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Galilean Invariant Lee Model for All Spins and Parities*

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Abstract. Using the minimal $6S+1$ components required to describe a nonrelativistic particle of spin S within the framework of a first order formalism, the Galilean invariant Lee model as formulated by Levy-Leblond is generalized to the case in which all particles are allowed to have arbitrary spins and parities. It is found that when the product of the parities of the three fields is even the coupling term is unique and the physics essentially identical to the spin zero case. A more interesting situation occurs when one considers the odd parity case where the criterion that derivatives not be explicitly used in writing the interaction requires that the dependent components enter into the coupling term. In this case one finds that except for certain degenerate cases there exist three similar but distinct interactions. Upon selecting one of these couplings and eliminating the dependent components by means of the constraint equations one finds the existence of a P wave $VN\theta$ coupling as well as direct S wave interactions between pairs. The V particle propagator is derived and it is found that unlike the even parity case the wave function renormalization constant is divergent. The P wave phase shift is obtained and found to satisfy the exact effective range formula $q^3 \cot\delta = -1/a + \frac{1}{2} r_0 q^2$.

I. Introduction

One of the most difficult obstacles in recent years to further progress in quantum field theory has been the absence of a consistent set of rules for the quantization and renormalization of field theories describing particles with spin greater than unity. A conspicuous example of the type of difficulty encountered is contained in the demonstration by Johnson and Sudarshan [1] that the equal time anticommutators of a spin 3/2 field become indefinite in the presence of an external electromagnetic field. This inconsistency emerges as a consequence of the secondary constraints upon the fields, the existence of which is guaranteed for all half-integral spins greater than one-half. Although one suspects (with good reason) that these difficulties are a consequence of the rigid requirements of Lorentz invariance, a complete confirmation of this conjecture is dependent upon the presentation of a consistent theory of

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interacting higher spin fields within the framework of a less exacting group of spacetime transformations. In the present paper such an objective is achieved by carrying out an extension of Levy-Leblond's Galilean invariant version of the Lee model to the case in which all particles have arbitrary spins and parities. This is equivalent to solving the problem of determining the most general Galilean invariant trilinear interaction subject to the assumptions that (a) the fields should involve the minimal number of components compatible with a theory involving only first order derivatives and (b) the couplings must not explicitly contain derivatives.

In the following section a brief review is given of Galilean free fields beginning with the spin one-half case and subsequently generalizing this to arbitrary spin by means of the formalism of totally symmetric multispinors. After constructing the most general Lagrangian in this formulation it is noted that an enormous simplification ensues upon expressing the multispinor in terms of a spherical basis. The Lagrangian is then displayed and its Galilean invariance demonstrated in terms of the new spherical tensor fields. As a byproduct it is shown that this leads to an alternative derivation of the g factor.

Section three presents the derivation of the coupling appropriate to the case in which the product of the parities of the three fields is even. In this case the interaction is unique and requires only the demonstration that the multispinor basis (in which the uniqueness is proved) reduces to the spherical basis with the insertion of the appropriate Clebsch-Gordan coefficient. The more interesting odd parity case is developed in the concluding section. It is shown that in general there are three distinct trilinear couplings, for each of which the V particle propagator is more singular than in the even parity case by two powers of the Galilean invariant momentum.

II. Galilean Free Fields

The problem of deriving a wave equation (or field theory) for a particle which transforms as a spin one-half object under the homogeneous Galilei group has been solved by Levy-Leblond [2] and it is upon this result that we proceed to construct a theory of higher spin fields. The equation which he derives for the four component spinor ψ is of the form

$$G\psi = 0$$

where

$$G = \frac{1}{2}(1+\varrho_3)i\hbar\frac{\partial}{\partial t} + \varrho_1 \boldsymbol{\sigma}\cdot\frac{\hbar}{i}\boldsymbol{\nabla} + m(1-\varrho_3)$$

and we have used the two independent sets of Pauli matrices ϱ^i and σ_i to span the 4×4 dimensional spinor space. Under the Galilean transformation

$$x' = Rx + vt + a$$

$$t' = t + b$$

$\psi(x, t)$ obeys the transformation law

$$\psi'(x', t') = e^{i\hbar^{-1} f(x,t)} \Delta^{\frac{1}{2}}(v, R) \psi(x, t)$$

where

$$f(x, t) = m v \cdot R x + \tfrac{1}{2} m v^2 t$$

and

$$\Delta^{\frac{1}{2}}(v, R) = \begin{pmatrix} 1 & 0 \\ \dfrac{-\sigma \cdot v}{2} & 1 \end{pmatrix} D^{\frac{1}{2}}(R)$$

with $D^{\frac{1}{2}}(R)$ being the usual two dimensional representation of spin one-half. Using the decomposition

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

into two component spinors ϕ and χ one has the equations

$$E\phi + \sigma \cdot p \chi = 0$$

$$\sigma \cdot p \phi + 2m\chi = 0$$

as well as the Schrödinger equation for all four components.

The generalization of this result to arbitrary spin has been given in the multispinor formalism [3, 4] with the result that if one chooses to describe a spin S object by a symmetric multispinor of rank $2S$ (i.e. $\psi_{a_1 \dots a_{2S}}$), the most general Lagrangian is of the form

$$\mathscr{L} = \left\{ \frac{1}{2S} \psi^+_{a_1 \dots a_{2S}} \sum_{i=1}^{2S} \Gamma_{a_1 a'_1} \dots \Gamma_{a_{i-1} a'_{i-1}} G_{a_i a'_i} \Gamma_{a_{i+1}, a'_{i+1}} \dots \Gamma_{a_{2S} a'_{2S}} \psi_{a'_1 \dots a'_{2S}} + \text{h.c.} \right\} \tfrac{1}{2} \tag{2.1}$$

where Γ is the matrix $\frac{1}{2}(1 + \varrho_3)$ which is invariant in the sense that

$$\Delta^{\frac{1}{2}+}(v, R) \Gamma \Delta^{\frac{1}{2}}(v, R) = \Gamma .$$

The simple property of Γ that it is nonzero only when each of its indices refer to "upper" components means that there is an extraordinarily large number of the $\frac{1}{6}(2S+1)(2S+2)(2S+3)$ independent components of $\psi_{a_1 \dots a_{2S}}$ which do not occur in the equations of motion. More specifically from the fact that Γ acts on all indices but one in the equation

$$\sum_{i=1}^{2S} \Gamma_{a_1 a'_1} \ldots \Gamma_{a_{i-1} a'_{i-1}} G_{a_i a'_i} \Gamma_{a_{i+1} a'_{i+1}} \ldots \Gamma_{a_{2S} a'_{2S}} \psi_{a'_1 \ldots a'_{2S}} = 0$$

one infers that the only relevant components of $\psi_{a_1 \ldots a_{2S}}$ are a) the $2S+1$ components for which all indices of $\psi_{a_1 \ldots a_{2S}}$ are 1 or 2 and b) the $4S$ components for which $2S-1$ indices take the values 1 or 2 and the one remaining index is 3 or 4. This description of a spin S particle by a $6S+1$ component formalism may be said to be "minimal" in that this is the smallest number of components compatible with a formalism which allows only first derivatives in the wave equation. If one now introduces the notation

$$\psi_{a_1 \ldots a_{2S}} = \phi_{a_1 \ldots a_{2S}} \qquad \text{for} \quad a_i = 1, 2$$

$$\psi_{a_1 \ldots a_{2S-1} r} = \chi^{r-2}_{a_1 \ldots a_{2S-1}} \qquad \text{for} \quad a_i = 1, 2; \; r = 3, 4$$

(2.1) can be rewritten as

$$\mathscr{L} = \left\{ \phi^{+}_{a_1 \ldots a_{2S}} E \phi_{a_1 \ldots a_{2S}} + \frac{1}{2S} \sum_{i=1}^{2S} [\phi^{+}_{a_1 \ldots a_{2S}} \boldsymbol{\sigma}_{a_i r} \cdot \boldsymbol{p} \chi^{r}_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_{2S}} \right. \tag{2.2}$$

$$\left. + \chi^{r+}_{a_1 \ldots a_{i-1} a_{i+1} \ldots a_{2S}} \boldsymbol{\sigma}_{r a_i} \cdot \boldsymbol{p} \phi_{a_1 \ldots a_{2S}} + 2m \chi^{r+}_{a_1 \ldots a_{2S-1}} \chi^{r}_{a_1 \ldots a_{2S-1}} + \text{h.c.} \right\} \frac{1}{2}$$

where all spinor summations are now over two-valued indices. The Lagrangian (2.2) has been used to show [4] that the g factor for this minimal component theory is $1/S$.

In view of the fact that the number of components required for a Galilean invariant theory increases only linearly with S one strongly suspects that the enumeration of these components can be accomplished more conveniently as well as more explicitly by using a basis of spherical tensors rather than a multispinor approach. Such fields can be introduced by the definition

$$\phi_{a_1 \ldots a_{2S}} = \sum_{m=-S}^{S} \left[\frac{(S+m)!\,(S-m)!}{2S!} \right]^{\frac{1}{2}} \varepsilon^{m}_{a_1 \ldots a_{2S}} \phi^{m}_{S} \tag{2.3}$$

where the square root factor has been inserted to normalize the coefficient of the time derivative term in the Lagrangian to unity. The quantity $\varepsilon^{m}_{a_1 \ldots a_{2S}}$ is defined to be totally symmetric in its lower indices and in the usual representation of the σ matrices is equal to unity when $S+m$ of its $2S$ indices take the value one and $S-m$ take the value two and zero otherwise. The contravariant components ϕ^{m}_{S} transform according to the $2S+1$ dimensional irreducible representation of the rotation group. Covariant components may be defined by use of the metric tensor

$$g^{S}_{mm'} = (-1)^{S+m} \delta_{-m,m'}$$

while the transition from covariant to contravariant indices is accomplished via the tensor

$$g_S^{mm'} = (-1)^{S-m} \delta_{-m,m'}.$$

The use of covariant and contravariant indices affords a great simplification in this development as it allows one to make use of Wigner's [5] tensorial notation for the $O(3)$ group. Thus repeated upper and lower indices are summed and the transformation properties of a given expression can be determined merely from an examination of non-repeated indices.

With the definition (2.3) the time derivative term in $\mathscr{L}$ becomes $\phi_S^{m+} E \phi_S^m$. It is to be noted that the fields ϕ_S^{m+} transform as covariant components and one consequently must generalize the summation convention by adopting the additional rule that two repeated upper indices are summed provided that one of the two objects involved is the adjoint of a contravariant field.

Since the field χ transforms as the direct product of $S-1/2$ and $1/2$ under $O(3)$ we introduce a spherical tensor in this case by defining

$$\chi^r_{a_1 \dots a_{2S-1}} = \sum_{m=-S+\frac{1}{2}}^{S-\frac{1}{2}} \sum_{m'=-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{(S-\frac{1}{2}+m)!\,(S-\frac{1}{2}-m)!}{(2S-1)!} \right]^{\frac{1}{2}} \varepsilon^m_{a_1 \dots a_{2S-1}} \delta_{m',\frac{3}{2}-r} \chi^{m,m'}_{S-\frac{1}{2},\frac{1}{2}}$$

where the somewhat unusual form of the Kronecker delta is necessary to transform the range of $r(1, 2)$ into the corresponding range of $m'(\frac{1}{2}, -\frac{1}{2})$ as appropriate to a spherical basis. This definition allows one to rewrite the mass term in $\mathscr{L}$ as $2m\chi^{m,m'}_{S-\frac{1}{2},\frac{1}{2}}{}^{+} \chi^{m,m'}_{S-\frac{1}{2},\frac{1}{2}}$ and leaves only the somewhat more difficult task of writing the spatial derivative expression in terms of spherical tensors.

In order to achieve this objective we introduce covariant spherical components for x and σ

$$x_{\pm 1} = \mp 2^{-\frac{1}{2}}(x \pm iy) \qquad x_0 = z$$
$$\sigma_{\pm 1} = \mp 2^{-\frac{1}{2}}(\sigma_x \pm i\sigma_y) \quad \sigma_0 = \sigma_z$$

and corresponding contravariant components for p

$$\nabla^{\pm 1} = \frac{\partial}{\partial x_{\pm 1}} = 2^{-\frac{1}{2}}\left(\mp \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$
$$\nabla^0 = \frac{\partial}{\partial x_0} = \frac{\partial}{\partial z}$$

such that

$$\boldsymbol{\sigma} \cdot \boldsymbol{p} = \sigma^\mu p_\mu.$$

Furthermore one notes that in terms of the three-j symbols

$$\left[\frac{(S+m)!\,(S-m)!}{2S!}\right]^{\frac{1}{2}}\left[\frac{(S-\frac{1}{2}+m')!\,(S-\frac{1}{2}-m')!}{(2S-1)!}\right]^{-\frac{1}{2}} = (2S+1)^{\frac{1}{2}}(-1)^{-S+1-m}\begin{pmatrix} S-\frac{1}{2} & \frac{1}{2} & S \\ m' & m-m' & -m \end{pmatrix}$$

while

$$[\sigma_\mu]_{m,m'} = (-1)^{\frac{1}{2}-m}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -m & \mu & m' \end{pmatrix}(\tfrac{1}{2}\|\sigma\|\tfrac{1}{2})$$

where $(\frac{1}{2}\|\sigma\|\frac{1}{2}) = \sqrt{6}$ is the reduced matrix element of σ. Upon combining these results and freely raising and lowering indices of the three-j symbols by means of the metric tensor one obtains the desired reduction of the spatial derivative term and thus the complete Lagrangian

$$\mathscr{L} = \Big\{\phi_S^{m+} E\phi_S^m + 2m\chi_{S-\frac{1}{2},\frac{1}{2}}^{m,m'+}\chi_{S-\frac{1}{2},\frac{1}{2}}^{m,m'} + 6^{\frac{1}{2}}(2S+1)^{\frac{1}{2}}\Big\{\phi_S^{m+} p^\nu \begin{pmatrix} m & \mu & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \mu & \nu & m'' \end{pmatrix}\chi_{S-\frac{1}{2},\frac{1}{2}}^{m',m''} + \chi_{S-\frac{1}{2},\frac{1}{2}}^{m,m'} p^\nu \begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \mu \\ \frac{1}{2} & \nu & \frac{1}{2} \end{pmatrix}\phi_S^{m''}\Big\} + \text{h.c.}\Big\}\tfrac{1}{2}. \tag{2.4}$$

As a consequence of the tensor notation this expression is manifestly invariant under rotations and it is only necessary to demonstrate the invariance under pure Galilean transformations. This somewhat tedious calculation is given in Appendix A.

The equations of motion implied by (2.4) are

$$E\phi_S^m + 6^{\frac{1}{2}}(2S+1)^{\frac{1}{2}} p^\nu \begin{pmatrix} m & \mu & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \mu & \nu & m'' \end{pmatrix}\chi_{S-\frac{1}{2},\frac{1}{2}}^{m',m''} = 0$$
$$2m\chi_{S-\frac{1}{2},\frac{1}{2}}^{m,m'} + 6^{\frac{1}{2}}(2S+1)^{\frac{1}{2}} p^\nu \begin{pmatrix} m & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu & m'' \end{pmatrix}\begin{pmatrix} m' & 1 & \mu \\ \frac{1}{2} & \nu & \frac{1}{2} \end{pmatrix}\phi_S^{m''} = 0 \tag{2.5}$$

while the commutation (anticommutation) relations implied by the action principle for the independent components ϕ are

$$[\phi_S^m(\boldsymbol{x},t), \phi_S^{m'+}(\boldsymbol{x}',t)]_\mp = \delta_{m'}^m\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,.$$

Eqs. (2.5) can be shown to imply the Schrödinger equation upon performing the same type of manipulations as those required in Appendix A to demonstrate the invariance of $\mathscr{L}$. It is of interest to point out that the introduction of a coupling to an external electromagnetic field allows one to calculate the g factor in this spherical basis and thereby provide a useful addition to the earlier calculation [4] in the multispinor formalism. Minimal coupling is seen from (2.5) to imply

$$(E-e\varphi)\phi_S^m - 6(2S+1)\frac{1}{2m}\Pi^\nu \Pi^{\nu'}\begin{pmatrix} m & \frac{1}{2} & S-\frac{1}{2} \\ S & \mu & m' \end{pmatrix}\begin{pmatrix} \mu & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & m'' \end{pmatrix}$$
$$\begin{pmatrix} m' & \mu' & S \\ S-\frac{1}{2} & \frac{1}{2} & m''' \end{pmatrix}\begin{pmatrix} m'' & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu' & \mu' \end{pmatrix}\phi_S^{m'''}=0 \tag{2.6}$$

where

$$\Pi^\nu = p^\nu - eA^\nu .$$

This reduces upon use of the six-j symbol to the form

$$\left(E-e\varphi-\frac{\Pi^2}{2m}\right)\phi_S^m + \frac{ie\hbar}{2m}3(2S+1)\begin{Bmatrix} S & 1 & S \\ \frac{1}{2} & S-\frac{1}{2} & \frac{1}{2} \end{Bmatrix}\begin{pmatrix} 1 & 1 & m'' \\ \nu & \nu' & 1 \end{pmatrix}\begin{pmatrix} m & 1 & S \\ S & m'' & m''' \end{pmatrix}$$
$$(-1)^{2S}\left[\partial^\nu A^{\nu'} - \partial^{\nu'} A^\nu\right]\phi_S^{m'''}=0 .$$

Introducing the Levi-Civita tensor ($\varepsilon^{10-1}=+1$) one can write

$$\partial^\nu A^{\nu'} - \partial^{\nu'} A^\nu = i\varepsilon^{\nu\nu'\nu''}H_{\nu''}$$

and

$$\begin{pmatrix} 1 & 1 & m'' \\ \nu & \nu' & 1 \end{pmatrix} = (-1)^{-m''}\varepsilon_{\nu\nu'-m''}6^{-\frac{1}{2}}$$

so that (2.6) becomes

$$\left(E-e\varphi-\frac{\Pi^2}{2m}\right)\phi_S^m$$
$$-\frac{e\hbar}{2m}6^{\frac{1}{2}}(2S+1)(-1)^{2S+1}H^{m''}\begin{pmatrix} m & 1 & S \\ S & m'' & m''' \end{pmatrix}\begin{Bmatrix} S & 1 & S \\ \frac{1}{2} & S-\frac{1}{2} & \frac{1}{2} \end{Bmatrix}\phi_S^{m'''}=0 .$$

If one now gives up a strict adherence to a tensorial notation and uses

$$\begin{Bmatrix} S & 1 & S \\ \frac{1}{2} & S-\frac{1}{2} & \frac{1}{2} \end{Bmatrix} = (-1)^{2S+1}\frac{S+1}{6^{\frac{1}{2}}[S(S+1)(2S+1)]^{\frac{1}{2}}}$$
$$\begin{pmatrix} m & 1 & S \\ S & m'' & m''' \end{pmatrix} = \langle S,m|S_{m''}|S,m'''\rangle\frac{1}{(S\|S\|S)}$$

where the reduced matrix element is given by

$$(S\|S\|S) = [(2S+1)S(S+1)]^{\frac{1}{2}},$$

one verifies by inspection the result

$$\left(E-e\varphi-\frac{\Pi^2}{2m}\right)\phi_S^m + \frac{e\hbar}{2mS}\langle S,m|\boldsymbol{S}\cdot\boldsymbol{H}|S,m'\rangle\phi_S^{m'}=0$$

thereby displaying the $1/S$ value of the g factor for the $6S+1$ component theory.

Before concluding this section it is necessary to comment briefly upon the exceptional spin zero case. Inasmuch as one cannot construct spin zero by the multispinor method as here described one must give an explicit statement as to how the Lagrangian of a scalar particle is to be written. It being desirable to retain the $6S+1$ component description we choose to describe the spinless case by a single field ϕ and write

$$\mathscr{L}=\left\{\phi^+ E\phi+\frac{1}{2m}(\boldsymbol{p}\phi^+)(\boldsymbol{p}\phi)+\text{h.c.}\right\}\frac{1}{2}$$

despite the fact that this requires that one give up the condition that only first derivatives occur in the equations of motion. This is primarily a technical point and will have no great bearing upon the remainder of this paper.

III. Interactions in the Even Parity Case

We now seek to write down the most general trilinear interaction which is consistent with Galilean invariance and does not require the explicit appearance of derivatives in the coupling term. Since this is equivalent to the problem of generalizing the Levy-Leblond version of the Lee model to the case of all three particles having arbitrary spin it will be convenient to use the notation of the Lee model and refer to the fields by the usual V, N and θ labels. Since the mass superselection rule of the physical Galilei group is consistent with the interaction $V\rightleftarrows N+\theta$ only if

$$m_V=m_N+m_\theta\,, \tag{3.1}$$

we require (3.1) to hold throughout the remainder of this paper.

The most systematic approach to the problem of the construction of interactions is to begin in the multispinor formalism and then transform to spherical tensors only after the derivation of the most general invariant form. To determine the possible invariants one notes that in order that only the $6S+1$ minimal components of each field be coupled, the matrices contracted between the three fields must in general connect only upper components to upper components. More specifically, if one writes the entire $2(S_V+S_N+S_\theta)$ index object which is to be contracted with the product of the V, N and θ field operators as the sum over various direct products of ϱ and σ matrices then only those combinations are allowed which involve at most one "upper-lower" matrix (e.g. ϱ_1) or "lower-lower" matrix (e.g. $(1-\varrho_3)$). The reason for this is that if, for example, two such matrices were to occur then with only three fields participating in the interaction both of these matrices would have to act on the same field operator. This would have the consequence of bringing in components of that field which are not admissible in the $6S+1$ component formalism.

In order to apply this specifically to the $V^+ N\theta$ interaction we note trivially that $S_V \leqq S_N + S_\theta$ and that except for the single case $S_V = S_N + S_\theta$ there must be matrix contractions between the indices of N and θ. This clearly requires that one obtain all matrices I which are invariant in the sense

$$\Delta^{\frac{1}{2}T}(\boldsymbol{v}, R)\, I\, \Delta^{\frac{1}{2}}(\boldsymbol{v}, R) = I \tag{3.2}$$

appropriate to these matrix contractions between N and θ as well as invariants I' such that

$$\Delta^{\frac{1}{2}+}(\boldsymbol{v}, R)\, I'\, \Delta^{\frac{1}{2}}(\boldsymbol{v}, R) = I' \tag{3.3}$$

which correspond to the matrices which connect V^+ to N and V^+ to θ. The problem of determining all I and I' for which (3.2) and (3.3) hold is an elementary one and it is sufficient to state that as a result of the calculation one finds

$$I = \alpha \Gamma \sigma_2 + \beta \varrho_1 \sigma_2 \tag{3.4}$$

$$I' = \alpha' \Gamma + \beta' \varrho_2 \tag{3.5}$$

where α, α', β, β' are arbitrary. It should be pointed out that this only enumerates the matrices which are scalar in the sense of (3.2) and (3.3) and leaves open the possibility of constructing invariants from vectors of the form $\frac{1}{2}(1+\varrho_3)\boldsymbol{\sigma}$. Although such additional invariants can indeed be constructed, it can be shown in relatively straightforward fashion that they do not generate any new couplings above and beyond those implied by (3.4) and (3.5). Combining this result with our previous observations concerning the nature of the couplings one can now infer that the most general invariant is constructed from the product of $2S_V$ matrices of the form (3.5) and $S_N + S_\theta - S_V$ matrices of the form (3.4) with the restriction, however, that only one β or β' be different from zero.

A final simplification occurs if one imposes parity conservation on the theory. It is trivial to show that the equations of motion possess the appropriate transformation properties under the parity operation provided that

$$\psi'_{a_1 \ldots a_{2S}}(-\boldsymbol{x}, t) = e^{i\eta} [\varrho_3]_{a_1 a'_1} \ldots [\varrho_3]_{a_{2S} a'_{2S}} \psi_{a'_1 \ldots a'_{2S}}(\boldsymbol{x}, t)\,.$$

This result immediately illustrates the fact that the invariant matrices ϱ_2 and $\varrho_1\sigma_2$ are, by virtue of their anticommutation with ϱ_3, the analogue of the pseudoscalar matrix γ_5 of the Dirac theory. One thus is led to consider the two possibilities

$$\exp i[\eta_N + \eta_\theta - \eta_V] = \pm 1$$

for which invariant couplings can be constructed. The upper sign may be referred to as the even parity case and will be the object of our immediate attention while the lower sign or odd parity case will be deferred to the following section.

The even parity case is thus seen to require that all β and β' be zero and one uniquely writes the coupling term as

$$\begin{aligned}\mathscr{L}' = & -g_0 \int d^3 r\, f(r)\, \psi^+_{a_1 \ldots a_{2S_V}}(\boldsymbol{R}, t) \\ & \cdot \psi_{a'_1 \ldots a'_{2S_N}}\left(\boldsymbol{R} - \frac{m_\theta}{m_V}\boldsymbol{r}, t\right) \psi_{a''_1 \ldots a''_{2S_\theta}}\left(\boldsymbol{R} + \frac{m_N}{m_V}\boldsymbol{r}, t\right) \\ & \cdot \prod_{i=1}^{S_N + S_\theta - S_V} [\Gamma \sigma_2]_{a'_i a''_i} \prod_{j=1}^{S_V - S_\theta + S_N} \Gamma_{a_j, a'_{S_N + S_\theta - S_V + j}} \prod_{k=1}^{S_V - S_N + S_\theta} \Gamma_{a_{S_V - S_\theta + S_N + k} a''_{S_N + S_\theta - S_V + k}} \\ & + \text{h.c.}\end{aligned}$$

where following Levy-Leblond we include the nonlocal form factor $f(r)$ allowed by Galilean invariance. The appearance of Γ in all indices means that all spinor summations can be limited to two-valued indices and one readily finds upon transforming to spherical tensors by means of (2.3) that

$$\begin{aligned}\mathscr{L}' = & -g_0 \int d^3 r\, f(r)\, \phi^{m_V +}_{S_V}(\boldsymbol{R}, t) \\ & \cdot \phi^{m_N}_{S_N}\left(\boldsymbol{R} - \frac{m_\theta}{m_V}\boldsymbol{r}, t\right) \phi^{m_\theta}_{S_\theta}\left(\boldsymbol{R} + \frac{m_N}{m_V}\boldsymbol{r}, t\right) i^{S_N + S_\theta - S_V} \qquad (3.6) \\ & \Sigma (-1)^\delta \delta_{m_V, m_N + m_\theta} \frac{(S_N + S_\theta - S_V)!\,(S_N + S_V - S_\theta)!\,(S_\theta + S_V - S_N)!}{\delta!\,(S_N + S_\theta - S_V - \delta)!\,(S_V - S_\theta - m_N + \delta)!\,(S_N + m_N - \delta)!} \\ & \cdot \frac{1}{(S_V - S_N + m_\theta + \delta)!\,(S_\theta - m_\theta - \delta)!} + \text{h.c.}\end{aligned}$$

The summation over δ merely defines the expected Clebsch-Gordan coefficient and one can thus write (3.6) in terms of the three-j symbols in the manifestly Galilean invariant form

$$\begin{aligned}\mathscr{L}' = & -g'_0 \int d^3 r\, f(r)\, \phi^{m_V +}_{S_V}(\boldsymbol{R}, t) \\ & \phi^{m_N}_{S_N}\left(\boldsymbol{R} - \frac{m_\theta}{m_V}\boldsymbol{r}, t\right) \phi^{m_\theta}_{S_\theta}\left(\boldsymbol{R} + \frac{m_N}{m_V}\boldsymbol{r}, t\right) \begin{pmatrix} S_\theta & S_N & m_V \\ m_\theta & m_N & S_V \end{pmatrix}\end{aligned}$$

where we have defined

$$g'_0 = (-1)^{S_N - S_\theta - S_V}$$
$$\cdot \left[\frac{(S_N + S_\theta - S_V)!\,(S_N - S_\theta + S_V)!\,(S_\theta - S_N + S_V)!\,(S_N + S_\theta + S_V + 1)!}{(2S_V)!\,(2S_N)!\,(2S_\theta)!} \right]^{\frac{1}{2}}$$

and replaced $i^{S_i} \phi_{S_i}^{m_i}$ by $\phi_{S_i}^{m_i}$.

The theory can now be solved in the lowest lying sectors as in the usual Lee model. Again the first nontrivial sector is that of a single V particle and one finds in terms of the Galilean invariant $\Omega \equiv E - p^2/2m_V$ that the unrenormalized propagator

$$\Delta_{m'_V}^{m_V}(\mathbf{x}, t; \mathbf{x}', t') = -i\hbar^{-1} \theta_+(t - t') \langle 0 | \phi_{S_V}^{m_V}(\mathbf{x}, t)\, \phi_{S_V}^{m'_V +}(\mathbf{x}', t') | 0 \rangle$$

has the momentum space representation

$$[\Delta_V^{-1}]_{m'_V}^{m_V} = \delta_{m'_V}^{m_V} \left[\Omega - U_0 + \frac{g_0'^2}{2S_V + 1} \int \frac{d^3 q}{(2\pi)^3} \frac{|f(\omega)|^2}{\omega - \Omega} \right] \tag{3.7}$$

where $\omega = q^2/2\mu$ with μ being the reduced mass

$$\mu = \frac{m_N m_\theta}{m_V}.$$

In writing (3.7) we have included a bare internal energy term by making the replacement

$$i\hbar \frac{\partial}{\partial t} \rightarrow i\hbar \frac{\partial}{\partial t} - U_0$$

in the Lagrangian. The renormalized internal energy is given by the expression

$$U = U_0 - \frac{g_0'^2}{2S_V + 1} \int \frac{d^3 q}{(2\pi)^3} \frac{|f(\omega)|^2}{\omega - U}$$

which is seen to be divergent in the local limit $f(\omega) = 1$ while the wave function renormalization constant is found to be

$$Z^{-1} = 1 + \frac{g_0'^2}{2S_V + 1} \int \frac{d^3 q}{(2\pi)^3} \frac{|f(\omega)|^2}{(\omega - U)^2}$$

which has the remarkable property of being finite in that limit. The extraction of the remaining physical content of the model can be continued in a straightforward fashion. However, as the results are qualitatively identical to those obtained by Levy-Leblond in the spin zero case we terminate this discussion of the even parity case in favor of the more interesting possibility referred to earlier in this section.

IV. Interactions in the Odd Parity Case

As a result of the discussion leading up to the construction of the interaction for the even parity case one can now directly undertake the problem of writing down the interaction for the odd parity situation. In this case one can obviously generate an invariant interaction by replacing one of the Γ's which multiply the V field by ϱ_2 or the replacement of a $\Gamma\sigma_2$ contracted between N and θ by $\varrho_1\sigma_2$. This procedure in general allows for more than one interaction with special cases as follows:

i) If all of the fields have zero spin clearly no parity conserving interaction is possible.

ii) If only one of the fields has zero spin then a ϱ_2 (or $\varrho_1\sigma_2$) can be contracted between the two remaining fields and the coupling is unique.

iii) If all spins are nonzero but one of the spin values is the arithmetic sum of the other two then there exist no matrix contractions between these two fields and consequently no ϱ_2 or $\varrho_1\sigma_2$ can be inserted in that pair. This leads to the existence of two invariant couplings and is clearly the situation which applies whenever the smallest spin of the three fields is one-half (e.g., $S_N=S_\theta=1/2$, $S_V=1$).

iv) In all other cases there exist three independent couplings, it being trivial to see from the above that all particles must have at least one unit of spin in this case and that the simplest example here is $S_N=S_\theta=S_V=1$.

Rather than deal separately with each of the three cases ii), iii), and iv) separately we proceed by assuming that all spins are nonzero and that they satisfy condition iv). One of the three possible interactions will be considered in detail and it will subsequently be indicated how the other two can be constructed. Since cases ii) and iii) can be obtained by taking limits of the three interactions implied in iv), the former two need not be separately considered.

We choose as the specific example the case in which ϱ_2 is contracted between V^+ and N, i.e.

$$\begin{aligned}\mathscr{L}' = &-g_0 \int d^3r\, f(r)\, \psi^+_{a_1\ldots a_{2S_V}}(\boldsymbol{R},t)\\ &\cdot \psi_{a'_1\ldots a'_{2S_N}}\left(\boldsymbol{R}-\frac{m_\theta}{m_V}\boldsymbol{r},t\right)\psi_{a''_1\ldots a''_{2S_\theta}}\left(\boldsymbol{R}+\frac{m_N}{m_V}\boldsymbol{r},t\right)\\ &\cdot [\varrho_2]_{a_1a'_1}\prod_{i=2}^{S_V-S_\theta+S_N}\Gamma_{a_ia'_i}\prod_{j=1}^{S_V+S_\theta-S_N}\Gamma_{a_{S_V-S_\theta+S_N+j}a''_j}\\ &\cdot\prod_{k=1}^{S_\theta+S_N-S_V}[\Gamma\sigma_2]_{a'_{S_V-S_\theta+S_N+k}a''_{S_V+S_\theta-S_N+k}}\\ &+\text{h.c.}\end{aligned}$$

This clearly brings in the "lower" components of both V and N and the labor required to carry out the reduction to a spherical basis is considerably more lengthy than in the even parity case. As a result of this tedious calculation one finds that

$$\mathcal{L}' = -g_0' \int d^3r\, f(r)$$

$$\cdot \left\{ \phi_{S_V}^{m_V\,+}(\boldsymbol{R}, t)\, \chi_{S_N-\frac{1}{2},\frac{1}{2}}^{m,m'}\left(\boldsymbol{R} - \frac{m_\theta}{m_V}\boldsymbol{r}, t\right) \phi_{S_\theta}^{m_\theta}\left(\boldsymbol{R} + \frac{m_N}{m_V}\boldsymbol{r}, t\right)\right.$$

$$\cdot \begin{pmatrix} S_\theta & S_V-\frac{1}{2} & S_N-\frac{1}{2} \\ m_\theta & \mu & m \end{pmatrix} \begin{pmatrix} \mu & \frac{1}{2} & m_V \\ S_V-\frac{1}{2} & m' & S_V \end{pmatrix} (2S_V+1)^{\frac{1}{2}} + \chi_{S_V-\frac{1}{2},\frac{1}{2}}^{m,m'\,+}(\boldsymbol{R}, t)$$

$$\cdot\, \phi_{S_N}^{m_N}\left(\boldsymbol{R} - \frac{m_\theta}{m_V}\boldsymbol{r}, t\right) \phi_{S_\theta}^{m_\theta}\left(\boldsymbol{R} + \frac{m_N}{m_V}\boldsymbol{r}, t\right) \begin{pmatrix} S_\theta & m & S_N-\frac{1}{2} \\ m_\theta & S_V-\frac{1}{2} & \mu \end{pmatrix}$$

$$\left. \cdot \begin{pmatrix} m' & S_N & \mu \\ \frac{1}{2} & m_N & S_N-\frac{1}{2} \end{pmatrix} (2S_N+1)^{\frac{1}{2}} \right\} + \text{h.c.} \tag{4.1}$$

where again we have pulled a factor of i^{S_i} into each field and define

$$g_0' = (-1)^{2S_N+1}$$

$$\cdot \left[\frac{(S_N+S_\theta-S_V)!\,(S_N-S_\theta+S_V-1)!\,(S_\theta-S_N+S_V)!\,(S_N+S_\theta+S_V)!}{(2S_V-1)!\,(2S_N-1)!\,(2S_\theta)!} \right]^{\frac{1}{2}}.$$

Using the transformation laws for ϕ_S^m and $\chi_{S-\frac{1}{2},\frac{1}{2}}^{m,m'}$ as given in Appendix A one can demonstrate the invariance of (4.1) under pure Galilean transformations while the rotational invariance is manifest as a consequence of the tensorial notation employed here.

It is of interest to note that as a result of the appearance of the χ components in (4.1), the dependent components of the V and N fields will now contain $N\theta$ and $V\theta^+$ terms respectively in addition to the usual derivative of ϕ. This means that the interaction (4.1) will, upon elimination of the dependent components, yield direct interactions of the form $(\phi_N^+\phi_N)(\phi_\theta^+\phi_\theta)$ and $(\phi_V^+\phi_V)(\phi_\theta^+\phi_\theta)$. Although these are not without interest, inasmuch as their structure can be altered by allowing explicit quadrilinear interactions (which, of course, are not forbidden), we shall not explicitly consider these direct or S wave interactions. Since they do not affect the V particle propagator, the disregard of these interactions need not be considered an approximation in the following discussion.

The V particle propagator can be found by straightforward calculation to have the form

$$[\Delta_V^{-1}]_{m_V'}^{m_V} = \delta_{m_V'}^{m_V}(\Omega - U_0)$$
$$+ \frac{g_0'^2}{2m_N^2}(2S_N+1)(2S_V+1)\begin{pmatrix} m & \frac{1}{2} & S_V-\frac{1}{2} \\ S_V & \mu_1 & m_1 \end{pmatrix}\begin{pmatrix} \mu_1 & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & \mu_2 \end{pmatrix}$$
$$\begin{pmatrix} S_\theta & m_1 & S_N-\frac{1}{2} \\ m_\theta & S_V-\frac{1}{2} & m_3 \end{pmatrix}\begin{pmatrix} \mu_2 & S_N & m_3 \\ \frac{1}{2} & m_4 & S_N-\frac{1}{2} \end{pmatrix}\begin{pmatrix} S_V & \mu_3 & m_6 \\ m' & \frac{1}{2} & S_V-\frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \nu & \mu_4 \\ \mu_3 & 1 & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} S_N-\frac{1}{2} & \frac{1}{2} & m_4 \\ m_5 & \mu_4 & S_N \end{pmatrix}\begin{pmatrix} m_\theta & S_V-\frac{1}{2} & m_5 \\ S_\theta & m_6 & S_N-\frac{1}{2} \end{pmatrix}\int \frac{d^3q}{(2\pi)^3}\,\frac{q^2|f(\omega)|^2}{\omega-\Omega}.$$

The lengthy succession of three-j symbols can be reduced and eventually one arrives at the form

$$[\Delta_V^{-1}]_{m_V'}^{m_V} = \delta_{m_V'}^{m_V}\left\{\Omega - U_0 + g_0'^2\,\frac{2S_N+1}{32\,S_N S_V m_N^2}\right.$$
$$\left.\left[1+\frac{2}{3}\,\frac{S_\theta(S_\theta+1)-S_N^2-S_V^2+\frac{1}{2}}{(2S_N+1)(2S_V+1)}\right]\int\frac{d^3q}{(2\pi)^3}\,\frac{q^2|f(\omega)|^2}{\omega-\Omega}\right\} \tag{4.2}$$

which thus yields for the internal energy of the V particle (which we assume to be stable in the sense $U<0$)

$$U = U_0 - g_0'^2\,\frac{2S_N+1}{32\,S_N S_V m_N^2}\,\lambda^2\int\frac{d^3q}{(2\pi)^3}\,\frac{q^2|f(\omega)|^2}{\omega-U}$$

where we have abbreviated the square bracket in (4.2) by the (positive definite) parameter λ^2. Similarly one has

$$Z^{-1} = 1 + g_0'^2\,\frac{2S_N+1}{32\,S_N S_V m_N^2}\,\lambda^2\int\frac{d^3q}{(2\pi)^3}\,\frac{q^2|f(\omega)|^2}{(\omega-U)^2}.$$

An important contrast with the even parity case is seen to consist in the more divergent character of the odd parity coupling, a circumstance which arises here from the occurrence of a q factor at each vertex in the fundamental bubble graph of the V propagator. Thus the internal energy and the wave function renormalization constant are cubically and linearly divergent respectively in the local limit $f(\omega)=1$.

It is interesting to note that the $N\theta$ pair which couple to the V particle are in a P wave as a consequence of the derivative coupling. Consequently the phase shift which is computed from

$$e^{2i\delta} = \Delta_V/\Delta_V^*$$

is the P wave phase shift which one expects to have the low energy expansion

$$q^3 \cot\delta = -1/a + \tfrac{1}{2} r_0 q^2 + O(q^4)\,.$$

As in the case of even parity where Levy-Leblond has observed that for $f(\omega)=1$ the S wave effective range formula is exact, we find here that the $O(q^4)$ terms are absent and that

$$-1/a = -\mu U \sqrt{-2\mu U} - \frac{64\pi m_N^2 S_N S_V U}{Z g_0^2 (2S_N+1)\lambda^2 \mu}$$

$$r_0 = -3\sqrt{-2\mu U} - \frac{64\pi m_N^2 S_N S_V}{Z g_0^2 (2S_N+1)\lambda^2 \mu^2}\,.$$

It is important to note that the S wave phase shift in the present theory is also nonvanishing as a consequence of the direct interactions, even though we shall not calculate it here. We do note, however, that there exists the possibility of having in addition to the P wave V particle an S wave bound $N\theta$ state if the coupling constant is sufficiently large. Thus there can exist two opposite parity stable particles in the V sector.

Although the propagator is seen from (4.2) to have a remarkably trivial dependence upon the spin value, it is interesting to note that if one employs a formalism which is not restricted to a calculation of matrix elements of only the independent components a more complicated structure is found. Thus in Appendix B we consider the odd parity case with $S_V = S_N = 1/2$, $S_\theta = 0$ and use the full four component spinors for the N and V fields. One finds that

$$\Delta_V^{-1} = [\Gamma E + \varrho_1 \boldsymbol{\sigma}\cdot\boldsymbol{p} + m_V(1-\varrho_3)]\left(1 + \frac{g_0^2}{2m_V}\int \frac{d^3q}{(2\pi)^3}\frac{|f(\omega)|^2}{\Omega-\omega}\right) \tag{4.3}$$
$$-\Gamma\left[U_0 + \Omega\frac{g_0^2}{2m_V}\int\frac{d^3q}{(2\pi)^3}\frac{|f(\omega)|^2}{\Omega-\omega} + \frac{g_0^2}{4m_N^2}\int\frac{d^3q}{(2\pi)^3}\frac{q^2|f(\omega)|^2}{\Omega-\omega}\right].$$

Inasmuch as this result is shown in the appendix to yield (4.2) for the specified values of the spins it serves the useful function of displaying the possible complexity of the Green's functions when the dependent components are explicitly included.

Finally we point out that the task of writing down the other two interactions possible in the odd parity case is entirely straightforward. One of these is trivially obtained by interchanging all N and θ labels in the above. The other (in which $\varrho_1\sigma_2$ is contracted between N and θ) requires that one repeat the whole calculation in order to get all the spin factors right. Inasmuch as all these factors can be included in g_0, the physics of the P wave V sector is unchanged and we consequently shall not display the explicit results.

Appendix A

Starting from the multispinor transformation law it can be shown that under the transformation $\boldsymbol{x}'=\boldsymbol{x}+\boldsymbol{v}t$ one has

$$(\phi_S^m)'\,(\boldsymbol{x}',t)=e^{i\hbar^{-1}f(\boldsymbol{x},t)}\,\phi_S^m(\boldsymbol{x},t)$$

$$(\chi^{m,m'}_{S-\frac{1}{2},\frac{1}{2}})'\,(\boldsymbol{x}',t)=e^{i\hbar^{-1}f(\boldsymbol{x},t)}\left[\chi^{m,m'}_{S-\frac{1}{2},\frac{1}{2}}(\boldsymbol{x},t)-(3/2)^{\frac{1}{2}}(2S+1)^{\frac{1}{2}}\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu & m'''\end{pmatrix}\begin{pmatrix} m' & 1 & \mu\\ \frac{1}{2} & \nu & \frac{1}{2}\end{pmatrix}v^\nu\,\phi_S^{m'''}(\boldsymbol{x},t)\right].$$

We consider separately the three terms in $\mathscr{L}$

i) $\phi_S^{m+}\,E\,\phi_S^m\rightarrow\phi_S^{m+}(E+\boldsymbol{v}\cdot\boldsymbol{p}+\frac{1}{2}mv^2)\,\phi_S^m$

ii)
$$2m\,\chi^{m,m'+}_{S-\frac{1}{2},\frac{1}{2}}\,\chi^{m,m'}_{S-\frac{1}{2},\frac{1}{2}}\rightarrow 2m\left[\chi^{m,m'+}_{S-\frac{1}{2},\frac{1}{2}}-(3/2)^{\frac{1}{2}}(2S+1)^{\frac{1}{2}}\begin{pmatrix} m'' & \mu & S-\frac{1}{2}\\ S & \frac{1}{2} & m\end{pmatrix}\cdot\begin{pmatrix}\frac{1}{2} & 1 & \frac{1}{2}\\ m' & \nu & \mu\end{pmatrix}v^\nu\,\phi_S^{m''+}\right]$$
$$\cdot\left[\chi^{m,m'}_{S-\frac{1}{2},\frac{1}{2}}-\left(\frac{3}{2}\right)^{\frac{1}{2}}(2S+1)^{\frac{1}{2}}\begin{pmatrix} m & \frac{1}{2} & S\\ S-\frac{1}{2} & \mu' & m'''\end{pmatrix}\begin{pmatrix} m' & 1 & \mu'\\ \frac{1}{2} & \nu' & \frac{1}{2}\end{pmatrix}v^{\nu'}\,\phi_S^{m'''}\right].\tag{A.1}$$

Upon making use of the six-j symbol by means of the relation

$$\begin{pmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ \nu & \mu & m''\end{pmatrix}\begin{pmatrix}\mu' & 1 & m''\\ \frac{1}{2} & \nu' & \frac{1}{2}\end{pmatrix}=-\sum_j(2j+1)\begin{Bmatrix}1 & 1 & j\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}\begin{pmatrix}1 & 1 & m'\\ \nu & \nu' & j\end{pmatrix}\begin{pmatrix}\mu' & \frac{1}{2} & j\\ \frac{1}{2} & \mu & m'\end{pmatrix}$$

one finds for the $\phi^+\,\phi$ term

$$-3m(2S+1)\,v^\nu v^{\nu'}\,\phi_S^{m+}\sum_j(2j+1)\begin{Bmatrix}1 & 1 & j\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}\begin{pmatrix}1 & 1 & m''\\ \nu & \nu' & j\end{pmatrix}\begin{pmatrix}m & j & S\\ S & m'' & m'''\end{pmatrix}(-1)^{j+2S}\,\phi_S^{m'''}.\tag{A.2}$$

Since the part of $\begin{pmatrix}1 & 1 & m''\\ \nu & \nu' & j\end{pmatrix}$ which is symmetric in ν and ν' is non-vanishing only for $j=0,2$ while for $j=2$ one has $\begin{Bmatrix}1 & 1 & j\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{Bmatrix}=0$, it follows that only the $j=0$ term remains. Using the known form of the relevant six-j symbol (A.2) reduces after some arithmetic to

$$\tfrac{1}{2}mv^2\,\phi_S^{m+}\,\phi_S^m.\tag{A.3}$$

It may be noted that the $\chi^+ \phi$ and $\phi^+ \chi$ terms require no simplification in order to carry through the proof of invariance.

$$\text{iii)}\ 6^{\frac{1}{2}}(2S+1)^{\frac{1}{2}} \phi_S^{m+} p^\nu \begin{pmatrix} m & \frac{1}{2} & S-\frac{1}{2} \\ S & \mu & m' \end{pmatrix} \begin{pmatrix} \mu & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & m'' \end{pmatrix} \chi_{S-\frac{1}{2},\frac{1}{2}}^{m',m''}$$

$$\to 6^{\frac{1}{2}}(2S+1)^{\frac{1}{2}} \phi_S^{m+} (p+mv)^\nu \begin{pmatrix} m & \mu & S-\frac{1}{2} \\ S & \frac{1}{2} & m' \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ \mu & \nu & m'' \end{pmatrix} \left[\chi_{S-\frac{1}{2},\frac{1}{2}}^{m',m''} \right.$$

$$\left. - \left(\frac{3}{2}\right)^{\frac{1}{2}} (2S+1)^{\frac{1}{2}} \begin{pmatrix} m' & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu' & m''' \end{pmatrix} \begin{pmatrix} m'' & 1 & \mu' \\ \frac{1}{2} & \nu' & \frac{1}{2} \end{pmatrix} v^{\nu'} \phi_S^{m'''} \right]. \tag{A.4}$$

The $\phi^+ \chi$ term generated by the transformation is seen to cancel the corresponding term in ii) and one consequently need examine only the $\phi^+ \phi$ term. Extracting this part from (A.4) one finds

$$3(2S+1)\phi_S^{m+}(p+mv)^\nu \begin{pmatrix} m & \frac{1}{2} & S-\frac{1}{2} \\ S & \mu & m' \end{pmatrix} \begin{pmatrix} \mu & 1 & \frac{1}{2} \\ \frac{1}{2} & \nu & m'' \end{pmatrix} \begin{pmatrix} m' & \frac{1}{2} & S \\ S-\frac{1}{2} & \mu' & m''' \end{pmatrix}$$

$$\cdot \begin{pmatrix} m'' & 1 & \mu' \\ \frac{1}{2} & \nu' & \frac{1}{2} \end{pmatrix} v^{\nu'} \phi_S^{m'''} .$$

Upon repeating the type of operations used in ii) and combining this with the $\chi^+ \phi$ term one finally obtains the $\phi^+ \phi$ part as

$$\phi_S^{m+}[-(\boldsymbol{v}\cdot\boldsymbol{p}+mv^2)]\,\phi_S^m$$

which upon reference to (A.1) and (A.3) is seen to complete the demonstration of the invariance of (2.4).

Appendix B

We consider the case $S_V = S_N = 1/2$, $S_\theta = 0$ with the interaction term

$$\mathscr{L}' = -g_0 \int d^3 r\, f(r)\, \psi_V^+(\boldsymbol{R},t)\varrho_2 \psi_N\left(\boldsymbol{R} - \frac{m_\theta}{m_V}\boldsymbol{r}, t\right) \psi_\theta\left(\boldsymbol{R} + \frac{m_N}{m_V}\boldsymbol{r}, t\right) + \text{h.c.}$$

The Green's function is defined to be

$$\Delta_V(\boldsymbol{x},t;\boldsymbol{x}',t') = -\frac{\delta}{\delta\varrho_2 K(\boldsymbol{x}',t')} \langle\psi_V(\boldsymbol{x},t)\rangle|_{K=0} \tag{B.1}$$

where we include a source K in the Lagrangian via the coupling

$$\psi_V^+ \varrho_2 K + K^+ \varrho_2 \psi_V .$$

From (B.1) and the field equations one thus infers

$$\Delta_V(\boldsymbol{x}, t; \boldsymbol{x}', t') = -i/\hbar\,\theta_+(t-t')\,\langle 0|\psi_V(\boldsymbol{x}, t)\psi_V^+(\boldsymbol{x}', t')|0\rangle + (1-\varrho_3)\frac{1}{4m_V}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t')\,.$$

The explicit form of Δ_V^{-1} is readily found to be

$$\Delta_V^{-1} = [\Delta_V^{-1}]_0 - \Gamma U_0 - \frac{g_0^2}{2m_N}\int\frac{d^3q}{(2\pi)^3}\,\frac{|f(\omega)|^2}{E - \dfrac{p^2}{2m_V} - \omega} \cdot\left[\frac{1}{2}(1+\varrho_3)\left(\frac{p^2}{2m_V} + q^2\frac{m_V}{2m_N^2}\right) + \varrho_1\boldsymbol{\sigma}\cdot\boldsymbol{p} + m_V(1-\varrho_3)\right] \tag{B.2}$$

where

$$[\Delta_V^{-1}]_0 = E\Gamma + \varrho_1\boldsymbol{\sigma}\cdot\boldsymbol{p} + m_V(1-\varrho_3)\,.$$

The result (B.2) is readily brought to the form (4.3) thereby leaving only the task of demonstrating consistency with (4.2). This is accomplished by projecting out the independent components via

$$\Delta_V \to \Gamma \Delta_V \Gamma$$

and carrying out an inversion of Δ_V in this nonsingular subspace. The result is

$$(\Gamma\Delta_V\Gamma)^{-1} = \Omega - U_0 + \frac{g_0^2}{4m_N^2}\int\frac{d^3q}{(2\pi)^3}\,\frac{q^2|f(\omega)|^2}{\omega-\Omega}$$

in agreement with (4.2) for $S_V = S_N = \frac{1}{2}$, $S_\theta = 0$.

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