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Constraints Imposed upon a State of a System that Satisfies the K.M.S. Boundary Condition

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Abstract. Using the uniqueness of the K.M.S. automorphism, we investigate the set of automorphisms that commutes with it. The results are applied to gauge invariant quasi-free states of a fermion system.

§ 1. Introduction

The purpose of this note is to investigate some properties of K.M.S. states especially with respect to other possible symmetries.

Let us first recall briefly some definitions and basic features of a K.M.S. state. We refer to [1, 2] for more details.

Definition 1.1. *For a given state ω, the representation $t \to \alpha_t$ of the additive group of real numbers in the $*$-automorphisms group of $\mathscr{A}$ with the property that $\omega(A\alpha_t B)$ is a continuous function of t, is called an evolution.*

Usually such an evolution is a grand-canonical evolution in the sense that it contains the chemical potential. Typically for a system in a finite box V, with Hamiltonian H_V and particle number operator N_V, one has

$$\alpha_t^V(A) = \exp i(H_V - \mu N_V)\, t A \exp - i(H_V - \mu N_V)\, t$$

for any bounded operator on Fock space relative to the finite box V.

Definition 1.2. *A state ω_β of a c^*-algebra $\mathscr{A}$ is said to be a K.M.S. state with respect to an evolution $t \to \alpha_t$ of $\mathscr{A}$ at the inverse temperature $\beta > 0, \beta < \infty$ if*

$$\int_{-\infty}^{+\infty} f(t - i\beta)\, \omega(A\alpha_t B)\, dt = \int_{-\infty}^{+\infty} f(t)\, \omega(\alpha_t B . A)\, dt$$

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$\forall A, B \in \mathscr{A}$ *and for every function* f *the Fourier transform of which is in* $\mathscr{D}$.

Given a K.M.S. state ω_β then one has the following structure

i) a representation R_β of $\mathscr{A}$ with a cyclic vector Ω_β;

ii) Ω_β is separating for $R_\beta(\mathscr{A})''$;

iii) an antiunitary operator J such that

$$J R_\beta(\mathscr{A})'' J = R_\beta(\mathscr{A})',$$

$$J\Omega_\beta = \Omega_\beta;$$

iv) a unitary strongly continuous group $t \to U_\beta(t)$ such that

$$U_\beta(t)\,\Omega_\beta = \Omega_\beta$$

and which implements $t \to \alpha_t$;

v) the K.M.S. boundary condition extends to the weak-closure $R_\beta(\mathscr{A})''$ of $R_\beta(\mathscr{A})$ with respect to the mapping $A \to U_\beta(t)\,A\,U_\beta(-t)$, $A \in R_\beta(\mathscr{A})''$;

vi) Let $\hat{H}$ be the generator of $U_\beta(t)$ and let $T = \exp\left(-\frac{\beta}{2}\hat{H}\right)$ then

$$R^*\Omega_\beta = J\,T R \Omega_\beta \quad \forall R \in R_\beta(\mathscr{A})'';$$

vii) The center of $R_\beta(\mathscr{A})''$ is pointwise invariant. Let E be a central projection, then we have

$$\omega_\beta(A) = \|(1-E)\,\Omega_\beta\|^2 \cdot \frac{((1-E)\,\Omega_\beta, R_\beta(A)\,\Omega_\beta)}{\|(1-E)\,\Omega_\beta\|^2} + \|E\,\Omega_\beta\|^2 \frac{(E\,\Omega_\beta, R_\beta(A)\,\Omega_\beta)}{\|E\,\Omega_\beta\|^2}$$

i.e. we can decompose ω_β into two states that give rise to two disjoint representations and which again satisfy the K.M.S. boundary condition at the same temperature as ω_β with respect to the same evolution.

§ 2. Consequences of the Uniqueness of the K.M.S. Automorphism

With respect to the K.M.S. automorphism we have the following.

Theorem 2.1. *Let* ω *be a state of a simple* C^**-algebra or a faithful state of an arbitrary* C^**-algebra that satisfies the K.M.S. boundary condition for two evolutions* $t \to \alpha_t^1$ *(respectively* $t \to \alpha_t^2$*) at two temperatures* β_1 *(respectively* β_2*) then*

$$\alpha_t^1 = \alpha^2_{\frac{\beta_2}{\beta_1}t}.$$

The proof of this theorem can be accomplished by using the structure properties as compiled in § 1 ([3] Chapters III. 1, III. 4; [2, 4]). In an appendix, we shall give a proof without appealing directly to the structure as described in § 1.

The following lemma is an easy consequence of the K.M.S. boundary condition:

Lemma 2.2. *If ω is a K.M.S. state of $\mathscr{A}$ with respect to an evolution $t \to \alpha_t$ at a temperature β, if furthermore α is an arbitrary $*$-automorphism of $\mathscr{A}$, then $\omega \circ \alpha$ is a K.M.S. state at the temperature β with respect to the evolution $\alpha^{-1}\alpha_t\alpha$.*

The previously discussed uniqueness has strong consequences on the symmetries linked to a given evolution. Let us give first two definitions:

Definition 2.3. *Let ω be a K.M.S. state of a C^*-algebra $\mathscr{A}$ with respect to an evolution $t \to \alpha_t$; then we define the two subgroups of $*$-automorphisms of $\mathscr{A}$*

i) *the stabilizer of ω, S_ω*

$$S_\omega = \{\alpha \in *\text{-aut}\,\mathscr{A};\ \omega \circ \alpha = \omega\}\,,$$

ii) *the commutant of the evolution, $\mathscr{C}$*

$$\mathscr{C} = \{\alpha \in *\text{-aut}\,\mathscr{A};\ \alpha\alpha_t = \alpha_t\alpha\}\,.$$

Then we have the following result:

Theorem 2.4. *Let ω be a faithful K.M.S. state of a C^*-algebra $\mathscr{A}$ with respect to an evolution $t \to \alpha_t$; then*

$$\mathscr{S}_\omega \subset \mathscr{C}\,.$$

The proof is an application of the two previous theorems. Let $\alpha \in S_\omega$ then according to Lemma 2.2 $\omega \circ \alpha = \omega$ is a K.M.S. state with respect to $\alpha^{-1}\alpha_t\alpha$; but due to the uniqueness of the K.M.S. automorphism $\alpha^{-1}\alpha_t\alpha = \alpha_t$.

Corollary 2.4. *If ω is a faithful uniquely defined K.M.S. state of a C^*-algebra $\mathscr{A}$, then*

$$S_\omega = \mathscr{C}$$

which means that the symmetries of the system are not broken when we have a uniquely defined K.M.S. state.

Corollary 2.5. *If α is a $*$-automorphism which connects two K.M.S. states with respect to the same evolution at the same temperature, then $\alpha \in \mathscr{C}$; consequently it is a permutation of the K.M.S. states.*

Combining these results with those that can be found in [2, 3], one has immediately the following corollary:

Corollary 2.6. *Let U_x be a strongly continuous group of unitaries implementing some group of automorphisms of $\mathscr{A}$ and such that $U_x\Omega = \Omega$ then*

i) $U_x U_t = U_t U_x$,

ii) *For the generator P of U_x one has $JPJ = -P$.*

U_t denotes the group of unitaries which implements the time evolution.

§3. Some Applications to the Clifford Algebra

In this section we want to apply the results of the previous section to the class of quasi-free states of the Clifford algebra. Such states have been studied in [5, 6]. For the sake of completeness let us give some definitions and easy results.

Let H be a real separable Hilbert space equipped with a scalar product $S(\varphi, \psi)$, $\varphi, \psi \in H$ and let J be a complex structure in H, viz an R linear operator such that

$$J^2 = -1 \qquad J^+ = -J. \tag{3.1}$$

There exists a mapping from H onto a complex Hilbert space H^J with scalar product

$$(\Gamma_J \varphi, \Gamma_J \psi)_J = S(\varphi, \psi) + iS(J\varphi, \psi); \tag{3.2}$$

it is clear that

$$\Gamma_J(a + bJ) = (a + ib)\,\Gamma_J \tag{3.3}$$

for any a and b real.

Since the dimensions of the various complex Hilbert spaces, obtained from different J's by the given procedure are all the same (namely $\frac{1}{2}\dim H$), there exist isometries $U_{J'J}$ from H^J onto $H^{J'}$ such that

$$(U_{J'J}\Gamma_J \varphi, U_{J'J}\Gamma_J \psi)_{J'} = (\Gamma_J \varphi, \Gamma_J \psi)_J. \tag{3.4}$$

Moreover V defined through:

$$U_{J'J}\Gamma_J = \Gamma_{J'} V \tag{3.5}$$

is an orthogonal operator given by

$$V = \Gamma_{J'}^{-1} U_{J'J} \Gamma_J \qquad J' = VJV^+. \tag{3.6}$$

Notice that Γ_J is invertible from (3.2).

The Clifford algebra is the C^*-algebra generated by the selfadjoint elements $B(\varphi)$, $\varphi \in H$ that satisfy:

$$[B(\varphi), B(\psi)]_+ = 2S(\varphi, \psi)\,1.$$

With respect to a given complex structure J one defines creation and annihilation operators

$$a^{\mp}(\Gamma_J \varphi) = \tfrac{1}{2}(B(\varphi) \pm i B(J\varphi))$$

for which:

$$[a^{+}(\Gamma_J \varphi), a^{-}(\Gamma_J \psi)]_{+} = (\Gamma_J \varphi, \Gamma_J \psi)_J \, 1 .$$

Every isometry $U_{J'J}$ defines a $*$-automorphism of the Clifford algebra [7, 8],

$$\alpha_{J'J} a^{-}(\Gamma_J \varphi) = a^{-}(U_{J'J} \Gamma_J \varphi) = a^{-}(\Gamma_{J'} V \varphi) .$$

Such transformations are indeed generalized Bogoliubov transformations viz

$$\alpha_{J'J} B(\varphi) = B(V\varphi)$$

or

$$\alpha_{J'J} a^{-}(\Gamma_J \varphi) = a^{-}(V_1 \Gamma_J \varphi) + a^{+}(V_2 \Gamma_J \varphi) .$$

V_1 and V_2 are respectively linear and antilinear operators on H^J with the properties:

$$V_1^* V_2 + V_2^* V_1 = 0 , \qquad V_2 V_1^* + V_1 V_2^* = 0 ,$$

$$V_1^* V_1 + V_2^* V_2 = 1 , \qquad V_1 V_1^* + V_2 V_2^* = 1 .$$

Conversely every generalized Bogoliubov transformation gives rise to an orthogonal operator V on H that satisfies (3.5) and (3.6).

One can define by means of the complex structure J a gauge group (viz a one parameter abelian compact group of $*$-automorphisms) β_α^J:

$$\beta_\alpha^J a^{+}(\Gamma_J \varphi) = e^{i\alpha} a^{+}(\Gamma_J \varphi) .$$

A state ω of Clifford algebra is said to be gauge invariant whenever it is invariant under at least one gauge group β_α^J. This definition is a slight generalization of the conventional one, namely one has:

$$\omega\left(\prod_{i=1}^{n} a^{+}(\Gamma_J \varphi_i) \prod_{j=1}^{m} a^{-}(\Gamma_J \psi_j)\right) = 0$$

whenever $n \neq m$.

A state ω of the Clifford algebra, when restricted to the products $B(\varphi)\, B(\psi)$, gives rise to a unique antisymmetric real linear operator A bounded by one on H such that

$$\omega(B(\varphi)\, B(\psi)) = S(\varphi, \psi) + i S(A\varphi, \psi) .$$

Whenever ω is gauge invariant A has to commute with at least one complex structure J and consequently the kernel of A must be of even or possibly infinite dimension.

A quasi-free state ω_A is a state such that the truncated functions are zero except $\omega(B(\varphi)\,B(\psi))^T = \omega(B(\varphi)\,B(\psi))$ and

$$\omega\left(\prod_{i=1}^{2n} B(\varphi_i)\right) = \sum_{i_1 < i_2 \ldots < i_n} \chi_\sigma \prod_{k=1}^{n} \omega_A(B(\varphi_{i_k})\,B(\varphi_{j_k}))$$

where χ_σ is the parity of the permutation σ

$$\sigma = \begin{pmatrix} 1\;2\;\ldots\;2n-1 & 2n \\ i_1 j_1 \ldots i_n & j_n \end{pmatrix}.$$

Since quasi-free states are uniquely defined once the operator A is given, we see that a quasi-free state is gauge invariant iff $\ker A$ is not of odd dimension; so that translationally invariant quasi-free states are gauge invariant [6].

Let us now come to the first result of this section.

Theorem 3.1. *Let ω_A be a quasi-free state and assume it is gauge invariant, then it is primary.*

Proof. Let us assume first that $\ker(1+A^2) = \{0\}$, then by [9], Theorem 1, there exists a unique one parameter abelian group of $*$-automorphisms $t \to \alpha_t$ such that ω_A is the unique K.M.S. state of $t \to \alpha_t$ at the temperature $\beta = 1$. Hence the result follows immediately since any non-trivial projection E in the center of the von Neumann algebra generated would give two disjoint K.M.S. states (§ 1, vii) contradicting the uniqueness.

For the case when $\ker(1+A^2) \neq \{0\}$, one gets the result by using the first part of this proof combined with [6] Lemma 2.3.12.

Theorem 3.2. *Let ω_A be a gauge invariant quasi-free state. Assume moreover that* $\ker(1+A^2) = \{0\}$*; then the stabilizer of this state is just the commutant of the group of $*$-automorphisms generated by the orthogonal operators*

$$T_t = \exp(Zt) \qquad t \in R\,,$$
$$Z = \operatorname{Arcth}(A)\,.$$

The result is obvious from Corollary 2.4 and [9], (Theorems 1 and 2).

Appendix: Proof of Theorem 2.1

Theorem 2.1. *Let ω a state of a C^*-algebra $\mathscr{A}$ be a K.M.S. state for the evolution $t \to \alpha_t^1$ (resp. $t \to \alpha_t^2$) at the inverse temperature β_1 (resp. β_2) then, for $\mathscr{A}$ simple or ω faithful, we have*

$$\alpha_t^1 = \alpha_{\frac{\beta_2}{\beta_1} t}^2\,.$$

Proof. Up to a trivial change of scale, we can assume that $\beta_1 = \beta_2$; indeed if ω is K.M.S. with respect to $t \to \alpha_t$ at the inverse temperature β then ω is also a K.M.S. state with respect to $\alpha_{\lambda t}(0 < \lambda < \infty)$ at the inverse temperature $\lambda^{-1}\beta$.

Consider now f and g in $\mathscr{S}$ with Fourier transform in $\mathscr{D}$. According to the invariance of the K.M.S. state ω,

$$t, u \to \omega(A\alpha_t^1 \alpha_{-u}^2 B)$$

is a continuous and bounded function, moreover

$$\begin{aligned} \int f(t - i\beta)\, g(u - i\beta)\, \omega(A\alpha_t^1 \alpha_{-u}^2 B)\, dt\, du & \\ &= \int f(t)\, g(u - i\beta)\, \omega(\alpha_t^1 \alpha_{-u}^2 B \cdot A)\, dt\, du \\ &= \int f(t)\, g(u - i\beta)\, \omega(\alpha_{-u}^2 B \cdot \alpha_{-t}^1 A)\, dt\, du \\ &= \int f(t)\, g(u)\, \omega(\alpha_{-t}^1 A \cdot \alpha_{-u}^2 B)\, dt\, du \\ &= \int f(t)\, g(u)\, \omega(A\alpha_t^1 \cdot \alpha_{-u}^2 B)\, dt\, du \end{aligned}$$

through repeated use of the K.M.S. boundary equation and invariance of the state. Using now the density of $\mathscr{D}_R \otimes \mathscr{D}_R$ in $\mathscr{D}_{R^2}$ the previous equation tells us that the Fourier transform of

$$t, u \to \omega(A\alpha_t^1 \alpha_{-u}^2 B)$$

has its support on the vector subspace $\omega_1 + \omega_2 = 0$ so that $\omega(A\alpha_t^1 \alpha_{-u}^2 B)$ depends on $t - u$ (by a theorem of Schwartz (cf. [10], p. 101) and by the boundedness of the function). We have therefore:

$$\omega(A\,B) = \omega(A\alpha_t^1 \alpha_{-t}^2 B),$$

so that *outside the kernel* of π_ω, which is the two-sided ideal of elements of A such that

$$\omega(A^* A) = 0,$$

the automorphism for which a given state is a K.M.S. state at a certain temperature is uniquely defined.

In the cases where $\omega(A^* A) = 0$ implies $A = 0$ (e.g. in the case of a simple algebra) the K.M.S. automorphism is uniquely defined.

In particular this is true for the quasi-local algebra (which is simple [11]), or the Clifford algebra, or in the case where one discusses the modular automorphism of a von Neumann algebra with a cyclic and separating vector (cf. [3, 2, 4]).

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