sqrt{-g}\, r^2\, dr, \quad \sqrt{-g} = \sqrt{U/V}. \tag{9}$$

Assuming the Euclidian character of the metrics at the infinity it is natural to suppose that when $r > r_0$, where r_0 is the radius of the spherically symmetric gravitational field source, the functions $U(r)$ and $V(r)$ can be taken in a form

$$U(r) = 1 - \sum_{n=2}^{\infty} a_n/r^n, \qquad V(r) = 1 - \sum_{n=1}^{\infty} b_n/r^n. \tag{10}$$

In this case we obtain a following expansion for a quantity

$$R = g^{ik}(\partial_{l1}\Gamma^l_{ik} - \partial_k \Gamma^l_{il} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma_{km}{}^{l}),$$

$$R = C_1/r^4 + C_2/r^5 + C_3/r^6 + \cdots \tag{11}$$

where the coefficients C_i are related in a simple way to the coefficients a_i, b_i, for example $C_1 = 2(b_2 - a_2) + a_1(b_1 - a_1)/2$. At the same time from equality (9) we obtain the following expansion

$$R = M_1/r + M_2/r^2 + M_3/r^3 + \cdots, \tag{12}$$

$$M_1 = (A + 3B)^{-1} \int_0^{r_0} T\sqrt{-g}\, r^2\, dr, \qquad M_2 = -(a_1 + b_1)\, M_1/4, \ldots \tag{13}$$

The expansion (12) is consistent with the Euclidian character of the metrics at $r \to \infty$, i.e. with equality (11) only when $M_1 = 0$. This leads to the equality $T = 0$ on assumption of the sign determinacy of the quantity T.

We conclude that under the condition $T \neq 0$ there is no a nontrivial static spherically-symmetric Euclidean at $r \to \infty$ solution of Eqs. (5).

The result obtained indicates definitely enough against the possibility of using Lagrangians (1) for the description of the real gravitational field. However to consider this case completely it is necessary to analyse the case $T = 0$. If we showed that Eqs. (5) did not describe the external gravitational field of a particle in this case also it would evidence in favour of the uniqueness of the Einstein field equations.

4. $T = 0$

The condition $T = 0$ is not consistent with the conventional properties of the matter energy-momentum tensor. But on this occasion let us refer to Einstein [42] "... although it might seem that the quantity T^i_i is positive for the whole system in fact the quantity $T^i_i + t^i_i$ is only positive". The possibility of $T = 0$ for a real massive body has been also discussed in the same paper.

The fact that the contribution of the proper gravitational field to the total energy-momentum tensor of a point particle has the same order of magnitude as the contribution of a "bare" particle is also the result of

ADM group [43–44]. This group has shown that the total mass of the neutral point particle is equal to zero. The possibility for the total mass of a closed system being equal to zero has been also considered by various authors [33, 45–48].

Using the commutation rules for the covariant derivatives in the expression $\nabla_n \nabla^i R^{kn}$ and the condition $T=R=0$ we can write Eqs. (5) in the form (we suppose that $A=1$)

$$T_i^k = R^k{}_{lin} R^{ln} - \tfrac{1}{4}\delta_i^k R_{ln} R^{ln} + \tfrac{1}{2}\nabla_n \nabla^n R_i^k . \tag{14}$$

For metrics (7) Eqs. (14) take a form

$$\begin{aligned} T_0^0 &= U^{-1} r^{-3} 2VU'(1-V) \\ &\quad + \tfrac{1}{8} r^{-2}(10U^{-1}VU'V' + 9(U^{-1}VU')^2 - 3V'^2) + \tfrac{1}{2}\nabla_n \nabla^n R_0^0 \\ T_1^1 &= -2V'(1-V)\, r^{-3} \\ &\quad + \tfrac{1}{8} r^{-2}(10U^{-1}VU'V' - 3(U^{-1}VU')^2 + 9V'^2) + \tfrac{1}{2}\nabla_n \nabla^n R_1^1 , \\ T_2^2 &= (U^{-1}VU' + V')(1-V)\, r^{-3} \\ &\quad + \tfrac{1}{8} r^{-2}(-10U^{-1}VU'V' - 3(U^{-1}VU')^2 - 3V'^2) + \tfrac{1}{2}\nabla_n \nabla^n R_2^2 . \end{aligned} \tag{15}$$

Besides we have the equation

$$\begin{aligned} R &= 2Vr^{-1}(V^{-1}V' + U^{-1}U') \\ &\quad + 2(V-1)\, r^{-2} + U^{-1}VU'' + \tfrac{1}{2}U^{-1}V'U' - \tfrac{1}{2}U^{-2}U'^2 = 0 . \end{aligned} \tag{16}$$

Only three equations from the four ones (15–16) are independent. Therefore it is necessary one additional equation to determine four independent functions T_0^0, T_1^1, U, V. Usually such the equation is the equation of the state of matter. In the present case it is only the principal possibility or nonpossibility of the existence of a solution that we are interested in. Therefore it is sufficient to analise an example with a truthful equation of the state. Let us choose this equation in the form

$$T_1^1 = T_2^2 . \tag{17}$$

Let us search for the solution in the form

$$\begin{aligned} &U = a\left(1 + br^2 + \sum_{n=2}^{\infty} b_{2n} r^{2n}\right), \qquad V = d - \beta r^2 + \sum_{n=2}^{\infty} \gamma_{2n} r^{2n}, \qquad r < r_0 , \\ &\qquad\qquad V = U = 1 - 2M/r , \qquad r > r_0 . \end{aligned} \tag{18}$$

From the conditions (16–17) we obtain that

$$\begin{aligned} U &= a(1 + br^2 + \tfrac{1}{5} b^2 r^4 - \tfrac{3}{35} b^3 r^6 + \cdots), \\ V &= 1 - br^2 + \tfrac{6}{5} b^2 r^4 - \tfrac{32}{35} b^3 r^6 + \cdots . \end{aligned} \tag{19}$$

From the boundary conditions $U(r_0) = V(r_0) = 1 - 2M/r_0$ we obtain the constants a and b

$$\begin{aligned} b &= r_0^{-2}\left(1 + \tfrac{6}{5}\,\delta + \tfrac{344}{175}\,\delta^2 + \tfrac{696}{875}\,\delta^3 + \cdots\right), \qquad \delta = 2M/r_0\,, \\ a &= (1-\delta)\left(1 + br_0^2 + \tfrac{1}{5}\,b^2 r_0^4 - \tfrac{3}{35}\,b^3 r_0^6 + \cdots\right)^{-1}. \end{aligned} \tag{20}$$

Then we determine the tensor $T_i^k(r)$ corresponding to the solution (20),

$$T_0^0 = 12b^2 - 54b^3 r^2 + \cdots; \qquad T_1^1 = T_2^2 = \tfrac{1}{3}\,T_0^0\,. \tag{21}$$

5. Conclusions

We have shown that the description of the gravitational field with the quadratic Lagrangian theory leads to the condition $T = R = 0$. In this case the nontrivial solution with the external Schwarzschild one exists. If we shall not consider the condition $T = 0$ as unacceptable it is necessary to admit that the variant of the gravitation theory with quadratic Lagrangians is consistent with the known experimental data.

However the very strong limitations imposed on the matter-gravitational field system in the quadratic Lagrangians theory (the condition $T = 0$, the single-valued form of the function $T_i^k(r)$) apparently give evidence against a possibility of use the quadratic Lagrangians for a description of a real gravitational field.

References

1. Poincaré, H.: Dernieres Penseis, ed. E. Flammarion. Paris 1913.
2. Weil, H.: Math. Z. **2**, 384 (1918); Ann. Physik **59**, 101 (1919).; Phys. Z. **22**, 473 (1921); Raum, Zeit, Materie. Berlin: Springer 1918.
3. Pauli, W.: Phys. Z. **20**, 457 (1919).
4. Weitzenboeck, R.: Wien. Ber. **129**, 683 (1920).
5. Bach, R.: Math. Z. **9**, 110 (1921).
6. Eddington, A.: Proc. Roy. Soc. **99**, 104 (1921). – The mathematical theory of relativity. Cambridge: The University Press, 1923. – Space, time and gravitation: an outline of the General Relativity Theory. New York: Harper's torchbooks 1959.
7. Juettner, F.: Math. Ann. **87**, 270 (1922).
8. Einstein, A.: Berl. Ber. 1928.
9. Veblen, O., Hoffman, B.: Phys. Rev. **36**, 810 (1930).
10. Dazig, D. van: Math. Ann. **106**, 400 (1932).
11. Lanczos, C.: Phys. Rev. **39**, 716 (1932); – Ann. Math. **39**, 842 (1938); – Phys. Rev. **61**, 713 (1942); – Rev. Mod. Phys. **21**, 497 (1949); **29**, 337 (1957); – J. Math. Phys. **10**, 1057 (1969).
12. Schouten, J., Haantjes, J.: Physica **1**, 869 (1934); – Proc. Kon. Akad. v. Wet. **39**, 1059 (1936); – Rev. Mod. Phys. **21**, 421 (1949).
13. Veblen, O.: Proc. Nat. Acad. Sci. **21**, 421 (1935).
14. Rosen, N.: Phys. Rev. **57**, 147 (1940).
15. Haantjes, J.: Proc. Kon. Acad. v. Wet. **43**, 1288 (1940).

16. Infeld, L., Shild, A.: Phys. Rev. **68**, 250 (1945); **70**, 410 (1946).
17. McVittie, G.: Proc. Roy. Soc. Edinburgh **62**, 147 (1945).
18. Walker, A.: Proc. Roy. Soc. Edinburgh **62**, 164 (1946).
19. Gregory, C.: Phys. Rev. **72**, 72 (1947).
20. Hill, E.: Phys. Rev. **72**, 143 (1947).
21. Hoffman, B.: Phys. Rev. **72**, 458 (1947); **73**, 30 (1948); **73**, 1042 (1948); **89**, 49 (1953).
22. Milne, E.: Kinematic Relativity. Oxford: Clarendon Press 1948.
23. Buchdal, H.: Quart. J. Math. Oxford **19**, 150 (1948); – Proc. Nat. Acad. Sci. **34**, 66 (1948); – J. London math. Soc. **26**, 139, 150 (1951); – Nuovo Cimento **23**, 141 (1962).
24. Kohler, M.: Z. Physik **131**, 571 (1952); **134**, 286 (1953).
25. Ingraham, R.: Nuovo Cimento **12**, 825 (1954).
26. Kantor, W., Szekeres, G.: Phys. Rev. **104**, 831 (1956).
27. Arnowitt, R.: Phys. Rev. **105**, 735 (1957).
28. Dicke, R.: Rev. Mod. Phys. **29**, 363 (1957); – Phys. Rev. **125**, 2163 (1962).
29. Stephenson, G.: Nuovo Cimento **9**, 263 (1958); – Lett. Nuovo Cim. **1**, 97 (1969).
30. Higgs, P.: Nuovo Cimento **11**, 816 (1959).
31. Motz, L.: Phys. Rev. **119**, 1102 (1960).
32. Fulton, T., Rohrlich, F., Witten, L.: Bull. Am. Phys. Soc. **6**, 346 (1961); – Rev. Mod. Phys. **34**, 442 (1962); – Nuovo Cimento **26**, 652 (1962).
33. Mandelstam, S.: Ann. Phys. **19**, 25 (1962).
34. Rohrlich, F.: Ann. Phys. **22**, 169 (1963).
35. Greenberger, D.: Ann. Phys. **25**, 290 (1963).
36. Milner, S.: Generalised electrodynamics, and the structure of matter, ed. by C. W. Kilmister. London: University of Sheffield 1963.
37. Polievktov-Nikoladze, N.: Lett. JETP **2**, 551 (1965); **6**, 531 (1967); **6**, 963 (1967); – JETP **52**, 1360 (1967); **57**, 2010 (1969).
38. Pechlander, E., Sexl, R.: Commun. math. Phys. **2**, 165 (1966).
39. Davis, W.: Ann. Phys. **22**, 77 (1968).
40. Lapedra, R.: Ann. Inst. H. Poincare A **11**, 277 (1969).
41. Ruzmaikina, T., Ruzmaikin, A.: JETP **57**, 680 (1969).
42. Einstein, A.: Coll. Scientific Papers, Vol. 1, p. 436. Moscow: Publishing House "Nauka" 1965.
43. Arnowitt, R., Deser, S., Misner, C.: Phys. Rev. **118**, 1100 (1960); **120**, 313 (1960).
44. Witten, L.: Gravitation: Introd. to Current Research. New York: John Wiley and Sons 1962.
45. Lorentz, H.: Ammst. Versl. **25**, 468 (1916).
46. Levi-Civita, T.: Rend. Acad. Linc. **26**, 381 (1917).
47. Souriao, J.: C. R. Acad. Sci. (France) **245**, 958 (1957).
48. Folomeshkin, V.: Sov. Nucl. Phys. **6**, 1080 (1967); – Acta Phys. Polon. **36**, 439 (1969); B **1**, 13 (1970).

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