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On the Spectrum of the Space Cut-Off :$P(\varphi)$: Hamiltonian in Two Space-Time Dimensions*

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Abstract. We use the method of the asymptotic fields to study the spectrum of the space cut-off :$P(\varphi)$: Hamiltonian in two space time dimensions, where P is a semibounded polynomial.

I. Introduction

Let H_0 be the free energy of a boson field $\varphi(x)$ of strictly positive mass m, in one space dimension, so that $x \in R$. We consider the operator $H = H_0 + V$ where

$$V = \int_R g(x) : P(\varphi(x)) : dx . \tag{1.1}$$

Here $g(x)$ is a smooth positive function of compact support, $P(\alpha)$ is a polynomial bounded below, and :$P(\varphi(x))$: is the corresponding Wick ordered polynomial in the field. H_0 and H are operators on the boson Fock space $\mathscr{F}$ which is the direct sum $\mathscr{F} = \sum_{n=0}^{\infty} \mathscr{F}^n$ where $\mathscr{F}^n$ is the space of symmetric square integrable functions of n (momentum) variables. The field $\varphi(x)$ is given in terms of annihilation-creation operators on $\mathscr{F}$ by

$$\varphi(x) = (4\pi)^{-\frac{1}{2}} \int e^{ikx} [a^*(k) + a(-k)] \mu(k)^{-\frac{1}{2}} dk , \tag{1.2}$$

where $\mu(k) = (k^2 + m^2)^{\frac{1}{2}}$. The annihilation-creation operators satisfy

$$[a(k), \ a^*(k')] = \delta(k - k') . \tag{1.3}$$

Glimm [1] proved that H is bounded below, and Rosen [8] proved that H is essentially self-adjoint. Later on Rosen [9] also gave the

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following higher order estimate involving the particle number operator N:

$$H_0^2 \leqq a(H+b)^2 \quad \text{and} \quad N^j \leqq a(H+b)^j \tag{1.4}$$

where $j=1,2,\ldots$ and a and b may depend on j.

The method we use for studying the spectrum of H, is the method of asymptodic fields. This method has been used by the autor in a series of papers, and is in fact an adaption of Cook's method to field theory (see Refs. [3–7]).

For h in $\mathscr{F}^1$ we write $a^\#(h)=\int a^\#(k)h(k)dk$, where $a^\#$ stands for a or a^*. The commutation relation (1.3) takes then the form

$$[a(\bar{h}), a^*(g)] = (h, g) \tag{1.5}$$

It is well known that $a^*(h)$ and $a(\bar{h})$ are closed operators on $\mathscr{F}$, which are the adjoint of each other. Let us now define

$$a_t^\# = e^{-itH} e^{itH_0} a^\#(h) e^{-itH_0} e^{itH}, \tag{1.6}$$

and making use of the well known relation

$$e^{itH_0} a^\#(h) e^{-itH_0} = a^\#(h_{\pm t}) \tag{1.7}$$

where $h_t(k)=e^{it\mu(k)}h(k)$, and $+$ goes with a^* and $-$ with a; (1.6) takes the form

$$a_t^\#(h) = e^{-itH} a^\#(h_{\pm t}) e^{itH}. \tag{1.8}$$

Making use of the fact that the domain of $a^\#(h)$ contains the domain of $N^{\frac{1}{2}}$ we see by (1.4) that $a_t^\#(h)$ are closed operators with domains containing the domain of $(H+b)^{\frac{1}{2}}$. From (1.4) we also get that for ψ in the domain of $(H=b)^{j/2}$ we have the following estimate

$$\|a_t^\#(h_1)\ldots a_t^\#(h_j)\psi\| \leqq a_j\|h_1\| \ldots \|h_j\| \, \|(H+b_j)^{j/2}\psi\|. \tag{1.9}$$

In the next section we shall see that (1.4) implies that the limits $a_\pm^\#(h)$ of $a_t^\#(h)$, as t tends to $\pm\infty$ exist; and this result then gives us the information on the spectrum of H.

II. The Asymptotic Fields

Let D_0 be the domain of H_0 and D_j the domain of $(H+b)^j$ for $j>0$. Since $P(\alpha)$ is a polynomial which is bounded below its degree is even, say $2n$.

Lemma 1. *Let ψ_1 and ψ_2 be in D_n, and let h be in $D_0\cap\mathscr{F}^1$. Then $(\psi_1, a_t^\#(h)\psi_2)$ is a differentiable function of t with a continuous derivative*

$$\frac{d}{dt}(\psi_1, a_t^\#(h)\psi_2) = (i\,Ve^{itH}\psi_1, a^\#(h_{\pm t})e^{itH}\psi_2) + (a^\#(h_{\pm t})^* e^{itH}\psi_1, i\,Ve^{itH}\psi_2). \tag{2.1}$$

Proof. Since $e^{itH}\psi_k$, $k=1,2$ both are strongly differentiable, and $a^{\#}(h_{\pm t})$ is strongly differentiable on the domain of $N^{\frac{1}{2}}$, we get by (1.4) that $(\psi_1, a_t^{\#}(h)\psi_2)$ is differentiable, with derivative equal to

$$(iHe^{itH}\psi_1, a^{\#}(h_{\pm t})e^{itH}\psi_2)+(a^{\#}(h_{\pm t})^* e^{itH}, iHe^{itH}\psi_2) + (e^{itH}\psi_1, a^{\#}(\pm i\omega k_{\pm t})e^{itH}\psi_2). \tag{2.2}$$

By (1.4) we get that $e^{itH}\psi_1$ and $e^{itH}\psi_2$ is in the intersection of the domains of H_0 and V, hence $He^{itH}\psi_1 = H_0 e^{itH}\psi_1 + Ve^{itH}\psi_1$ by the essential self-adjointness, and this then gives (2.1). To see that the derivative is continuous we have only to point out that $e^{itH}\psi_1$ and $e^{itH}\psi_2$ are strongly continuous in D_n with its natural norm and by (1.4) V maps D_n into $\mathscr{F}$ continuously, and $a^{\#}(h_{\pm t})$ is a normcontinuous and uniformly bounded family of operators from D_n into $\mathscr{F}$. This proves the lemma.

Lemma 2. *Let ψ be in D_n, and let h be in $\mathscr{F}^1 \cap D_0$. Then*

$$a_t^{\#}(h)\psi = a^{\#}(h)\psi - i\int_0^t e^{-isH}[V, a^{\#}(h_{\pm s})]e^{isH}\psi\, ds,$$

where the integral is a strong integral.

Proof. Using that the strong integral is equal to the weak integral whenever the first exists, we get this lemma by integrating (2.1) if we can prove that the integrand is strongly continuous. It is easy to see however that $[V, a^{\#}(h_{\pm s})]$ is a uniformly bounded mapping from D_n with its natural norm into $\mathscr{F}$, which depends normcontinuously on s. Using now that e^{isH} is uniformly bounded and strongly continuous on D_n with its natural norm, as well as on $\mathscr{F}$ we see that $e^{-isH}[V, a^{\#}(h_{\pm s})]e^{isH}\psi$ is uniformly bounded and strongly continuous. This proves the lemma.

In the proof of the lemma above we saw that $\|[V, a^{\#}(h_{\pm s})](H+b)^{-n}\|$ is uniformly bounded. It is easy to see however that if h is in $C_0^{\infty}(R)$ and zero in a neighborhood of the origin that $\|[V, a^{\#}(h_{\pm s})](N+1)^{-n}\|$ hence by (1.4) also $\|V, a^{\#}(h_{\pm s})](H+b)^{-n}\|$ converge to zero faster than any inverse power of s.

But from Lemma 2 we get that for ψ in D_n and h in $D_0 \cap \mathscr{F}$,

$$\|(a_{t_1}(h)-a_{t_2}(h))\psi\| \leq \int_{t_1}^{t_2} \|[V, a^{\#}(h_{\pm s})](H+b)^{-n}\|\, ds\, \|(H+b)^n\psi\|. \tag{2.3}$$

We formulate this in a theorem

Theorem 1. *Let h be in $\mathscr{F}^1$ and ψ in the domain of $(H+b)^{\frac{1}{2}}$. Then $a_t^{\#}(h)\psi$ converge strongly to $a_{\pm}^{\#}(h)\psi$ as t tends to $\pm\infty$. $a_{\pm}^{\#}(h)$ satisfy the same commutation relations on the domain of H as do $a^{\#}(h)$ on the domain* *of* H_0. *H and* $a^{\#}_{\pm}(h)$ *satisfy the same commutation relations as do* H_0 *and* $a^{\#}(h)$ *in the sense that*

$$e^{itH}\, a^{\#}_{\pm}(h)\, e^{-itH} = a^{\#}_{\pm}(h_{\pm t})\,.$$

Proof. We have already seen that for h in $C_0^\infty(R)$ and zero in a neighborhood of zero and ψ in D_n then $a^{\#}_t(h)\,\psi$ converge strongly. But the uniform estimate (1.9) then gives strong convergence for h in $\mathscr{F}^1$ and ψ in $D_{\frac{1}{2}}$. From the fact that $a^{\#}_t(h)$ and $a_t(\bar{h})$ are adjoints and the strong convergence we see that $a^{\#}_{\pm}(h)$ have densely defined adjoints and are therefore closable. Denoting by $a^{\#}_{\pm}(h)$ also their closure we see that $a^{*}_{\pm}(h)$ and $a_{\pm}(\bar{h})$ are adjoints. The commutation relations for $a^{\#}_{\pm}(h)$ follows from the strong convergence by utilizing the uniform estimate (1.9) for $j=2$. The commutation relation for H and $a^{\#}_{\pm}(h)$ follows immediately from the strong convergence. For more details we refer the reader to the proof of the corresponding theorem in Ref. [6] or Ref. [7].

The following theorem is a consequence of Theorem 1, and we refer the reader to Ref. [7], Section 3 for the proof of this. We introduce first some notations and then will state the theorem.

Let Ω be the vacuum for H, and let $V_{\pm}$ be the subspaces annihilated by $a_{\pm}(h)$ for all h in $\mathscr{F}^1$. It is easy to see that all eigenvectors for H are in $V_{\pm}$. It is also easy to see that $V_{\pm}$ is in the domain of any polynomial in the $a^{\#}_{\pm}(h)$. Let $\mathscr{F}_{\pm}$ be the subspaces generated by applying polynomials in $a^{*}_{\pm}(h)$ to Ω. $\mathscr{F}_{\pm}$ are then naturally isomorphic to the Fock spaces for $a^{\#}_{\pm}$. For convenience let us add a constant to H such that the eigenvalue for Ω is zero.

Theorem 2. $V_{\pm}$ *and* $\mathscr{F}_{\pm}$ *are invariant subspaces for H. Let* $H^0_{\pm}$ *be the restriction of H to* $V_{\pm}$, *and let* $H_0^{\pm}$ *be the restriction of H to* $\mathscr{F}_{\pm}$. *Then* $H^0_{\pm}$ *is positive, and its spectral projection on the interval* $[0, m-\varepsilon]$ *is finite dimensional for* $\varepsilon > 0$. *If* $\mathscr{F}_{\pm}$ *are identified in a natural way with the Fock spaces for* $a^{\#}_{\pm}$, *then* $H_0^{\pm}$ *are the free energy with mass m in* $\mathscr{F}_{\pm}$. $\mathscr{F}$ *decompose as a tensor product* $\mathscr{F} = \mathscr{F}_{\pm} \otimes V_{\pm}$ *and relative to this decomposition H has the form*

$$H = H_0^{\pm} \otimes 1 + 1 \otimes H^0_{\pm}\,.$$

Proof. The proof goes in the same way as the proof of Theorem 4 of Ref. [7], apart from the spectral projection on $[0, m-\varepsilon]$ being finite dimensional. But this follows from Theorem 3 of Ref. [8].

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