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On Representations of the Canonical Commutation Relations*

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Abstract. In the measure space construction of a representation of the canonical commutation relations, the strong continuity of any one parameter subgroup is proved.

All multipliers for the separable case are expressed in a constructive manner and an irreducibility criterion for a subset of multipliers is obtained.

§ 1. Introduction

For a pair of a linear space V_ϕ and a subspace V_π of its algebraic dual V_ϕ^*, a representation of CCR (canonical commutation relations) is unitary operators $U(f)$ and $V(g)$ for each $f \in V_\phi$ and $g \in V_\pi$ satisfying

$$U(f_1)\, U(f_2) = U(f_1 + f_2)\,, \tag{1.1}$$

$$V(g_1)\, V(g_2) = V(g_1 + g_2)\,, \tag{1.2}$$

$$U(f)\, V(g) = V(g)\, U(f) e^{-ig(f)}\,. \tag{1.3}$$

It is usually required that $U(\lambda f)$ and $V(\lambda g)$ are strongly continuous in the real parameter λ for each fixed $f \in V_\phi$ and $g \in V_\pi$.

Let μ be a V_π-quasi-invariant probability measure on (V_ϕ^*, B_ϕ), where B_ϕ is the σ-algebra generated by cylinder sets. The standard representation of CCR on $H_\mu = L_2(V_\phi^*, B_\phi, \mu)$ is given by $U_\mu(f)$ and $V_\mu(g)$ defined as follows:

$$[U_\mu(f)\, \Psi]\, (\xi) = e^{i\xi(f)}\, \Psi(\xi)\,, \tag{1.4}$$

$$[V_\mu(g)\, \Psi]\, (\xi) = [d\mu(\xi + g)/d\mu(\xi)]^{1/2}\, \Psi(\xi + g)\,. \tag{1.5}$$

Here $\Psi \in H_\mu$ and $\xi \in V_\phi^*$ [1, 7].

The continuity of $U_\mu(\lambda f)$ in λ is easily proved but the continuity of $V_\mu(\lambda g)$ in λ is not known in the literature for non-separable space (cf. [9, 10]). We shall prove continuity of $V_\mu(\lambda g)$ in λ in Section 2.

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When the representation space is separable, we may restrict our attention to those V_ϕ and V_π which consists of finite linear combinations of countable numbers of f_j and g_j, $j=1,2,\ldots$ satisfying $g_j(f_k)=\delta_{jk}$ (Section 3 and [6]). We shall call such V_ϕ and V_π as separable.

Any representation of CCR is a direct sum of cyclic representations. Any cyclic representation of CCR for separable V_ϕ and V_π is on a separable Hilbert space and is a direct sum of representations given by

$$U(f)=U_\mu(f)\otimes 1 \tag{1.6}$$

and $V(g)$ on

$$H=H_\mu\otimes M \tag{1.7}$$

where μ is V_π-quasi-invariant measure on V_ϕ^*, H_π and $U_\mu(f)$ are as before and M is a separable Hilbert space.

Let us consider $U(f)$ given by (1.6) on H of (1.7) and arbitrary $V(g)$. Let

$$M_0=[\{U(f); f\in V_\phi\}\cup B(M)]'', \tag{1.8}$$

$$R_\phi=\{U(f); f\in V_\phi\}'', \tag{1.9}$$

where $B(M)$ denotes the set of all bounded linear operators on M. R_ϕ is known to be maximal abelian in $B(H_\mu)\otimes 1$ and hence

$$M_0=R'_\phi. \tag{1.10}$$

Let $\tau(g)$ be the $*$-automorphism of M_0 defined by

$$\tau(g)A=(V_\mu(g)\otimes 1)\,A(V_\mu(-g)\otimes 1). \tag{1.11}$$

We set

$$W(g)=V(g)\,[V_\mu(-g)\otimes 1]. \tag{1.12}$$

Then we obtain immediately the following properties for W:

(1) $W(\lambda g)$ is a unitary operator depending continuously on λ due to our result in Section 2.

(2) $W(g)\in R'_\phi=M_0$.

(3) $W(g_1)\{\tau(g_1)\,W(g_2)\}=W(g_1+g_2)$.

Conversely, any operator $W(g)$, $g\in V_\pi$, satisfying (1) $\sim$ (3) defines a representation of CCR by $U(f)=U_\mu(f)\otimes 1$ and

$$V(g)=W(g)\,(V_\mu(g)\otimes 1). \tag{1.13}$$

Such $W(g)$ is called a multiplier.

We shall give a constructive formula which exhausts all multipliers in Section 3. Our method is taken from the work of Gårding and Wightman [5]. We shall obtain some irreducibility criterion for some class of multipliers in Section 4.

We shall call a representation of CCR ϕ-cyclic if R_ϕ has a cyclic vector. This is the case if and only if M has a dimension 1. This is due to the fact that R'_ϕ contains mutually non-commuting elements $1 \otimes A$, $A \in B(M)$ if $\dim M > 1$ and hence is not maximal abelian, while R_ϕ must be maximal abelian if it has a cyclic vector. Our result shows the existence of irreducible representations with $\dim M > 1$.

In Section 5, we shall prove that any R^n-quasi-invariant measure on a product space $R^n \times Y$ is equivalent to the product of the Lebesgue measure on R^n and the restriction of the given measure to the cylinder sets with the base on Y, as a simple corollary of our continuity theorem.

§ 2. Proof of the Continuity

Lemma 2.1. *Let (Y, B_2) be a Borel space, (R, B_1) be the real line with the σ-field of Borel sets, (Z, B) be the product Borel space $(R, B_1) \times (Y, B_2)$ and μ be a probability measure on (Z, B), which is quasi-invariant under the R-translation:*

$$(x, \eta) \to (x + x', \eta), \quad x \in R, \quad \eta \in Y.$$

Let $H = L_2(Z, B, \mu)$, $M = L_\infty(Z, B)$, $M_1 = L_\infty(Y, B_2)$ as a subalgebra of M, and $U(s)$ be the multiplication operator of e^{isx} on $\Psi(x, \eta) \in H$. Here L_∞ denotes the set of all bounded Borel functions as multiplication operators. Let Ω be a vector with $\Omega(x, \eta) = 1$ and $F \neq 0$ be a projection operator in M_1.

Then $U(s)$ is strongly continuous in s and the finite measure ν_F defined by

$$(\Omega, U(s) F \Omega) = \int e^{isp} \, d\nu_F(p) \tag{2.1}$$

is equivalent to the Lebesgue measure.

Proof. First we give the proof of the continuity of $U(s)$: Let

$$\Delta_k = \{(x, \eta) \in Z; \; k \leqq |x| < k + 1\} \in B. \tag{2.2}$$

Since the union of Δ_k for $k = 0, 1, \ldots$ is Z, we have $\sum_{k=0}^{\infty} \mu_k = \|\Psi\|^2$ for

$$\mu_k = \int_{\Delta_k} |\Psi(x, \eta)|^2 \, d\mu(x, \eta) \tag{2.3}$$

where $\Psi \in H$. Given ε such that $4\|\Psi\| > \varepsilon > 0$, there exists a k such that $\sum_{j=k}^{\infty} \mu_j < \varepsilon/4$. For this k, take $\delta < k^{-1} 2 \sin^{-1}(\varepsilon/4\|\Psi\|)$. Then

$$\|[U(s) - 1]\Psi\| < \varepsilon \tag{2.4}$$

for $|t| < \delta$, which proves the continuity of $U(s)$ at $s = 0$. Since $U(s)$ is a one parameter group of unitaries, it is strongly continuous at arbitrary s.

Next we prove the quasi-invariance of ν_F, which then implies that it is equivalent to the Lebesgue measure ([3], § 1, No. 9, Proposition 11).

Since F is a projection in M_1, there exists a Borel set S in Y such that F corresponds to the characteristic function of S. Let ν'_F be the measure on (R, B_1) induced by $\nu'_F(\Delta) = \mu(\Delta \times S)$, $\Delta \in B_1$. For any $f(x) \in L_\infty(R, B_1)$, we have $(\Omega, f F \Omega) = \int f(x)\, d\nu'_F(x)$ by the Fubini theorem. Since the Fourier transform determines the probability measure, $\nu_F = \nu'_F$. By the quasi-invariance of μ, ν'_F is also quasi-invariant and so is ν_F. Q.E.D.

Lemma 2.2. *Let $H, M, M_1, U(s)$ and Ω be as in Lemma 2.1. Let $\{E_j\} \equiv \mathscr{E}$ be a finite partition of 1 by orthogonal projections in M_1. ($E_j E_k = 0$ for $j \neq k$ and $\sum E_j = 1$.) Then there exists a one parameter family of operators $V_{\mathscr{E}}(t)$, $t \in R$ satisfying the following properties:*

(1) $[E_j, V_{\mathscr{E}}(t)] = 0$ *for all* $E_j \in \mathscr{E}$.

(2) *Let $P(\mathscr{E})$ be the orthogonal projection on the closure of $N_{\mathscr{E}} \Omega$ where $N_{\mathscr{E}}$ is the von Neumann algebra generated by $\mathscr{E}$ and $\{U(s); s \in R\}$. Then*

$$V_{\mathscr{E}}(t)\, V_{\mathscr{E}}(t)^* = V_{\mathscr{E}}(t)^*\, V_{\mathscr{E}}(t) = P(\mathscr{E}) = V_{\mathscr{E}}(0). \tag{2.5}$$

(3) *Canonical commutation relation with $U(s)$:*

$$V_{\mathscr{E}}(t)\, U(s) = U(s)\, V_{\mathscr{E}}(t)\, e^{ist}. \tag{2.6}$$

(4) *$V_{\mathscr{E}}(t)$ is strongly continuous in t.*

(5) *For any unitary operator $V(t)$ in M'_1 satisfying the CCR $V(t)\, U(s) = U(s)\, V(t)\, e^{ist}$, it satisfies*

$$|(\Omega, A^* A\, V(t)\, \Omega)| \leqq (\Omega, A^* A\, V_{\mathscr{E}}(t) \Omega) \tag{2.7}$$

for any A in $N_{\mathscr{E}}$, where the right hand side is non-negative.

Proof. We have $P(\mathscr{E}) H = \bigoplus_j E_j P(\mathscr{E}) H$ and $\{U(s); s \in R\}'' \equiv N_1$ has a cyclic vector $E_j \Omega$ on $E_j P(\mathscr{E}) H$. Let ν_F of Lemma 2.1 for $F = E_j$ be denoted as ν_j. Then $E_j P(\mathscr{E}) H$ can be identified with $L_2(R, B_1, \nu_j)$, $E_j \Omega$ with the constant function 1 and $U(s)$ with the multiplication operator e^{isx}, $x \in R$. (Note that $E_j \neq 0$ implies $E_j \Omega \neq 0$ because Ω is cyclic for $M = M'$ and hence separating for M.)

Let $V_{\mathscr{E}j}(t)$ be defined on $E_j P(\mathscr{E}) H$ by

$$[V_{\mathscr{E}j}(t)\, \Psi]\,(x) = [d\nu_j(x+t)/d\nu_j(x)]^{1/2}\, \Psi(x+t).$$

Let $V_{\mathscr{E}}(t) = \bigoplus_j V_{\mathscr{E}j}(t)$.

The properties (1) ~ (4) are immediate from the definition. We shall prove (2.7). Since E_j commutes with A, $V(t)$, and $V_{\mathscr{E}}(t)$, and since it is a partition of 1, we may check (2.7) for each $E_j \Omega$ instead of Ω. (2.7) will then result by addition and the triangle inequality for the absolute value.

Let $W(t) = V(t)\, V_{\mathscr{E}}(t)^*$. ($t$ will be fixed in the entire discussion.) Let $\Psi_t \equiv V_{\mathscr{E}}(t) \Omega$. Since $V(t)\, P(\mathscr{E}) = W(t)\, V_{\mathscr{E}}(t)$, we have $\Phi_t \equiv V(t) \Omega = W(t)\, \Psi_t$. Let $P(\mathscr{E})\, W(t)\, P(\mathscr{E}) = W_t$. We have $W_t \in N'_1$, $[P(\mathscr{E}), W_t] = 0$ and $\|W_t\| \leqq 1$.

Since W_t leaves each $E_j P(\mathscr{E}) H$ invariant and $N_{\mathscr{E}} E_j P(\mathscr{E}) = N_1 E_j P(\mathscr{E})$ is maximal abelian there, its restriction to this space can be represented by a Borel function $W_t(x)$ with $|W_t(x)| \leqq 1$. Hence $|[E_j \Phi_t](x)| \leqq |[E_j \Psi_t](x)|$. Since $[E_j \Psi_t](x) = [d\nu_j(x+t)/d\nu_j(x)]^{1/2} \geqq 0$, we obtain

$$\begin{aligned} |(E_j\Omega, A^* A\, V(t) E_j \Omega)| &\leqq \int |A_j(x)|^2\, |[E_j \Phi_t](x)|\, d\nu_j(x) \\ &\leqq \int |A_j(x)|^2\, [E_j \Psi_t](x)\, d\nu_j(x) = (E_j \Omega, A^* A\, V_{\mathscr{E}}(t)\, E_j \Omega) \end{aligned}$$

where $A \in N_{\mathscr{E}}$ is represented by $A_j(x)$ on $E_j P(\mathscr{E}) H$. ($A = \sum E_j A_j$, $A_j \in N_1$.)
Q.E.D.

Lemma 2.3. *Let H, $U(s)$ and Ω be as in Lemma 2.1. Let $V(t)$ be a one parameter family of operators defined by*

$$[V(t)\Psi](x, \eta) = [d\mu(t+x, \eta)/d\mu(x, \eta)]^{1/2}\, \Psi(x+t, \eta)\,. \tag{2.8}$$

It is unitary, satisfies $V(t_1)\, V(t_2) = V(t_1 + t_2)$ and $V(t)\, U(s) = U(s)\, V(t)\, e^{ist}$ and is continuous in t.

Proof. The unitarity and the commutation relations are an immediate consequence of the definition. We now prove the continuity.

We order the family of all finite partitions of 1 by projections in M_1 by $\mathscr{E} \subset \mathscr{F}$ if each $F_k \in \mathscr{F}$ satisfies $E_j F_k = F_k$ for some $E_j \in \mathscr{E}$. For a finite family $\mathscr{E}_l$, $l = 1, \ldots, n$, of finite partitions, let $\mathscr{F}$ be the finite partition consisting of all nonzero $\prod_{l=1}^{n} E_{j_l}^{(l)}$, $E_{j_l}^{(l)} \in \mathscr{E}_l$. Then $\mathscr{F} \supset \mathscr{E}_l$ for all l. We now consider the net $V_{\mathscr{E}}(t)$ for each fixed t and proves $\lim_{\mathscr{E}\uparrow} V_{\mathscr{E}}(t) = V(t)$.

We first note that projections in M_1 generates M_1 and hence $\left(\bigcup_{\mathscr{E}} N_{\mathscr{E}}\right)'' = M$.

Let $V'(t) \in \bigcap_{\mathscr{E}'} \left[\bigcup_{\mathscr{E} > \mathscr{E}'} V_{\mathscr{E}}(t)\right]^{-w}$ where $-w$ denotes the weak closure. By the weak compactness of the unit ball of $B(H)$ for any H, there exists at least one $V'(t)$. It has the following properties.

$$\|V'(t)\| \leqq 1\,, \tag{2.9}$$

$$V'(t) \in M_1'\,, \tag{2.10}$$

$$V'(t)\, [V(t)^*\, U(s)\, V(t)] = U(s)\, V'(t)\,, \tag{2.11}$$

$$0 < (\Omega, A^* A\, V(t) \Omega) \leqq (\Omega, A^* A\, V'(t) \Omega)\,, \tag{2.12}$$

where $A \neq 0$, $A \in M$.

Here (2.11) is shared by all $V_{\mathscr{E}}(t)$ and hence holds for $V'(t)$. Since $[V_{\mathscr{E}}(t), E] = 0$ for any projection E in M_1 if $\mathscr{E} \supset \mathscr{E}' \ni E$, we have (2.10). The first inequality of (2.12) follows from the definition of $V(t)$ and $d\mu(x+t, \eta)/d\mu(x, \eta) > 0$ for μ-almost all (x, y) by the quasi-invariance.

The second inequality holds for $A \in N_{\mathscr{E}'}$ for all $V_{\mathscr{E}}(t)$, $\mathscr{E} \supset \mathscr{E}'$ and hence holds for $V'(t)$, for such A. Since $(\cup N_{\mathscr{E}})'' = M$, it holds for arbitrary $A \in M$.

Let $W = V'(t)\, V(t)^*$ (t fixed). From (2.10) and (2.11), we have $W \in M' = M$. Hence it is represented by a multiplication of a Borel function $W(x, \eta)$.

Let $[V(t)\Omega](x, \eta) = \Psi_1(x, \eta)$ and $[V'(t)\Omega](x, \eta) = \Psi_2(x, \eta)$. From (2.12), we have $0 < \Psi_1(x, \eta) \leqq \Psi_2(x, \eta)$ for μ-almost all (x, η). Since $W\Psi_1 = \Psi_2$, we have $W(x, \eta) = \Psi_2(x, \eta)/\Psi_1(x, \eta) \geqq 1$. From (2.9), we also have $|W(x, \eta)| \leqq 1$. Hence $W(x, \eta) = 1$ for μ-almost all (x, η). Therefore $W = 1$ and $V(t) = V'(t)$, which means $w - \lim_{\mathscr{E}\uparrow} V_{\mathscr{E}}(t) = V(t)$. [Since $V(t)$ is unitary,

$$\|[V(t) - V_{\mathscr{E}}(t)]\, \Psi\|^2 \leqq 2\,\|\Psi\|^2 - 2\,\mathrm{Re}(V(t)\Psi, V_{\mathscr{E}}(t)\Psi)$$

and hence $\lim_{\mathscr{E}\uparrow} V_{\mathscr{E}}(t) = V(t)$ strongly.]

We now have, from (2.7) and the positivity of $(\Omega, A^* A\, V(t)\Omega)$,

$$(\Omega, A^* A\; V(t)\Omega) = \inf_{\mathscr{E}}(\Omega, A^* A\; V_{\mathscr{E}}(t)\Omega) \tag{2.13}$$

for all $A \in M$ and $t \in R$. Since $(\Omega, A^* A\, V_{\mathscr{E}}(t)\,\Omega)$ is continuous in t, $(\Omega, A^* A\, V(t)\Omega)$ is upper semi-continuous, and hence measurable. Any $C \in M$ can be written as

$$C = A_1^* A_1 - A_2^* A_2 + i(A_3^* A_3 - A^* A\;\;)$$

(the decomposition of a Borel function in $L_\infty(Z, B)$) and since Ω is cyclic for M, we have the weak measurability of $V(t)\Omega$.

Since

$$V(t)\; U(s) A\,\Omega = e^{-ist}\; U(s)\; A\;\, V(t)\Omega \tag{2.14}$$

for $A \in M_1$ and $\{U(s)\; A\,\Omega\}$ is total in H, we obtain the weak measurability of $V(t)$.

We now, have, from the group property,

$$(Q_f(o)^*\, \Psi, V(t)\Phi) = (\Psi, Q_f(t)\Phi) \tag{2.15}$$

where f is a continuous L_1 function on R and

$$Q_f(t) = \int V(s)\, f(s-t)\, ds\,. \tag{2.16}$$

Since

$$\|(\Phi, V(s)\,\Psi)\| \leqq \|\Phi\|\;\|\Psi\| \quad \text{and} \quad \|A\| = \sup \|\Phi\|^{-1}\;\|\Psi\|^{-1}|(\Phi, A\,\Psi)|\,,$$

we obtain

$$\|Q_f(t) - Q_f(t')\| \leqq \int |f(s-t) - f(s-t')|\, ds\,. \tag{2.17}$$

Given $\varepsilon>0$, and $t'\in R$, we choose K such that $\int_{|s|>K} |f(s)|\,ds<\varepsilon/4$. We then choose $0<\delta<1$ such that $|f(t-s)-f(t'-s)|<[4(K+2|t'|+2)]^{-1}\varepsilon$ for all $|s|<K+|t'|+1$ and $|t-t'|<\delta$. We then have $\|Q_f(t)-Q_f(t')\|<\varepsilon$ for $|t-t'|<\delta$ from (2.17) and $Q_f(t)$ is norm continuous in t.

From (2.15), we now see that $(X, V(t)\Phi)$ is continuous in t for all $\Phi\in H$ and $X=Q_f(0)^*\Psi$, $\Psi\in H$. Since $\|V(t)\Phi\|\leqq\|\Phi\|$ is uniformly bounded, it remains to show that X is total in H.

Let X_0 be the specific X with $f_0(s)=\exp-s^2$ and $\Psi=\Omega$. Let $F_A(t)\equiv(\Omega, A^*A\,V(t)\Omega)$ and $\Delta_n\equiv\{t; F_A(t)<1/n\}$. Due to the first inequality of (2.12), $\bigcup_n \Delta_n=R$. Since each Δ_n is Borel, at least one of them has a Lebesgue measure non zero. Hence $\int F_A(t) f_0(t)\,dt>0$ for every $A\neq 0$. This implies that $X_0(x,\eta)>0$ for μ-almost all (x,η), and hence X_0 is a cyclic vector of M.

Since $A_1 U(s) X_0=Q_f(0)^* A_1 U(s)\Omega$ with $f(t)=e^{-ist}e^{-t^2}$, it is another X and hence X is total. Q.E.D.

Theorem 2.4. *Let V_Φ^* be the algebraic dual of V_Φ, B_Φ be the σ-field generated by cylinder sets of V_Φ^* and μ be a probability measure on (V_Φ^*, B_Φ). Let $V_\pi\subset V_\Phi^*$ and assume that μ is V_π-quasi-invariant. Then $U_\mu(f)$ and $V_\mu(g)$ defined on $H_\mu=L_2(V_\Phi^*, B_\Phi,\mu)$ by* (1.4) *and* (1.5) *have the property that $U(tf)$ and $V(tg)$ are continuous in t for each $f\in V_\Phi$ and $g\in V_\pi$.*

Proof. The proof of the continuity of $U(tf)$ in t is exactly the same as the proof of the continuity of $U(s)$ in Lemma 2.1. For given $0\neq g\in V_\pi$, there exists an $f_g\in V_\Phi$ such that $g(f_g)=1$. (Since $V_\pi\subset V_\Phi^*$, $g(f)=0$ for all $f\in V_\Phi^*$ means $g=0$.) Let $V_{\Phi g}\equiv\{f-g(f)f_g; f\in V_\Phi\}$, $Y=(V_{\Phi g})^*$, B_2 be the σ-field generated by cylinder sets of $V_{\Phi g}^*$. Then $(V_\Phi^*, B_\Phi)=(R,B_1)\times(Y,B_2)$. By Lemma 2.3, $V(tg)$ is continuous in t. Q.E.D.

§ 3. Multipliers

We first give a motivation for our choice of V_Φ and V_π.

We consider a representation of CCR on a separable Hilbert space H. The set of all unitary operators is then second countable in its strong operator topology. Choosing one $U(f)$ from each neighbourhood in the countable basis for the strong operator topology of unitaries, if that neighbourhood contains at least one $U(f)$, we obtain a countable subset $V_\Phi^0=\{f_j^0; j\in N\}$ of V_Φ such that $\{U(f); f\in V_\Phi{}^0\}$ is dense in $\mathscr{U}\equiv\{U(f); f\in V_\Phi\}$. Similarly we choose $V_\pi^0=\{g_j^0; j\in N\}$ in V_π such that $\{V(g); g\in V_\Phi^0\}$ is dense in $\mathscr{V}\equiv\{V(g); g\in V_\pi\}$.

Assume that V_Φ and V_π are separating each other. If $g(f)=0$ for all $g\in V_\pi^0$, then $U(f)$ commutes with all $V(g)$, $g\in V_\pi^0$ and hence with all

$V(g)$, $g \in V_\pi$, a contradiction with the assumption. Hence V_π^0 separates V_ϕ and V_ϕ^0 separates V_π.

We define f_j^1 and g_j^1, $j \in N$, inductively to satisfy the following properties: (1) $g_j^1(f_k^1) = \delta_{jk}$. (2) f_j^1 is a non-zero element with the smallest k among $f_k^0 - \sum_{l=1}^{j-1} g_l^1(f_k^0) f_l^1$. (3) $g_j^1 = g_j^2(f_j^1)^{-1} g_j^2$ and g_j^2 is an element with the smallest k among $g_k^0 - \sum_{l=1}^{j-1} g_k^0(f_l^1) g_l^1$ satisfying $g_k^2(f_j^1) \neq 0$. We start with $f_1^1 = f_1^0$ and this procedure determines f_j^1 and g_j^1 uniquely for $j = 1, 2, \ldots$.

Let V_ϕ^1 and V_π^1 be finite linear spans of $\{f_j^1; j \in N\}$ and $\{g_j^1; j \in N\}$. From the construction, they are subsets of V_ϕ and V_π such that $\{U(f); f \in V_\phi^1\}$ and $\{V(g); g \in V_\pi^1\}$ are dense in $\mathscr{U}$ and $\mathscr{V}$.

This discussion motivates our choice of V_ϕ and V_π:

$$V_\phi = \left\{ \sum_{j=1}^{n} \lambda_j f_j; \lambda_j \in R,\ n \in N \right\}, \tag{3.1}$$

$$V_\pi = \left\{ \sum_{j=1}^{n} \lambda_j g_j; \lambda_j \in R,\ n \in N \right\}, \tag{3.2}$$

$$g_j(f_k) = \delta_{jk}. \tag{3.3}$$

Lemma 3.1. *Let* $V_{\phi n} = \left\{ \sum_{j=1}^{n} \lambda_j f_j; \lambda_j \in R \right\}$ *and* $V_{\pi n} = \left\{ \sum_{j=1}^{n} \lambda_j g_j; \lambda_j \in R \right\}$ *Let* $U(f)$, $V_0(g)$ *and* $U(f)$, $V(g)$, $f \in V_{\phi n}$, $g \in V_{\pi n}$ *be two representations of CCR with the common* U *on the same space. Then there exists a unitary operator* D *commuting with* $U(f)$, $f \in V_{\phi n}$ *such that* $V(g) = D V_0(g) D^*$.

Proof. It is known [8] that any representation of CCR for $V_{\phi n}$ and $V_{\pi n}$ is a direct sum of irreducible representations, all irreducible representations are mutually unitarily equivalent and $\{U(f); f \in V_{\phi n}\}''$ is cyclic in each irreducible representation. Hence the multiplicity of the unique irreducible representation is the same as the multiplicity of $\{U(f); f \in V_{\phi n}\}''$ (which must be uniform) and hence is common for $U(f)\,V_0(g)$ and $U(f)\,V(g)$. Hence the two representations are unitarily equivalent and there exists a unitary D such that $DU(f)D^* = U(f)$, $f \in V_{\phi n}$ and $D V_0(g) D^* = V(g)$, $g \in V_{\pi n}$. Q.E.D.

Theorem 3.2. *Let* $U(f)$, $f \in V_\phi$ *and* $V_0(g)$, $g \in V_\pi$ *be a representation of CCR where* V_ϕ *and* V_π *are given by* (4.1) *and* (4.2). *Suppose that* $U(f)$ *and* $V(g) = W(g)\,V_0(g)$ *are also a representation of CCR on the same space. Let* $R_{\phi n}$ *denote the von Neumann algebra generated by* $U(f)$, $f \in V_\phi$ *and* $V_0(\lambda_j g_j)$, $j = 1, \ldots, n$, *where* $n = 0, 1, \ldots,$ *and* $R_{\phi 0}$ *is written as* R_ϕ. *Let* $\tau(g) A = V_0(g) A V_0(g)^*$ *for* $A \in R'_\phi$. *Then there exists a unitary* $C_n \in R'_{\phi(n-1)}$,

$n = 1, 2, \ldots,$ *such that*

$$W\left(\sum_{j=1}^{n} \lambda_j g_j\right) = (C_1 \ldots C_n)\,\tau\left(\sum_{j=1}^{n} \lambda_j g_j\right)(C_n^* \ldots C_1^*). \tag{3.4}$$

Conversely, any such $W(g)$ *gives rise to a representation* $U(f)$ *and* $V(g) = W(g)\,V_0(g)$ *of CCR.*

Proof. If C_j, $j \in N$ is given, then W defined by (3.4) is consistent $\left(\text{namely } W\left(\sum_{j=1}^{n+m} \lambda_j g_j\right) = W\left(\sum_{j=1}^{n} \lambda_j g_j\right) \textit{ if } \lambda_{n+1} = \cdots = \lambda_{n+m} = 0\right)$. *Since*

$$V(g) = W(g)\,V_0(g) = C_1 \ldots C_n V_0(g)\, C_n^* \ldots C_1^* \quad \text{for} \quad g \in V_{\pi n}$$

and $U(f) = C_1, \ldots, C_n U(f)\, C_n^*, \ldots, C_1^*$, $U(f)$, $V(g)$ are unitarily equivalent to $U(f)$, $V_0(g)$ if f, g are restricted to $V_{\phi n}$ and $V_{\pi n}$. Since n is arbitrary, $U(f)$, $V(g)$ are a representation of CCR.

Now assume that $V(g)$ is given. By Lemma 3.1, we have unitary D_n for each $n = 1, 2, \ldots$ such that $D_n \in R'_\phi$ and $V(g) = D_n V_0(g)\, D_n^*$ for $g = \sum_{i=1}^{n} \lambda_i g_i$. Let $C_n = D_{n-1}^* D_n$ with $D_0 = 1$. Then we have (3.4). Since $C_n \in R'_\phi$ and C_n commutes with $V_0(g)$, $g = \sum_{i=1}^{n-1} \lambda_i g_i$, as is seen from $C_n V_0(g)\, C_n^* = D_{n-1}^* V(g)\, D_{n-1} = V_0(g)$, C_n belongs to $R'_{\phi(n-1)}$. Q.E.D.

Remark 3.3. Let us call a sequence of unitary operators $C_n \in R'_{\phi(n-1)}$ as M-sequence. Consider the transformation $e_k(A)$ of M-sequence defined by $(e_k(A)\,C)_n = C_n$ if $n \neq k, k+1$, $(e_k(A)C)_k = C_k A^*$, $(e_k(A)\,C)_{k+1} = A\,C_{k+1}$ where A is a unitary operator in $R'_{\phi k}$. We equip C_n with the product topology of strong operator topology of unitaries. Let $E(C)$ be the smallest closed set containing C and stable under every $e_k(A)$. We then call $C^{(1)}$ and $C^{(2)}$ equivalent if $C^{(1)} \in E(C^{(2)})$. It can easily be shown that this is an equivalence relation and W corresponding to $C^{(1)}$ and $C^{(2)}$ coincides if and only if $C^{(1)} \sim C^{(2)}$.

We say that two multipliers are equivalent if the corresponding $U(f)$, $V(g)$ are unitarily equivalent for a common U and V_0. The set of all M-sequences yielding multipliers equivalent to a given M-sequence is the smallest closed set containing that M-sequence which is stable under $e_k(A_k)$ for all unitary $A_k \in R'_{\phi k}$, $k \in N$ and the transformation $C_1 \to A\,C_1$ for all unitary $A \in R'_\phi$.

§ 4. Irreducibility Criterion

We consider the following situation. V_ϕ and V_π are given by (3.1) $\sim$ (3.3). V_ϕ^* is then identified with a countably infinite topological product of R. Let μ_j, $j \in N$ be a probability measure on R equivalent to the Lebesgue measure and μ be the product measure of $\{\mu_j; j \in N\}$. Then μ is V_π-quasi-invariant and $H = L_2(V_\phi^*, B_\phi, \mu)$ can be identified with the incomplete infinite tensor product $\bigotimes_{j \in N} (H_j, \Omega_j)$ where $H_j = L_2(R, \mu_j)$ and $\Omega_j(x) = 1$. We shall denote the multiplication by x on H_j by q_j. We define $[V_{0j}(\lambda)\Psi](x) = (d\mu_j(x+\lambda)/d\mu_j(x))^{1/2}\,\Psi(x+\lambda)$ for $\Psi \in H_j$ and denote $V_{0j}(\lambda) = e^{i\lambda p_j}$. The corresponding operators on H_μ are denoted by ϕ_j and π_j: $\phi_j = q_j \otimes \left(\bigotimes_{k \neq j} 1_k\right)$, $\pi_j = p_j \otimes \left(\bigotimes_{k \neq j} 1_k\right)$. With these notations, we have

$$U(f) = \prod_{j=1}^{n} e^{i\lambda_j \phi_j} \text{ for } f = \sum_{j=1}^{n} \lambda_j f_j \text{ and } V(g) = \prod_{j=1}^{n} e^{i\lambda_j \pi_j} \text{ for } g = \sum_{j=1}^{n} \lambda_j g_j.$$

The total Hilbert space H is taken to be $H_\mu \otimes M$, $\dim M < \infty$.

We also restrict multipliers by assuming

$$C_n \in [1 \otimes B(M) \cup \{U(\lambda f_n);\ \lambda \in R\}]''. \tag{4.1}$$

In this case we may introduce a $B(M)$-valued Borel function $C_n(\lambda)$ of $\lambda \in R$ for each $n \in N$ such that $[C_n \Psi]_\xi = C_n(\xi(f_n))\Psi_\xi$ for $\Psi \in H$, $\Psi_\xi \in M$, $\xi \in V_\phi^*$. We wirte C_n as $C_n(\phi_n)$ on H and $C_n(q_n)$ on $H_j \otimes M$.

$\{U(f)\,V_0(g)\}$ is irreducible on H_μ due to $B(H_\mu) = \left\{\bigcup_j \left[B(H_j) \otimes \left(\bigotimes_{k \neq j} 1_k\right)\right]\right\}''$.
Hence μ is ergodic under V_π translation.

If an operator S is in the commutant of $R = \{U(f)\,V(g); f \in V_\phi, g \in V_\pi\}''$, then we have $S \in R'_\phi$ and

$$S_n \equiv C_n^* \ldots C_1^* S C_1 \ldots C_n \in R'_{\phi n} \tag{4.2}$$

where $R_{\phi n} = \left\{R_\phi \cup \left[B\left(\bigotimes_{j=1}^{n} H_j\right) \otimes 1\right]\right\}''$. This condition is necessary and sufficient due to $V(g) = C_1 \ldots C_n V_0(g)\, C_n^* \ldots C_1^*$.

For $S \in R'_\phi$, we define $\operatorname{tr} S \in R_\phi$ by $(\operatorname{tr} S)_\xi = \operatorname{tr} S_\xi$ where $(S\Psi)_\xi = S_\xi \Psi_\xi$ for $\Psi \in H$, $\Psi_\xi \in M$ and $S_\xi \in B(M)$. Let $S' = S - (\operatorname{tr} 1)^{-1} (\operatorname{tr} S) 1$. Then $\operatorname{tr} S' = 0$. We also have $\operatorname{tr} S_n = \operatorname{tr} S \in R'_{\phi n}$ for all n. Hence $\operatorname{tr} S \in R'_{\phi\infty}$. Since $R_\phi \cap R'_{\phi\infty}$ is trivial (μ is V_π ergodic), $\operatorname{tr} S$ is a multiple of identity. Since S' cannot be a multiple of identity due to $\operatorname{tr} S' = 0$ unless $S' = 0$, the irreducibility is equivalent to the statement that all S satisfying (4.2) and $\operatorname{tr} S = 0$ vanishes.

Lemma 4.1. *Let K_n be the Hilbert space (of dimension n^2) obtained by introducing the inner product $\langle A_1, A_2 \rangle = \operatorname{tr} A_1^* A_2$ in $B(M)$. Let $\alpha(U)$ be the unitary representation on K_n of the group of unitaries on M by $\alpha(U)A = UAU^*$.*

Then $\alpha(U)$ is irreducible on the orthogonal complement of 1, consisting of traceless A.

Proof. It is enough to show that an arbitrary $A \neq c1$ together with 1 are cyclic under $\alpha(U)$. There exists a unit vector $e \in M$ such that $(e, Ae) \neq 0$. We integrate $\alpha(U)A$ over all U leaving e invariant relative to the Haar measure. We then obtain $(e, Ae)E - c(1-E)$ where E is a projection on e and $c(\mathrm{tr}\,1) - c = (e, Ae)$. By subtracting $c \cdot 1$, we have a one-dimensional projection from which all A can be generated. Q.E.D.

We denote $\alpha(U)$ for $U = C_n^*(\lambda)$ by $\alpha_n(\lambda)$ and set

$$\alpha_n = (\Omega_n, \alpha_n(q_n)\Omega_n) = \int \alpha_n(\lambda)\, d\mu_n(\lambda)\,. \tag{4.3}$$

Lemma 4.2. *$U(f)$, $V(g)$ are irreducible if*

$$\lim_{n\to\infty} \alpha_{n+k} \dots \alpha_k A = 0\,, \qquad k = 1, 2, \dots \tag{4.4}$$

for every $A \in K_{\dim M}$, $\mathrm{tr}\,A = 0$.

Proof. Let Ψ and Φ be product unit vectors in H_μ with $\bigotimes_{j=k}^{\infty} \Omega_j$ as a factor for some k. Consider each $(S_l)_\xi \in B(M)$ as a vector in $K_{\dim M}$. From (4.2), we have

$$S_{k-1} = \alpha_k(\phi_k)^* \dots \alpha_{n+k}(\phi_{n+k})^* S_{n+k} \tag{4.5}$$

By taking $(\Psi, X\Phi)$ in the sense that $(\Psi, X\Phi) \in B(M)$, $(\psi, (\Psi, X\Phi)\phi) = (\Psi \otimes \psi, X[\Phi \otimes \phi])$ for $\psi, \phi \in M$, we obtain

$$(\Psi, S_{k-1}\Phi) = \alpha_k^* \dots \alpha_{n+k}^*(\Omega, S_{n+k}\Omega)\,. \tag{4.6}$$

Taking inner product with any A with $\mathrm{tr}\,A = 0$, we have

$$\langle A, (\Psi, S_{k-1}\Phi)\rangle = \langle \alpha_{n+k} \dots \alpha_k A,\ (\Omega, S_{n+k}\Omega)\rangle\,. \tag{4.7}$$

By (4.4), we have

$$\langle A, (\Psi, S_{k-1}\Phi)\rangle = 0 \tag{4.8}$$

for every $A \in B(M)$ with $\mathrm{tr}\,A = 0$. Note that $(\Omega, S_{n+k}\Omega)$ is bounded by the unitarity. Hence, if $\mathrm{tr}\,S = 0$, then $\mathrm{tr}\,S_{k-1} = 0$ and we have $(\Psi, S_{k-1}\Phi) = 0$. Since product vectors of the specified kind is total in H_μ, we have $S_{k-1} = 0$. Since C_j are unitary, $S = 0$.

For arbitrary S, we decompose $S = S' + (\mathrm{tr}\,1)^{-1}(\mathrm{tr}\,S)1$ with $\mathrm{tr}\,S' = 0$. We then have $S' = 0$ from the present argument. We have already seen (immediately before Lemma 4.1) that $\mathrm{tr}\,S$ is a constant and hence S is a multiple of the identity operator. Q.E.D.

Lemma 4.3. *If $\alpha_{n+k} \dots \alpha_k A$ does not tend to 0 as $n \to \infty$ for one $k \in N$ and one $A \in B(M)$ with $\mathrm{tr}\,A = 0$, then $U(f)$, $V(g)$ is not irreducible.*

Proof. Suppose $\alpha_{n+k} \ldots \alpha_k A$ does not tend to 0 as $n \to \infty$ for $A \in B(M)$ with $\operatorname{tr} A = 0$. Since $\alpha_j(q_j)$ is unitary, its average α_n satisfies

$$\|\alpha_j\| \leqq \sup_\lambda \|\alpha_j(\lambda)\| = 1 .$$

Hence there exists a subsequence $n(l)$, $l = 1, 2, \ldots$ such that

$$A_0 = \lim_{l \to \infty} \alpha_{n(l)+k} \ldots \alpha_k A \neq 0 . \tag{4.9}$$

We also have $\operatorname{tr} A_0 = \langle 1, A_0 \rangle = 0$. Let

$$A_m = \alpha_1(\phi_1)^* \ldots \alpha_m(\phi_m)^* A_0 \tag{4.10}$$

Since $\|A_n\| = \|A_0\|$, there exists a subsequence $m(j)$ of $k + n(l)$ such that

$$A_\infty = w - \lim_{j \to \infty} A_{m(j)} . \tag{4.11}$$

From (4.10) and (4.11), it follows that

$$C_n^* \ldots C_1^* A_\infty C_1 \ldots C_n \in R'_{\phi n} . \tag{4.12}$$

Hence $A_\infty \in R'$.

From (4.10), we have

$$\begin{aligned} A_{\infty(k-1)} &\equiv C_{k-1}^* \ldots C_1^* A_\infty C_1 \ldots C_{k-1} \\ &= w \lim_{j \to \infty} \alpha_k(\phi_k)^* \ldots \alpha_{m(j)}(\phi_{m(j)})^* A_0 . \end{aligned} \tag{4.13}$$

Hence for $\Omega = \Omega_j \in H_\mu$,

$$(\Omega, A_{\infty(k-1)} \Omega) = \lim_{j \to \infty} \alpha_k^* \ldots \alpha_{m(j)}^* A_0 . \tag{4.14}$$

Since $m(j)$ is taken to be a subsequence of $k + n(l)$, we have

$$\begin{aligned} \langle A, (\Omega, A_{\infty(k-1)} \Omega) \rangle &= \lim_{j \to \infty} \langle \alpha_{m(j)} \ldots \alpha_k A, A_0 \rangle \\ &= \langle A_0, A_0 \rangle > 0 . \end{aligned} \tag{4.15}$$

Hence $A_{\infty(k-1)} \neq 0$. Therefore $A_\infty = C_1 \ldots C_{k-1} A_{\infty(k-1)} C_{k-1}^* \ldots C_1^* \neq 0$ because C_j are unitary.

Due to $\operatorname{tr} A_0 = 0$, we have $\operatorname{tr} A_m = 0$. Hence $\operatorname{tr} A_\infty = 0$. Therefore A_∞ is not a multiple of identity. [$\operatorname{tr} A \in \mathrm{B}(H_\mu)$ is defined by

$$(\Psi, [\operatorname{tr} A] \Phi) = \sum_{j=1}^{\dim M} (\Psi \otimes a_j, A[\Phi \otimes a_j])$$

for an orthonormal basis a_j in M and $\Psi, \Phi \in H_\mu$. Hence $\operatorname{tr} A = 0$ is preserved by the weak limit.] Q.E.D.

Theorem 4.4. *$U(f)$, $V(g)$ are irreducible if and only if*

$$\lim_{n \to \infty} \alpha_{n+k} \ldots \alpha_k A = 0$$

for every $k \in N$ and $A \in K_{\dim M} = B(M)$, $\operatorname{tr} A = 0$.

Remark 4.5. α_k is an average of unitary $\alpha_k(q_k)$ and hence has a norm smaller than 1 in general. The norm approaches to 1 if either μ_k becomes concentrated to a single point or if $\alpha_k(\lambda)$ becomes independent of λ.

Example 4.6. We take $\dim M = 2$. Let $\sigma_1, \sigma_2, \sigma_3$ be Pauli matrices on M. Let $C_{3n+j}(\lambda) = \exp i\lambda\sigma_j$, $j = 1, 2, 3$. We have

$$e^{-i\lambda\sigma_j}\sigma_k e^{i\lambda\sigma_j} = \begin{cases} (\cos 2\lambda)\sigma_k + (\sin 2\lambda)\sum_{i=1}^{3} \varepsilon^{ijk}\sigma_i & (k \neq j), \\ \sigma_j & (k = j), \end{cases} \tag{4.16}$$

where $\varepsilon^{ijk} = 1$ for even permutation of (1 2 3), -1 for odd permutation and 0 otherwise.

We take $d\mu_j(\lambda) = \pi^{-1/2} e^{-\lambda^2} d\lambda$. Then, relative to the orthonormal system $\{\sigma_j\}$ in $K_2 \ominus 1$, we have

$$(\alpha_{3n+j})_{kl} = \delta_{kl}[e^{-1} + (1 - e^{-1})\delta_{jk}].$$

Hence

$$\alpha_{3n+3}\alpha_{3n+2}\alpha_{3n+1} = e^{-2}.$$

Therefore (4.4) holds and we have the irreducibility.

Example 4.7. Any incomplete infinite tensor product of a countably infinite number of copies of Example 4.6 gives an example where $U(f)$, $V(g)$ are irreducible and $\dim M = \infty$.

§ 5. Miscellaneous Discussions

Our Theorem 3.2 and the following lemma, contained in [7], yield a rather complete structure theory for the representation of CCR when V_ϕ and V_π are separable.

Lemma 5.1. *Any representation of CCR, for V_ϕ and V_π given by (3.1) ~ (3.3), is a direct sum of representations, each of which has a form $U_\mu(f) \otimes 1$ and $V(g)$ on $L_2(V_\phi^*, B_\phi, \mu) \otimes M$ where V_ϕ^* is the algebraic dual of V_ϕ, B_ϕ is the σ-algebra generated by cylinder sets, μ is a V_π-quasi-invariant probability measure on (V_ϕ^*, B), M is a Hilbert space, and $U_\mu(f)$ is a multiplication of $e^{i\xi(f)}$, $\xi \in V_\phi^*$.*

Proof. Any representation is a direct sum of cyclic representation, each of which is separable due to (3.1), (3.2), and the continuity of $U(tf)$ and $V(tg)$ in t. Let

$$R_\phi = \{U(f); f \in V_\phi\}''. \tag{5.1}$$

By the multiplicity theorem (for example, see [4], Proposition 2, p. 252), there exists a partition $\{E_\alpha; \alpha \in N \cup \{\infty\}\}$ of 1 by central projections of R_ϕ such that R_ϕ has a uniform multiplicity α on $E_n H$. $\{E_\alpha\}$ is a unitary invariant of R_ϕ, namely, every unitary W satisfying $WR_\phi W^* = R_\phi$ commutes with all E_α. Hence each $E_\alpha H$ is an invariant subspace of the representation.

$E_\alpha H$ can be identified with $H_\alpha \otimes M_\alpha$ and $U(f)|E_n H$ with $U(f)_n \otimes 1$, where $\dim M_\alpha = \alpha$ and $(R_\phi)_n = \{U(f)_n;\ f \in V_\phi\}''$ has a cyclic vector in H_α. H_α can be identified with $L_2(V_\phi^*, B_\phi, \mu_\alpha)$ for some probability measure μ_α. (See, for example, [1] Appendix.) It remains to show the quasi-invariance of μ_α. Let $\Omega(\xi) = 1 \in L_2(V_\phi^*, B_\phi, \mu_\alpha)$.

From the commutation relation, we have

$$V(g)\,[F(\xi) \otimes 1]\; V(g)^* = F(\xi + g) \otimes 1 \tag{5.2}$$

when $F(\xi) = e^{i\xi(f)}$, $f \in V_\phi$. Here $F(\xi)$ denotes the operator multiplying $F(\xi)$ on $\Psi(\xi)$. The following series of approximations by sequential pointwise limits of uniformly bounded functions and algebraic operations prove the validity of (5.2) for any bounded Borel function on (V_ϕ^*, B).

A periodic function by uniform limit of finite linear combinations of e^{it}, $t = \xi(f)$. A continuous function $f(t)$ with a compact support by $\lim_n \sum_{k \in Z} f(t + nk)$. The characteristic function of a bounded open interval (a, b) by monotonously increasing continuous functions. The characteristic function of any open rectangle in $\xi(f_1) \ldots \xi(f_n)$ by multiplication. The characteristic function of any Borel set in V_ϕ^* by finite addition, multiplication, subtraction from 1 and limit of monotone sequences, starting from cylinder sets whose bases are open rectangles. Any Borel function by limit of monotonously increasing simple functions.

Let $X_\Delta(\xi)$ be the characteristic function of a Borel set Δ. Then $V(g)^* X_\Delta(\xi)\, V(g) = X_\Delta(\xi - g) = X_{\Delta+g}(\xi)$. Hence

$$\mu(\Delta + g) = (\Omega, X_{\Delta+g}\Omega) = (V(g)\Omega, X_\Delta\, V(g)\Omega)\,. \tag{5.3}$$

Since R_ϕ has a uniform multiplicity on $E_\alpha H$, $\mu(\Delta) = 0$ implies $X_\Delta = 0$ as an operator and hence $\mu(\Delta + g) = 0$ from (5.3). Therefore μ is V_π-quasi-invariant. Q.E.D.

As an application of Theorem 2.4, we have the following measure theoretic consequence. Conversely, any other (possibly measure theoretic) proof of the following Lemma gives an alternative proof of Theorem 2.4, as is readily seen.

Lemma 5.2. *Let $X = R^n$ and Y be a Borel space. Let μ be R^n-quasi-invariant probability measure on $Z = R^n \times Y$. Let μ_2 be the measure induced on Y by $\mu_2(\Delta) = \mu(R^n \times \Delta)$. Then μ is equivalent to the product of the Lebesgue measure and μ_2.*

Proof. First consider the case $n = 1$. Let $H_\mu = L_2(Z, \mu)$, $\Omega(x, \eta) = 1$, $[U(s)\Psi](x, \eta) = e^{isx}\Psi(x, \eta)$ and $[V(t)\Psi](x, \eta) = [d\mu(x + t, \eta)/d\mu(t, \eta)]^{1/2}$

$\cdot \Psi(x+t,\eta)$. Let M_1 be the von Neumann algebra of all bounded Borel functions of $\eta \in Y$ independent of $x \in R^n$.

By the proof of Lemmas 2.1 and 2.3, $U(s)$ and $V(t)$ are a representation of CCR, continuous in s and t. Therefore $R = \{U(s)\, V(t);\ s \in R,\ t \in R\}''$ is a type I factor [8].

We can identify H_μ with $H_1 \otimes H_2$ and R with $B(H_1) \otimes 1$. Since $M = \{M_1 \cup M_0\}''$ is maximal abelian in H_μ where $M_0 = \{U(s);\ s \in R\}''$, M_1 is maximal abelian in $B(H_2)$.

Let the standard diagonal expansion ([2], Definition 2.1) of Ω be

$$\Omega = \sum_{j=1}^{\infty} \lambda_j (\Omega_j^1 \otimes \Omega_j^2), \qquad \lambda_j \geqq 0, \tag{5.4}$$

where Ω_j^1 and Ω_j^2 are orthonormal in H_1 and H_2. With the restriction $\lambda_j > 0$, the sum must be countable. Since Ω is separating M, $A\Omega_j^2 = 0$ for all j implies $A = 0$ for $A \in \mathrm{M}_1$.

Let $\bar{\Omega}_j^2 = E_{j-1}\Omega_j^2$ where E_j is a projection on $\left\{\bigcup_{k=1}^{j} M_1 \Omega_j^2\right\}^{\perp}$. Let $\Omega^2 = \sum_{j=1}^{\infty} 2^{-j} \bar{\Omega}_j^2$. Then $A\Omega^2 = 0$ implies $A = 0$ for $A \in M_1$. Since M_1 is maximal abelian, the separating vector Ω^2 of $M_1 = M_1'$ is cyclic for M_1. M_0 has always a cyclic vector, which we may take to be $\Omega^1 = \sum_{j=1}^{\infty} 2^{-j} \bar{\Omega}_j^1$. $\Omega^1 \otimes \Omega^2$ is obviously cyclic for M, and hence is separating for $M = M'$.

For a Borel set $\Delta \subset Z$, let X_Δ be the characteristic function of Δ. Then $X_\Delta \in M$ and $X_\Delta = 0$ if and only if $\mu(\Delta) = 0$. We define

$$\nu(\Delta) = (\Omega^1 \otimes \Omega^2,\ X_\Delta [\Omega^1 \otimes \Omega^2]) .$$

ν is a probability measure on Z and is a product measure of the Lebesgue measure on R and ν_2, the restriction of ν to $R^n \times \Delta_2$.

Since $\Omega^1 \otimes \Omega^2$ is separating for M, $\nu(\Delta) = 0$ is equivalent to $X_\Delta = X_\Delta^* \cdot X_\Delta = 0$ and hence is equivalent to $\mu(\Delta) = 0$. Hence ν is equivalent to μ and ν_2 is equivalent to μ_2. This proves the case $n = 1$.

Since $R^n = R^{n-1} \times R$, the general case $n > 1$ can be proved by trivial inductive argument. Q.E.D.

In connection with the notation $C_n(\phi_n)$, we have the following generalization (see Definition 5.5).

Lemma 5.3. *Let $H = L_2(Y, B, \mu) \otimes M$, $R_\phi = L_\infty(Y, B) \otimes 1$ where (Y, B) is a standard Borel space, $L_\infty(Y, B)$ is the set of bounded Borel functions and* $\dim H = \aleph_0$. *Let $W(\lambda)$ be a family of unitary operators in R_ϕ', weakly Borel in $\lambda \in R^n$. Then there exists a $B(M)$-valued weakly Borel function*

$W(\lambda)_\eta$ of $(\lambda, \eta) \in R^n \times Y$ such that

$$[W(\lambda)\Psi]_\eta = W(\lambda)_\eta \Psi_\eta \tag{5.5}$$

for almost all (λ, η) relative to $d\lambda \times \mu$. Here $\Psi \in H$ is represented by $\Psi_\eta \in M$, $\eta \in Y$.

Proof. Consider $\mathscr{H} = K \otimes H, K = L_2(R^n, d\lambda)$. Let $\Psi \in \mathscr{H}$ be represented by $\Psi(\lambda) \in H$, $(\Psi, \Phi) = \int (\Psi(\lambda), \Phi(\lambda))\, d\lambda$. Let W be defined by $[W\Phi](\lambda) = W(\lambda)\Phi(\lambda)$. Let $\hat{R}_\Phi = L_\infty(R^n \times Y) \otimes 1$ in $(K \otimes H_\mu) \otimes M = \mathscr{H}$. Then $W \in \hat{R}'_\Phi$.

Since $L_\infty(R^n \times Y)$ is maximal abelian on $K \otimes H_\mu = L_2(R^n \times Y)$ every $A \in \hat{R}'_\Phi$ can be represented by weakly Borel $B(M)$-valued function $A(\lambda, \eta)$. If A is unitary, $A(\lambda, \eta)$ is unitary for almost all (λ, η). Redefining A at (λ, η) for which $A(\lambda, \eta)\, A(\lambda, \eta)^* \neq 1$ or $A(\lambda, \eta)^*\, A(\lambda, \eta) \neq 1$, $A(\lambda, \eta)$ can be made unitary for all (λ, η). Let $W(\lambda)_\eta = W(\lambda, \eta)$. From

$$W(\lambda)_\eta \Psi_\eta f(\lambda) = (W\Phi)(\lambda, \eta) = [W(\lambda)\Psi]_\eta f(\lambda)$$

for $\Phi = f \otimes \Psi \in K \otimes H$, we have (5.5) for almost all (λ, η). Q.E.D.

Lemma 5.4. *Let (Y, B) be a standard Borel space, μ be a probability measure on (Y, B), $H = H_\mu \otimes M$, $H_\mu = L_2(Y, B, \mu)$ and $R_\Phi = L_\infty(Y, B) \otimes 1$. Assume that H is separable. Let A_j, $j = 1, \ldots, n$ be self-adjoint operators corresponding to multiplication of real-valued Borel functions $A_j(\eta)$, $\eta \in Y$. Let $W(\lambda)$ be a family of unitary operators in R'_Φ, weakly Borel in $\lambda \in R^n$. Then there exists a family of unitary operators in R'_Φ, weakly Borel in λ such that*

$$[V(\lambda)\Psi]_\eta = W(\lambda + A(\eta))_\eta \Psi_\eta \tag{5.6}$$

for almost all (λ, η) relative to $(d\lambda, \mu)$, where $W(\lambda)_\eta$ is taken from Lemma 5.3. Two such $V(\lambda)$ can differ at most for λ in a Null set.

Proof. Since $(\lambda, \eta) \to (\lambda + A(\eta), \eta)$ is an invertible Borel map of $R^n \times Y$, $W(\lambda + A(\eta))_\eta$ is weakly Borel and hence $V(\lambda)$ defined by (5.6) is weakly Borel where $A(\eta)$ denotes $\{A_j(\eta)\} \in R^n$. Since $W(\lambda + A(\eta))_\eta$ is unitary, $V(\lambda)$ is unitary for all λ. Any two such $V(\lambda)$ can obviously differ only at λ in a Null set for a fixed Ψ. Since H is separable, they differ as an operator only at λ in a Null set. Q.E.D.

Definition 5.5. *The operator $V(\lambda)$ in Lemma 5.4 is denoted by $W(\lambda + A)$. It is defined up to a Null-set of λ.*

Example 5.6. We take $Y = R^n$, μ equivalent to the Lebesgue measure, $W(\lambda) = D\tau(\lambda)D^*$, $(\tau(\lambda)A)_y = A_{y+\lambda} (y \in Y)$ and D is a unitary operator commuting with $L_\infty(Y, B)$. Let $D_y \in B(M)$ be such that $(D\Psi)_y = D_y \Psi_y$,

$y \in Y$, $\Psi_y \in M$. Let ϕ_j be the multiplication of y_j. We then have

$$W(\lambda - \phi) = D(D_\lambda^* \otimes 1) \tag{5.7}$$

for almost all λ. This shows that the dependence of $W(\lambda - \phi)$ on λ need not be continuous even if $W(\lambda)$ is continuous.

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References

1. Araki, H.: J. Math. Phys. **1**, 492—504 (1960).
2. — Woods, E. J.: Publ. Res. Inst. Math. Sci. Kyoto Univ. Ser. A **2**, 157—242 (1966).
3. Bourbaki, N.: Integration (Chapter 7 and 8). Paris: Hermann 1963.
4. Dixmier, J.: Les algèbres d'opérateurs dans l'espace Hilbertien. Paris: Gauthier-Villars 1957.
5. Gårding, L., Wightman, A. S.: Proc. Nat. Acad. Sci. U.S. **40**, 622—626 (1954).
6. Hegerfeldt, G. C.: Basis independence of the basis dependent approach to canonical commutation relations (preprint).
7. Lew, J. S.: Princeton Thesis (1960).
8. Neumann, J. von: Math. Ann. **104**, 570—578 (1931).
9. Mackey, G. W.: Ann. Math. **55**, 101—139 (1952).
10. Umemura, Y.: Publ. Res. Inst. Math. Sci. Kyoto Univ. A, **1**, 1—47 (1965).

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