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States and Automorphism Groups of Operator Algebras

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Abstract. Suppose that a group of automorphisms of a von Neumann algebra M, fixes the center elementwise. We show that if this group commutes with the modular (KMS) automorphism group associated with a normal faithful state on M, then this state is left invariant by the group of automorphisms. As a result we obtain a "noncommutative" ergodic theorem. The discrete spectrum of an abelian unitary group acting as automorphisms of M is completely characterized by elements in M. We discuss the KMS condition on the CAR algebra with respect to quasi-free automorphisms and gauge invariant generalized free states. We also obtain a necessary and sufficient condition for the CAR algebra and a quasi-free automorphism group to be η-abelian.

Introduction

Let A be a C^*-algebra, σ_t a one-parameter automorphism group of A, and φ a σ_t-invariant state on A. The state φ is said to satisfy the Kubo-Martin-Schwinger (KMS) boundary condition for $\beta > 0$ if to each $x, y \in A$, there corresponds a function $F(z)$ holomorphic in the strip: $0 < \operatorname{Im} z < \beta$ and bounded on $0 \leqq \operatorname{Im} z \leqq \beta$ with boundary values

$$F(t) = \varphi(\sigma_t(x)\, y) \quad \text{and} \quad F(t + i\beta) = \varphi(y\, \sigma_t(x)) .$$

This condition was first introduced into the "algebraic approach" in [5]. Since that time a great deal of work has been done on the boundary condition. We refer the reader to references [1, 6, 7, 9, 21].

The KMS condition seems to say a good deal about the structure of the algebra involved. For instance it was shown in [17] using Tomita's theory [20] that to every faithful normal state ψ on a von Neumann algebra M there exists a unique one-parameter automorphism group (the modular automorphism group) σ_t^ψ satisfying the KMS condition for $\beta = 1$. The structure link here is that M is semi-finite if and only if σ_t^ψ is inner [17]. More recently in [18] it was shown that if one has a β-KMS state, ψ, and a γ-KMS state, φ, on a C^*-algebra A, $\beta \neq \gamma$, then the corresponding representations π_φ and π_ψ are disjoint, provided one representation is of type III.

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In this paper we turn to the question of automorphism groups of a von Neumann algebra M which commute with the modular automorphism group associated with a given normal state on M. This situation has been considered, for example in [15] and [8]. Under minimal conditions we find that the given state must be invariant under this new automorphism group. This allows us to obtain a "non-commutative ergodic theorem".

In § 2 we analyze completely the discrete spectrum of an abelian unitary group inducing an automorphism group of M, in the case that a cyclic and separating vector for M is left fixed by the unitary group. We show that the discrete spectrum is intimately connected and in fact solely dependent upon "eigenvectors" in the algebra. Results obtained here extend those in [15] and [8].

As a concrete situation we choose the CAR-algebra, discusses such things as KMS, gauge invariant generalized free states and various notions of asymptotic abelianess in time. As a consequence we find it is possible to settle a problem of R. Kadison. Our results here partially overlap with those in [10].

Finally we interpret a result of Powers [12], vis à vis the KMS-condition.

By the KMS-state, we shall mean a state satisfying the KMS condition for $\beta=1$.

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§ 1. Automorphisms Commuting with Time Evolution

In this section we shall discuss automorphisms commuting with the modular automorphism group, σ_t^φ, arising from a KMS state φ on a C^*-algebra.

We shall show, in § 3, how to construct the CAR algebra $\mathscr{A}(\mathscr{H})$ a quasi-free automorphism satisfying the KMS condition for a gauge invariant generalized free state (see ahead for definitions). If one chooses $\mathscr{H}=\mathscr{L}^2(\mathbf{R}^n)$ then it is clear from this construction that it is possible to find such an automorphism which commutes with the automorphism induced by space translation.

Let φ be a normal faithful state on a von Neumann algebra M. Define $M_\varphi=\{x\in M\,|\,\varphi(yx)=\varphi(xy)$ for all $y\in M\}$. Then it is known that $M_\varphi=\{x\,|\,\sigma_t^\varphi(x)=x\}$ [17].

For a KMS state φ on a C^*-algebra A if we write M_φ we mean $M=\pi_\varphi(A)''$ and $\mathscr{Z}$ is the center of M. π_φ is the cyclic representation induced by φ. We show in § 3 that it is possible to have $M_\varphi=\mathscr{Z}$ and it is this case that will be of interest to us in this section. The existence of a state φ on a von Neumann algebra M such that $M_\varphi = \mathscr{Z}$ settles in the negative a problem made by R. Kadison at the Baton Rouge conference; to the effect that for every normal state ψ on a von Neumann algebra M, M_ψ contains a maximal abelian subalgebra of M.

To decide when $M_\varphi = \mathscr{Z}$ we have

Proposition 1. *Let A be a C^*-algebra and suppose φ on A is a KMS state. Then $M_\varphi = \mathscr{Z}$ iff every positive linear functional $\psi \leqq \varphi$, which is invariant under σ_t^φ is a KMS positive linear functional.*

Proof. Since $\psi \leqq \varphi$. We may consider ψ to be a normal state on $M = \pi_\varphi(A)''$. By the invariance of ψ, there exists [17] $h \in M_\varphi^+$ such that $\psi = h\varphi h$. If $M_\varphi = \mathscr{Z}$ then ψ is clearly a KMS state.

Conversely every element $h \in M_\varphi^+$ $0 \leqq h \leqq I$ gives rise to $\psi = h\varphi h \leqq \varphi$ where ψ is invariant. However if ψ is a KMS state then $\psi = k\varphi k$ with $k \in \mathscr{Z}$ by [18]. Since $\psi \leqq \varphi$ $k = h$ and so $M_\varphi = \mathscr{Z}$.

Lemma 1. *Let M be a von Neumann algebra and φ a normal faithful positive linear functional on M with modular automorphism group σ_t^φ. If σ is another automorphism of M, let $\psi = \varphi \circ \sigma$. Then σ_t^ψ, the modular automorphism group of ψ is given by*

$$\sigma_t^\psi = \sigma^{-1} \sigma_t^\varphi \sigma \qquad t \in \mathbf{R}.$$

Proof. Clearly ψ is normal and faithful. Thus there is a modular automorphism group σ_t^ψ. If $x, y \in M$ suppose $F_{x,y}(z)$ is a bounded function analytic in and continuous on the strip $0 \leqq \operatorname{Im} z < 1$ with boundary values:

$$F_{x,y}(t) = \varphi(\sigma_t^\varphi(x)\, y); \;\; F_{x,y}(t+i) = \varphi(y\, \sigma_t^\varphi(x)).$$

Then

$$\begin{aligned} F_{\sigma(x),\sigma(y)}(t) &= \varphi\,(\sigma_t^\varphi(\sigma(x))\, \sigma(y)) = \varphi \circ \sigma(\sigma^{-1} \sigma_t^\varphi \sigma(x)\, y) \\ &= \psi(\sigma^{-1} \sigma_t^\varphi \sigma(x)\, y). \end{aligned}$$

Similarly $F_{\sigma(x),\sigma(y)}(t+i) = \psi(y\, \sigma^{-1} \sigma_t^\varphi \sigma(x))$. So that $F_{\sigma(x),\sigma(y)}$ is the relevant function assuring the KMS boundary condition for ψ with respect to $\sigma^{-1} \sigma_t^\varphi \sigma$. The lemma then follows by the uniqueness of the modular automorphism.

We now can prove

Theorem 1. *Let M be a von Neumann algebra and φ a normal faithful positive linear functional on M with modular automorphism group σ_t^φ. Let σ be another automorphism of M leaving the center $\mathscr{Z}$ of M elementwise fixed. Then the following two statements are equivalent.*

(i) *φ is σ invariant i.e. $\varphi \circ \sigma = \sigma$.*
(ii) *σ commute with $\sigma_t^\varphi t \in \mathbf{R}$.*

Proof. (i)$\Rightarrow$(ii): By the lemma $\sigma_t^{\varphi \circ \sigma}$, the modular automorphism group associated with $\varphi \circ \sigma$ is given by

$$\sigma_t^{\varphi \circ \sigma} = \sigma^{-1} \sigma_t^\varphi \sigma .$$

Then the assumption (i) together with the unicity of the modular automorphism group yields the desired result.

(ii)$\Rightarrow$(i): Since σ and σ_t^φ are assumed to commute, one obtains via lemma that $\sigma_t^\psi = \sigma_t^\varphi$ where $\psi = \varphi \circ \sigma$. Thus ψ being σ_t^ψ-invariant is σ_t^φ-invariant. According to Theorem 15.2 of [17] there exists h, a positive self-adjoint operator affiliated with M_φ such that

$$\psi(x) = (x h \xi_0 | h \xi_0) .$$

Here we have assumed that we are on the representation space $\mathscr{H}_\varphi$ and thus φ is given by ω_{ξ_0}. We have seen in [18] that σ_t^ψ is given by

$$\sigma_t^\psi(x) = \sigma_t^\varphi(h^{2it} x h^{-2it}), \qquad x \in M .$$

Since $\sigma_t^\psi = \sigma_t^\varphi$ it follows that $x = h^{2it} x h^{-2it}$ and h is thus affiliated with $\mathscr{Z}$.

Suppose $h \neq I$. Then there exists $k \neq 0$, positive, $k \in \mathscr{Z}$ such that hk is bounded and $h^2 k^2 > k^2$ or $h^2 k^2 < k^2$. Then

$$\|h k \xi_0\|^2 = (h^2 k^2 \xi_0 | \xi_0) \neq (k^2 \xi_0 | \xi_0) = \|k \xi_0\|^2$$

but

$$\begin{aligned} \|h k \xi_0\|^2 &= (k h \xi_0 | k h \xi_0) = (k^2 h \xi_0 | h \xi_0) \\ &= \psi(k^2) = \varphi \circ \sigma(k^2) = \varphi(k^2) \\ &= (k^2 \xi_0 | \xi_0) = \|k \xi_0\|^2 . \end{aligned}$$

This contradiction shows $h = I$ and $\psi = \varphi$.

We also have the following addition to

Theorem 2. *Let φ and ψ be faithful normal states of a von Neumann algebra M. If σ_t^φ and σ_t^ψ are the modular automorphism groups of M associated with φ and ψ respectively, then the following are equivalent:*
(i) *ψ is σ_t^φ-invariant;*
(ii) *φ is σ_t^ψ-invariant;*
(iii) *σ_t^φ and σ_s^ψ commute for $s, t \in \mathbf{R}$;*
(iv) *φ and ψ commute. ($\varphi + i\psi$ and $\varphi - i\psi$ have the same absolute value* [17]).

Proof. The equivalence (i)$\Leftrightarrow$(iv)$\Leftrightarrow$(ii) is shown in [17]. The equivalence of (ii) and (iii) follow from Theorem 1 if we take σ_s^ψ for that σ.

To conserve the notation to be introduced we state Theorem 3 for η-abelian systems [4]. Clearly it can be stated for $\mathfrak{M}$-abelian systems as well [3].

A group $\mathcal{G}$ is said to be amenable if there exists an invariant mean over $\mathcal{G}$ viz; a state η on $C(\mathcal{G})$, the continuous functions on $\mathcal{G}$, such that $\eta\{f(\hat{g})\}=\eta\{f(h\hat{g})\}=\eta\{f(\hat{g}h)\}$ for a fixed $h\in\mathcal{G}$. We consider the system $\{A,\sigma\}$ consisting of a C^*-algebra A and an automorphism group $\{\sigma_g\}$. We suppose that $g\to\sigma_g$ is a homomorphism of $\mathcal{G}$ into the automorphism group of A such that $g\to\psi(\sigma_g(x))$ is a continuous function for each linear functional $\psi\in A^*$. $\mathcal{G}$ is assumed to be locally compact and amenable. Then $\{A,\sigma\}$ is said to be η-asymptotically abelian if for each $x,y\in A$ and φ a state on A

$$\eta\{\varphi([\sigma_g(x),y])\}=0\,.$$

As shown in [4] we can now define a map ε_η of A into the center of the double dual. If we deal with a invariant state φ one can clearly define the map ε_η as a map of $\pi_\varphi(A)$ into the center of $M=\pi_\varphi(A)''$. We note that in this case, $\varphi(x)=(\pi_\varphi(x)\,\xi_\varphi|\xi_\varphi)$ and there exists a strongly continuous unitary representation $g\to U^\varphi(g)$ such that

$$\pi_\varphi(\sigma_g(x))=U^\varphi(g)\,\pi_\varphi(x)\,U^\varphi(g)^{-1}\,;$$

$$U^\varphi(g)\,\xi_\varphi=\xi_\varphi\,.$$

If ω_{ξ_φ} is faithful, it is a standard argument to show that ε_η extends to a normal faithful projection of M onto the fixed points subalgebra $M^{\mathcal{G}}$ of M, fixed under the automorphism $x\to U^\varphi(g)\,x\,U^\varphi(g)^{-1}$. The fact that $\{A,\sigma\}$ is η-asymptotically abelian says then that $M^{\mathcal{G}}\subseteqq\mathcal{Z}$, the center of M. We remark that this latter condition is all that we actually need in what follows.

It is now possible to obtain what one might refer to as a non-commutative generalization of the classical ergodic theorem viz; the replacement of time averages by space averages.

First we need a lemma. Let $\{\pi,\mathcal{H}\}$ be a representation of a C^*-algebra A and $M_*(\pi)$ denote the Banach space of all σ-weakly continuous linear functionals on $\pi(A)''=M$. Let $V(\pi)$ denote the subspace ${}^t\pi(M_*(\pi))$ of A^* where ${}^t\pi$ is the transpose of π. $V(\pi)$ is an invariant subspace of A^* and thus there is a central projection $z(\pi)$ in the universal enveloping von Neumann algebra $\tilde{A}$ such that $V(\pi)=A^*z(\pi)$. A subspace $W\subseteqq A^*$ is said to be *invariant* if for $\varphi\in W$ and $x\in A$, $x\varphi$ and $\varphi x\in W$, where $x\varphi(y)=\varphi(yx)$ and $\varphi x(y)=\varphi(xy)$.

Lemma 2. *Let σ be an automorphism of A and π a representation of A. Suppose that for each invariant subspace $W\subseteqq V(\pi)\Rightarrow{}^t\sigma(W)=W$. There* *exists an automorphism σ' of M, leaving the center of M elementwise fixed, such that $\sigma'(\pi(x)) = \pi(\sigma(x))$, $x \in A$.*

Proof. Let $\tilde{A}$ be the universal enveloping von Neumann algebra of A. Then σ extends to an automorphism $\tilde{\sigma}$ of $\tilde{A}$ and π extends to a representation $\tilde{\pi}$ of $\tilde{A}$ such that $\tilde{\pi}(i(x)) = \pi(x)$ and $\tilde{\pi}(\tilde{A}) = \pi(A)''$ where i is the injection of A into $\tilde{A}$. The kernel of $\tilde{\pi}$ is $(I - z(\pi))\,\tilde{A}$, for if $x \in \tilde{A}$ and φ is a normal linear functional on $M = \pi(A)''$ one has

$$\langle \varphi, \tilde{\pi}((I - z(\pi))\,x)\rangle = \langle {}^t\pi(\varphi), (I - z(\pi))\,x\rangle = \langle (I - z\,(\pi))V(\pi), x\rangle = 0$$

since ${}^t\pi$ is an isometry of $M_*(\pi)$ onto $V(\pi)$. Moreover $\tilde{\sigma}$ preserves the kernel of $\tilde{\pi}$. If $x \in \ker\tilde{\pi}$ then with φ as above one sees

$$\begin{aligned}\langle \varphi, \tilde{\pi}(\tilde{\sigma}(x))\rangle &= \langle {}^t\pi(\varphi), \tilde{\sigma}(x)\rangle \\ &= \langle {}^t\sigma\,{}^t\pi(\varphi), x\rangle = \langle \psi, x\rangle\end{aligned}$$

where $\psi = {}^t\sigma\,{}^t\pi(\varphi) \in {}^t\sigma(V(\pi)) = V(\pi)$. Thus $\psi(x) = 0$ and $\tilde{\pi}(\tilde{\sigma}(x)) = 0$. We now conclude that there exists σ' extending σ on M.

By similar arguments, we can conclude $\sigma'(z) = z$ for every central projection in M.

Here $\tilde{T}$ shall denote a map corresponding to ε_η for an automorphic representation of the real line, $\mathbf{R}$.

Theorem 3. *Let A be a C^*-algebra and φ a state on A satisfying the KMS boundary condition with respect to the automorphism group σ_t; $t \in \mathbf{R}$. Suppose σ_g; $g \in \mathscr{G}$ is another automorphism group commuting with σ_t. Let both $g \to \psi(\sigma_g(x))$ and $t \to \psi(\sigma_t(x))$ be continuous for each linear functional ψ. Suppose that*

(i) *$\{A, \sigma_g\}$ is η-asymptotically abelian,*

(ii) *$M_\varphi = \mathscr{Z}$ and,*

(iii) *${}^t\sigma_g(W) = W$ for each invariant $W \subseteqq V_\varphi$ and $g \in \mathscr{G}$ then $\tilde{T}(x) = \varepsilon_\eta(x)$ for each $x \in M = \pi_\varphi(A)''$.*

Proof. Let $\varphi(x) = (\pi_\varphi(x)\,\xi_\varphi|\xi_\varphi)$, then ω_{ξ_φ} is a normal faithful state on M. By condition (iii) and Lemma 2 σ_g extends to an automorphism of M fixing its center. Theorem 1 applies and one has that $\omega_{\xi_\varphi} \circ \sigma_g = \omega_{\xi_\varphi}$ one immediately has that the projection $\tilde{T}$ and ε_η are normal faithful maps of M onto the respective fixed points of $\{\sigma_g : g \in \mathscr{G}\}$ and $\{\sigma_t : t \in \mathbf{R}\}$ in M. By the η-abelianess we have $M^{\mathscr{G}} \subseteqq \mathscr{Z}$. Condition (iii) yields $M^{\mathscr{G}} = \mathscr{Z}$. Thus the respective fixed points coincide. One could now appeal to [16] to finish the proof. However, one knows that $\varepsilon_\eta(x) \in \mathrm{co}[\sigma_g(x)]^-$ and $\tilde{T}(x) \in \mathrm{co}[\sigma_t(x)]^-$ viz the weakly (strongly) closed convex hulls of

$\{\sigma_g(x)|g\in\mathscr{G}\}$ and $\{\sigma_t(x)|t\in\mathbf{R}\}$ respectively. Now let $\xi_1,\xi_2\in\mathscr{H}$ then

$$(\varepsilon_\eta(\sigma_t(x))\,\xi_1|\xi_2)=\int_{\mathscr{G}}(\sigma_g\sigma_t(x)\,\xi_1|\xi_2)\,d\mu_\eta(g)$$

where μ_η is the mean on $\mathscr{G}$ corresponding to η. We have

$$\begin{aligned}(\varepsilon_\eta(\sigma_t(x))\,\xi_1|\xi_2)&=\int_{\mathscr{G}}(\sigma_t\sigma_g(x)\,\xi_1|\xi_2)\,d\mu_\eta(g)\\&=\int_{\mathscr{G}}(\sigma_g(x)\,U_{-t}\xi_1|U_{-t}\xi_2)\,d\mu_\eta(g)\\&=(\sigma_t(\varepsilon_\eta(x))\,\xi_1|\xi_2)\\&=(\varepsilon_\eta(x)\,\xi_1|\xi_2)\end{aligned}$$

since $M^{\mathscr{G}}\subsetneqq\mathscr{Z}=M_\varphi$. Then if $\sum_1^n\lambda_i\sigma_{t_i}(x)\to\tilde{T}(x)$ strongly, $\sum_1^n\lambda_i=1$. We have by the continuity of ε_η that

$$\varepsilon_\eta\left(\sum_1^n\lambda_i\sigma_{t_i}(x)\right)\to\varepsilon_\eta(\tilde{T}(x))=\tilde{T}(x)$$

the latter follows by assumption (ii) and the fact that $M^{\mathscr{G}}=\mathscr{Z}$. However the left hand side is by the previous remarks

$$\sum_1^n\lambda_i\varepsilon_\eta(x)=\varepsilon_\eta(x)$$

also

$$\varepsilon_\eta(x)=\tilde{T}(x)$$

as desired. In the former case of course it is known that $\varepsilon_\eta(x)=\tilde{T}(x)=\omega_{\xi_\varphi}(x)$.

Theorem 4. *Let A be a C^*-algebra and φ and ψ states on A satisfying the KMS boundary condition for commuting one parameter automorphisms groups σ_s and τ_t respectively. Suppose that $M_\psi=\{\lambda I\}$. If π_φ and π_ψ are quasi-equivalent, then $\varphi=\psi$.*

Proof. Let $\pi_\varphi(A)''=M_1$ and $\pi_\psi(A)''=M_2$ with the isomorphism Φ mapping M_1 onto M_2 such that $\pi_\psi=\Phi\cdot\pi_\varphi$. Further let

$$\varphi(x)=(\pi_\varphi(x)\,\xi_\varphi|\xi_\varphi)\quad\text{and}\quad\psi(x)=(\pi_\psi(x)\,\xi_\psi|\xi_\psi)\,.$$

Define a state of M_2 by $\varphi'(y)=\omega_{\xi_\varphi}\circ\Phi^{-1}(y)$, $y\in M_2$. The automorphism σ_s and τ_t extend to the modular automorphism of M_1 and M_2 respectively so that $\Phi\circ\sigma_s\circ\Phi^{-1}$ defines an automorphism $\tilde{\sigma}_s$ on M_2 which is clearly the modular automorphism for φ'. Further $\tilde{\sigma}_s$ commutes with τ_t on M_2.

Then by Theorem 15.2 of [17], $\varphi'(y)=(yh\xi_\psi|h\xi_\psi)$ where h is positive, self-adjoint and affiliated with the fixed point algebra M_ψ. However, this is the scalars so that $\varphi'=\psi$ on M_2 and thus $\varphi=\psi$ on A.

§ 2. Some Spectral Properties

We extend here results of [15] and [8] concerning the discrete spectrum of certain abelian unitary groups whose induced automorphism group commutes with the modular automorphism group. More specifically we have

Theorem. *Let M be a von Neumann algebra with cyclic and separating vector ξ_0. Suppose $g\to U_g$ is a strongly continuous representation of an abelian group $\mathcal{G}$ such that $\sigma_g(x)=U_g x U_g^{-1}$ is an automorphism of M. Further suppose $U_g\xi_0=\xi_0$. If, for a character χ of $\mathcal{G}$, $M_\chi=\{x\in M:\sigma_g(x)=\chi(g)\,x\}$ then $[M_\chi\xi_0]=\{\xi\in\mathcal{H}:U_g\xi=\chi(g)\,\xi\}$.*

Proof. We make M into a generalized Hilbert algebra [17] by taking $x\to x\xi_0$ with multiplication $(x\xi_0)(y\xi_0)=(xy)\,\xi_0$ and involution $(x\xi_0)^\#=x^*\xi_0$. Of course one now has the modular automorphism $\sigma_t(x)=\Delta^{it}x\Delta^{-it}$ associated with the normal faithful state ω_{ξ_0} and by Theorem 1.1 σ_t commutes with σ_g.

We apply the theory presented in [17] to our situation. Define for each $\xi\in\mathcal{H}$ two functionals

$$\varphi'_\xi(x')=(x'\xi|\xi_0)\qquad x'\in M'$$

and

$$\varphi_\xi(x)=(x\xi|\xi_0)\qquad x\in M\,.$$

Let $\mathcal{P}^\#(\mathcal{P}^\flat)$ be the set of $\xi\in\mathcal{D}^\#(\mathcal{D}^\flat)$ such that $\varphi'_\xi\geqq 0(\varphi_\xi\geqq 0)$. $\mathcal{P}^\#$ and $\mathcal{P}^\flat$ are dual cones [17]. We note that $\mathcal{D}^\#=\mathcal{D}(\Delta^{\frac{1}{2}})$ and $\mathcal{D}^\flat=\mathcal{D}(\Delta^{-\frac{1}{2}})$ as Hilbert spaces.

We prove first that $[M^{\mathcal{G}}\xi_0]=\{\xi\in\mathcal{H}:U_g\xi=\xi\}$ where $M^{\mathcal{G}}=\{x\in M:\sigma_g(x)=x\}$. Suppose now that for $\xi_1\in\mathcal{P}^\#$ and $U_g\xi_1=\xi_1$ we shall show that $\xi_1\in[M^{\mathcal{G}}\xi_0]$. Then if ξ is any invariant vector in $\mathcal{H}$ we form $\varphi'_\xi(x')=(x'\xi|\xi_0)$. φ'_ξ has the polar decomposition $\varphi'_\xi=u'\varphi'_{\xi_1}$ where $\xi_1\in\mathcal{P}^\#$ and $u'\in M'$, since if $\varphi'_\xi=u'\psi$ in the usual polar decomposition, then one has $\psi=u'^*\varphi'_\xi=\varphi'_{u'^*\xi}$. Defining $\xi_1=u'^*\xi$, $\xi_1\in\mathcal{P}^\#$ [17]. Now φ'_ξ is invariant under σ'_g so by the unicity of the polar decomposition $u'\in M'^{\mathcal{G}}$, the fixed points in M' for $\{\sigma'_g:g\in\mathcal{G}\}$, and $U_g\xi_1=\xi_1$. Now $J=\Delta^{-\frac{1}{2}}S$ where S is #-operation; so that J commutes with U_g, $g\in\mathcal{G}$.

Then $M'^{\mathcal{G}} = J M^{\mathcal{G}} J$. Moreover $\xi = u'\xi_1$ so that

$$\xi \in [M'^{\mathcal{G}} M^{\mathcal{G}} \xi_0] = [J M^{\mathcal{G}} J M^{\mathcal{G}} \xi_0]$$
$$= [M^{\mathcal{G}} \xi_0].$$

It remains to show that if $\xi_1 \in \mathcal{P}^\sharp$ and $U_g \xi_1 = \xi_1$ then $\xi_1 \in [M^{\mathcal{G}} \xi_0]$. We do this as follows: define the map h_0 by: $h_0 x'\xi_0 = x'\xi_1$. h_0 is densely defined and since ξ_0 is separating it is well defined. Further h_0 is a positive map since:

$$(h_0 x'\xi_0 | x'\xi_0) = (x'\xi_1 | x'\xi_0) = (x'^* x'\xi_1 | \xi_0)$$
$$= \varphi'_{\xi_1}(x'^* x') \geqq 0,$$

h_0 is thus a symmetric densely defined operator so we take its Friedrichs' extension h. It is straightforward to verify that h is affiliated with M. Moreover we see that $U_g h U_g^{-1} = h$ so that h is affiliated with $M^{\mathcal{G}}$ or the desired result $\xi_1 \in [M^{\mathcal{G}} \xi_0]$.

Now let $K = \{\xi \in \mathcal{H} : U_g \xi = \chi(g)\, \xi\}$ and M_χ be as in the statement of the theorem.

Clearly $[M_\chi \xi_0] \subseteqq K$. If $\xi \in K$ we form φ_ξ and write $\varphi_\xi = u \varphi_{\xi_1}$ its polar decomposition. Then

$$\varphi_\xi(\sigma_g(x)) = u \varphi_{\xi_1}(\sigma_g(x));$$

hence

$$(U_g x U_g^{-1} \xi | \xi_0) = (x \sigma_g^{-1}(u)\, U_g^{-1} \xi_1 | \xi_0),$$

so that

$$\overline{\chi(g)}\, \varphi_\xi(x) = \sigma_g^{-1}(u)\, \varphi_{U_g^{-1}\xi_1}(x)$$

Now $U_g^{-1}\xi_1 \in \mathcal{P}^\sharp$ so by the unicity of the polar decomposition we have $\xi_1 = U_g^{-1}\xi_1$ and $\sigma_g^{-1}(u) = \overline{\chi(g)}\, u$. Then $\xi_1 \in [M^{\mathcal{G}} \xi_0]$ and $u \in M_\chi$. This completes the proof.

Jadcyck [8] has obtained the part of this theorem concerning the fixed points of $\{U_g : g \in \mathcal{G}\}$ by a different method. He obtains the full theorem under more restrictive conditions.

Since $\mathcal{G}$ is amenable let μ be an invariant mean on $\mathcal{G}$. The map ε_χ defined by

$$(\varepsilon_\chi(x)\, \xi | \eta) = \int_{\mathcal{G}} \overline{\chi(g)}\, (\sigma_g(x)\, \xi | \eta)\, d\mu(g)$$

exists by the continuity of $g \to U_g$. Moreover ε_χ is strongly continuous. The matter follows because the map ε_0 given by

$$(\varepsilon_0(x)\, \xi | \eta) = \int_{\mathcal{G}} (\sigma_g(x)\, \xi | \eta)\, d\mu(g)$$

is the normal projection onto $M^{\mathscr{G}}$ and is thus strongly continuous. Thus, about the subspace M_χ, we may conclude that $\left\{\sum_{\chi \neq \chi_0} M_\chi\right\}^{-wk} \cap M_{\chi_0} = (0)$. A closer analysis of these subspaces would likely yield information concerning the various types of states defined in [3] and of course determine whether or not the discrete spectrum of $\{U_g\}$ is a subgroup. The latter is of course satisfied if in each M_χ one can find a unitary operator.

§ 3. The CAR-algebra

In order to support our hypothesis in § 1 we discuss, for the CAR-algebra, gauge invariant generalized free states satisfying the KMS condition and asymptotic abelianess with respect to time.

Example. Let $\mathscr{A} = \mathscr{A}(\mathscr{H})$ be the CAR-algebra over a separable, complex Hilbert space, $\mathscr{H}$. That is the C^*-algebra generated by the identity and elements $a(f)$ where $f \to a(f)$ is a linear map of $\mathscr{H}$ satisfying the anti-commutation relation

$$\begin{aligned} a(f)^* a(g) + a(g)\, a(f)^* &= (g|f)\, I\,, \\ a(f)\, a(g) + a(g)\, a(f) &= 0\,. \end{aligned} \tag{1}$$

(We follow the mathematical convention for the linearity of the inner product in the first variable.)

We construct a time automorphism σ_t of $\mathscr{A}$. On the even subalgebra, $\mathscr{A}_e$, of $\mathscr{A}$ this automorphism is asymptotically abelian in norm. Further it is shown that one can find a gauge invariant generalized free state ω_A satisfying the KMS condition on $\mathscr{A}$.

The even subalgebra, $\mathscr{A}_e$, is that subalgebra of $\mathscr{A}$ generalized by monomials in an even number of f's, alternatively the fixed points for the automorphism of $\mathscr{A}$ defined by $\alpha_I(a(f)) = -a(f)$. An element x is odd if $\alpha_I(x) = -x$.

Suppose now that U_t is a strongly continuous unitary representation of the real line on $\mathscr{H}$ with $(U_t f|g) \to 0$ as $t \to \infty$ for each $f, g \in \mathscr{H}$. This of course is satisfied in the particular case that the spectral measure of U_t is absolutely continuous with respect to Lebesgue measure. We define the quasi-free [10, 11] automorphism σ_t of $\mathscr{A}$ by $\sigma_t(a(f)) = a(U_t f)$.

Suppose that we have a fixed automorphism group σ_t of $\mathscr{A}$. Let us say that x and y *commute asymptotically* if

$$\lim_{t\to\infty} \|[\sigma_t(x), y]\| = 0 \quad \text{where} \quad [z, y] = zy - yz\,.$$

For a set $S \subseteq \mathscr{A}$ we write $S^{ac} = \{x \in \mathscr{A} | x$ commutes asymptotically with all $y \in S\}$. One easily sees

Lemma 1. *If $S = S^*$ then S^{ac} is a C^*-subalgebra of $\mathscr{A}$.*

Proposition 1. *For arbitrary* $y \in \mathscr{A}\ \{y\}^{ac} \supseteqq \mathscr{A}_e$.

Proof. Since $x \in \{y\}^{ac}$ iff $y \in \{x\}^{ac}$ it suffices by Lemma 1 to suppose that y is of the form $a(h)$ or $a(h)^*$ and x is of the form $a(f)\,a(g)$ or $a(f)\,a(g)^*$. One has the equality

$$\begin{aligned}[x_1 x_2, y] &= x_1 x_2 y - y x_1 x_2 \\ &= x_1 \{x_2, y\} - \{x_1, y\}\, x_2\end{aligned}$$

where $\{z, w\} = zw + wz$.

This fact together with the canonical anticommutation relations (CAR), (1), and the assumption that $(U_t f \mid g) \to 0$ shows that

$$\|[\sigma_t(x_1 x_2), y]\| \to 0$$

whenever

$$x_1 = a(f),\ x_2 = a(g) \quad \text{or} \quad a(g)^*$$

and

$$y = a(h) \quad \text{or} \quad a(h)^*.$$

For our KMS state we choose a gauge invariant generalized free state ω_A. That is one given by

$$\omega_A(a(f_n)^* \dots a(f_1)^*\, a(g_1) \dots a(g_m)) = \delta_{nm} \det(A g_i \mid f_j)$$

where $0 \leqq A \leqq I$. A state is defined by the above expression and $\omega_A(I) = 1$ [10, 11, 14]. The KMS condition for such states has been discussed in [10]. We do obtain a more specific form for the operator A satisfying the KMS boundary condition for $\beta = 1$.

Let $\mathfrak{J}$ be the analytic vectors viz; the manifold spanned by vectors of the form $\int \varphi(t)\, U_t f\, dt$ where $f \in \mathscr{H}$ and $\varphi(t)$ (the Fourier transform of φ) is infinitely differentiable with compact support. $\mathfrak{J}$ is dense in $\mathscr{H}$ and it suffices to define A there. Since U_t is assumed to be strongly continuous $U_t = e^{itH}$. We claim $A = e^{-H}(I + e^{-H})^{-1}$ works. Clearly $0 \leqq A \leqq I$. Now

$$\begin{aligned}F(t) &= \omega_A(a(U_t f)^*\, a(f)) \\ &= (A f \mid U_t f)\end{aligned}$$

and thus (since $f \in \mathfrak{J}$)

$$F(t + i) = (A f \mid e^{itH} e^{H} f).$$

{See axiom (v) page 3 of [17]}. Now the KMS condition implies that

$$\begin{aligned}F(t + i) &= \omega_A(a(f)\, a(U_t f)^*) \\ &= (f \mid U_t f) - (A f \mid U_t f)\end{aligned}$$

Setting $t=0$ and equating the two expressions, we obtain

$$e^{H} A = (I - A)$$

so that

$$A = e^{-H}(I + e^{-H})^{-1}$$

as stated. At this point one may appeal to [10] to show that this A does indeed give a KMS state. We give an independent proof of this fact below. The authors should like to express their thanks to Oscar Lanford for suggesting the above example.

Suppose now that $K = e^{-H}$. We verify the KMS condition on the dense subalgebra $\mathscr{A}_0(\mathfrak{I})$, where $\mathscr{A}_0(\mathfrak{I})$ is the subalgebra of $\mathscr{A}(\mathscr{H})$ generated algebraically by $a(f)$, and $a(f)^*, f \in \mathfrak{I}$. For each $\alpha \in \mathbb{C}$ define an automorphism σ_α of $\mathscr{A}_0(\mathfrak{I})$ by

$$\sigma_\alpha : a(f) \to a(K^{i\alpha} f)$$

$$\sigma_\alpha : a(f)^* \to a(K^{-i\alpha} f)^*$$

This is not, clearly, a $*$-automorphism unless α is real.

To verify the KMS condition we show that if $x, y \in \mathscr{A}_0(\mathfrak{I})$ then $\omega_A(xy) = \omega_A(y\sigma_i(x))$. Consider the set $B = \{x \in \mathscr{A}_0(\mathfrak{I}) | \omega_A(xy) = \omega_A(y\sigma_i(x))$ for all $y \in \mathscr{A}_0(\mathfrak{I})\}$. This set is clearly an algebra. Therefore we will be done if we verify the KMS condition for a dense set of elements, y, and arbitrary fields $a(f)$ and $a(f)^*$. A dense set is given by linear combination of elements of the form

$$y = a(f_1)^* \dots a(f_n)^* a(g_1) \dots a(g_m).$$

Note that in dealing with an expression such as $a(h_1) \dots a(h_j)$ one may assume the $\{h_k\}$ to be linearly independent and conclude that $a(h_1) \dots a(h_j) = (\det \gamma)\, a(k_1) \dots a(k_j)$ where $\{k_i\}$ is any other linearly independent set of vectors in $[h_1, \dots, h_j]$ and $h_i = \Sigma\, \gamma_{ij} k_j$.

Let $F = [f_1, \dots, f_n]$ and write $g_i = g_i' + g_i''$ where $g_i' \in F$ and $g_i'' \in F^\perp$. Expanding and using the CAR one obtains linear combinations of terms like $a(f_1)^* \dots a(f_n)^* a(g_i') \dots a(g_k') a(g_{k+1}'') \dots a(g_m'')$. Again we may assume that $g_1', \dots, g_k'$ are orthogonal. Then let $g_1', \dots, g_k'$ be part of a basis for $F = [f_1, \dots, f_n]$ so that by re-expressing $a(f_1)^* \dots a(f_n)^*$ and re-labeling our basic element becomes

$$a(f_1)^* \dots a(f_n)^* a(g_1)^* a(g_1) \dots a(g_j)^* a(g_j)\, a(h_1) \dots a(h_l)$$

where $[f_1, \dots, f_n]$, $[g_1, \dots, g_j]$, $[h_1, \dots, h_l]$ are orthogonal subspaces. We now proceed with our proof. There are several cases to be considered and we shall only work out one. The others can be carried through similarly. Consider then $a(k), k \in \mathfrak{I}$. $a(k)=a(k_1)+a(k_2)+a(k_3)+a(k_4)$ where $k_1 \in [f_1, \ldots, f_k]$ $k_2 \in [g_1, \ldots, g_j]$, etc. It suffices to deal with each one separately. Suppose then for definiteness that $k \in [f_1, \ldots, f_n]$ and we write our basic element

$$a(f_n)^* \ldots a(f_1)^* a(g_1)^* a(g_1) \ldots a(g_j)^* a(g_j)\, a(h_1) \ldots a(h_l)$$

Since $k \in [f_1, \ldots, f_n]$ we may assume $k=f_n$. Consider the expression

$$\omega_A(a(f_n)\, a(f_n)^* \ldots a(f_1)^* a(g_1)^* \ldots a(g_1)\, a(h_1) \ldots a(h_l))$$

and note that by form of ω_A this expression is zero unless $n=l+1$. Thus we have

$$\omega_A(a(f_n)\, a(f_n)^* \ldots a(h_{n-1})) = \omega_A((1-a(f_n)^* a(f_n))\, a(f_{n-1})^* \ldots a(h_{n-1}))$$

which, after repeated application of the CAR yields

$$\omega_A(a(f_{n-1})^* \ldots a(h_{n-1})) - \omega_A(a(f_n)^* \ldots a(f_1)^* a(g_1)^* a(g_1) \ldots a(g_j) \ldots a(h_{n-1})\, a(f_n))$$

This should equal $\omega_A(a(f_n)^* \ldots a(h_{n-1})\, a(K^{-1} f_n))$. The latter however equals the determinant of the following matrix

$$\begin{pmatrix} (Ag_1|g_1) \ldots (Ag_1|g_j)\,(Ag_1|f_1) \ldots (Ag_1|f_n) \\ \vdots \\ (Ag_j|g_1) \\ (Ah_1|g_1) \\ \vdots \\ (AK^{-1}f_n|g_1) \ldots (AK^{-1}f_n|g_j) \ldots (AK^{-1}f_n|f_n) \end{pmatrix}$$

$$A=K(I+K)^{-1} \quad \text{so} \quad AK^{-1}=I-A\,,$$

so using the orthonormality of the vectors in question the bottom row looks like

$$-(Af_n|g_1) \ldots -(Af_n|g_j) \ldots 1-(Af_n|f_n)\,.$$

Add to this the expression for $\omega_A(a(f_n)^* \ldots a(h_{n-1})\, a(f_n))$. The only difference in the two, occurs in the last row, where one has $(Af_n|g_1) \ldots (Af_n|g_j) \ldots (Af_n|f_n)$. Thus the new matrix is the one displayed above with all zeros in the last two except for one in the last column. Expansion about the last row clearly gives us $\omega_A(a(f_{n-1})^* \ldots a(h_{n-1}))$ and so we are done.

Lemma 2. *Suppose*

$$\frac{1}{2T}\int_{-T}^{T}|(U_t f|g)|^2\,dt\to 0 \quad \textit{as} \quad T\to\infty\,.$$

If ω *is an invariant state on* $\mathscr{A}$ *for the induced automorphism* σ_t *then* $\omega(x)=0$ *for all odd elements* x *in* $\mathscr{A}$.

Proof. Let x be an odd element with $\omega(x)\neq 0$. We can suppose that x is a monomial in the fields and that $\omega(x)=1$. Write $x=a(f_1)^*\dots a(f_{2j+1})$. Set $K=\max_{1\leqq i\leqq 2j+1}\|f_i\|$ and let $\|x\|=\delta>0$. If $\{e_i\}$ is a basis for $\mathscr{H}$, let $\mathscr{M}_n=[e_1,\dots,e_n]$. From x we construct a sequence of elements $\{x_n\}$ such that

(i) x_n is odd for all n, $x_n\in\mathscr{A}(\mathscr{M}_n^\perp)$,
(ii) $\frac{3}{2}\delta\geqq\|x_n\|$,
(iii) $|\omega(x_n)|\geqq\frac{1}{2}$.

This will be in contradiction with a result in [11] so $\omega(x)=0$. Since this argument does not appear in print we reproduce it in the appendix. With $\mathscr{M}_n$ as above, consider that

$$\frac{1}{2T}\int_{-T}^{T}\sum_{i=1}^{2j+1}\sum_{k=1}^{n}|(U_t f_i|e_k)|^2\,dt\to 0$$

so there exists a sequence $t_m\to\infty$ such that

$$\lim_{t_m\to\infty}|(U_{t_m}f_i|e_k)|\to 0 \quad \text{for} \quad i=1,\dots,2j+1;\ k=1,\dots,n\,.$$

Now pick m so large that

$$|(U_{t_m}f_i|e_k)|<\frac{\delta'}{2K(2^l-1)} \tag{†}$$

where $l=2j+1$ and $\delta'=\min\{\delta,1\}$. We have

$$\sigma_t(x)=a(U_t f_1)^*\dots a(U_t f_{2j+1})\,.$$

Let $U_{t_m}f_i=g_i+g_i'$ where g_i (g_i') is the projection of $U_{t_m}f_i$ onto $\mathscr{M}_n^\perp$ $(\mathscr{M}_n)$. Thus

$$\sigma_{t_m}(x)=a(g_1)^*\dots a(g_{2j+1})+z_1\equiv x_n+z_1$$

Since $\|\sigma_{t_m}(x)\|=\|x\|=\delta$ and $\omega(\sigma_{t_m}(x))=\omega(x)$ by hypothesis, the inequality (†) guarantees that z_1 is small enough in norm so that (i), (ii) and (iii) are satisfied. This completes the proof of the lemma.

Proposition 2. *The CAR-algebra is weakly asymptotically abelian with respect to the automorphism* σ_t *constructed above.*

Proof. We show, for any $\varphi \in \mathscr{A}^*$, that

$$\varphi([\sigma_t(x), y]) \to 0 \quad \text{as} \quad t \to \infty .$$

Writing $x = x_e + x_0$ (the even and odd parts of x respectively) we have

$$\varphi([\sigma_t(x), y]) = \varphi([\sigma_t(x_e), y]) + \varphi([\sigma_t(x_0), y]) .$$

The first term goes to zero since $\|[\sigma_t(x_e), y]\| \to 0$ as we have shown above. For the second term we write

$$\begin{aligned} \varphi([\sigma_t(x_0), y]) &= \varphi(\sigma_t(x_0)\, y) - \varphi(y\, \sigma_t(x_0)) \\ &= y\varphi(\sigma_t(x_0)) - \varphi\, y(\sigma_t(x_0)) \end{aligned}$$

Thus it suffices to show for any $\psi \in \mathscr{A}^*$ that $\psi(\sigma_t(x_0)) \to 0$. We note that this argument has its origins in [4]. The latter is accomplished by assuming that there exists a sequence $t_m \to \infty$ such that $|\psi(\sigma_{t_m}(x_0))| \geqq \delta > 0$. Since $(U_{t_m} f | g) \to 0$ we may apply the method of Lemma 2 to construct a sequence $\{x_n\}$ satisfying the conditions (i), (ii) and (iii) of that lemma.

We can now state

Theorem 1. *There exists a quasi-free automorphism σ_t of the CAR-algebra $\mathscr{A}(\mathscr{H})$ and a gauge invariant generalized free state ω_A on $\mathscr{A}(\mathscr{H})$ such that*

(i) *σ_t acts in a weakly asymptotically abelian fashion on $\mathscr{A}(\mathscr{H})$.*
(ii) *ω_A satisfies the KMS condition with respect to σ_t.*
(iii) *If $\sigma_t(a(f)) = a(U_t f)$ with $U_t = e^{itH}$, then $A = e^{-H}(I + e^{-H})^{-1}$.*

Corollary. *There is a factor M, and a normal faithful state φ on M such that $M_\varphi = \{\lambda I\}$.*

Proof. Let $M = \pi_{\omega_A}(\mathscr{A})''$ and $\varphi = \omega_{\xi_0}$ where $\omega_A(x) = (\pi_{\omega_A}(x)\, \beta_0 | \xi_0) = \omega_{\xi_0} \circ \pi_{\omega_A}(x)$. M is a factor [14]. Since σ_t acts in asymptotically abelian fashion $M_\varphi = \{x \in M | \sigma_t(x) = x\} = \{\lambda I\}$.

In our example we assumed that $(U_t f | g) \to 0$ as $t \to \infty$. This statement is stronger than saying that the spectral measure of U_t is non-atomic. The latter is equivalent, by a theorem of Wiener, [22], to

$$\frac{1}{2T} \int_{-T}^{T} |(U_t f | g)|^2 \, dt \to 0 \quad \text{as} \quad T \to \infty . \tag{†}$$

We should like to thank P. Koosis for pointing out this fact to us. Under this condition we show that the CAR algebra is η-asymptotically abelian [see § 1].

Theorem 2. *The spectral measure of U_t is non-atomic iff for every $\omega \in \mathscr{A}^*$ one has*

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \omega([x, \sigma_t(y)])\, dt = 0 \quad \textit{for all} \quad x, y \in \mathscr{A}. \tag{††}$$

Proof. Let $y = y_e + y_0$ and write

$$\frac{1}{2T} \int_{-T}^{T} \omega([x, \sigma_t(y)])\, dt = \frac{1}{2T} \int_{-T}^{T} \omega([x, \sigma_t(y_e)])\, dt \tag{1}$$

$$+ \frac{1}{2T} \int_{-T}^{T} \omega([x, \sigma_t(y_0)])\, dt$$

Consider the second term. It equals

$$\frac{1}{2T} \int_{-T}^{T} \omega(x\sigma_t(y_0))\, dt - \frac{1}{2T} \int_{-T}^{T} \omega(\sigma_t(y_0)\, x)\, dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \sigma_t' \psi_1(y_0) - \frac{1}{2T} \int_{-T}^{T} \sigma_t' \psi_2(y_0)\, dt$$

where

$$\psi_1(z) = \omega x(z) = \omega(xz),$$

$$\psi_2(z) = x\omega(z) = \omega(z\omega)$$

and σ_t' denotes the transpose of σ_t.

It is well known that $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_t' \psi(y_0) = \varphi(y_0)$ where φ is an invariant element of $\mathscr{A}^*$. Thus by Lemma 2, $\varphi(y_0) = 0$. Hence we need only concern ourselves with the first term in (1).

Arguing as we have previously, it is clear that if (††) holds for fixed y, and $x_1, x_2 \in \mathscr{A}$ and all $\psi \in \mathscr{A}^*$ that

$$\operatorname*{limit}_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \omega([x_1 x_2, \sigma_t(y)]\, dt = 0, \quad \text{for all} \quad \omega \in \mathscr{A}^*.$$

To complete the proof then we consider $x = a(f)$ or $a(f)^*$ and $y \in \mathscr{A}_e$. We assume y is an even monomial i.e. $y = y_1 y_2 \ldots y_{2n}$, where $y_j = a(f_j)$ or $a(f_j)^*, j = 1, \ldots, 2n$. One easily sees that

$$[x, \sigma_t(y)] = \sum_{i=1}^{2n} (-1)^{i-1} \{x, \sigma_t(y_i)\}\, \sigma_t[y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{2n}]$$

Thus

$$\left|\frac{1}{2T}\int_{-T}^{T}\omega[x,\sigma_t(y)]\,dt\right|$$
$$\leqq \sum_{i=1}^{2n}\frac{1}{2T}\int_{-T}^{T}|\{x,\sigma_t(y_i)\}|\,|\omega(\sigma_t[y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{2n}])|\,dt$$
$$\leqq \sum_{i=1}^{2n}\left(\frac{1}{2T}\int_{-T}^{T}|\{x,\sigma_t(y_1)\}|^2\right)^{\frac{1}{2}} K_i$$

where K_i is independent of t and depends only on the norms of the y_i $i=1,\ldots,2n$.

We now note that $\{x,\sigma_t(y_i)\}$ is either zero or of the form $(f\,|\,U_t f_j)$. Thus as $T\to\infty$ the terms in the summation go to zero by hypothesis, hence the result.

The converse of the theorem is easily verified.

§ 4. The KMS States for Gauge Transformations

In this final section we point out the connection between Power's result [12, 13] and the KMS condition on the CAR algebra.

As mentioned above the CAR algebra, is a UHF algebra of type $\{2^n\}$. If $\{e_{ij}^{(n)}, i,j=1,2\}$ are the matrix units for the factorization $\{N_n\}$, then the state introduced by Powers is ω where $0<\lambda<\frac{1}{2}$ and $\omega(e_{ij}^{(n)})=\delta_{ij}\lambda_i$ with $\lambda_1=\lambda$, $\lambda_2=1-\lambda$. Moreover $\omega=\omega|N_1\otimes\omega|N_2\otimes\ldots$.

Suppose now that we construct the gauge invariant generalized free state ω_λ on the CAR algebra for $0<\lambda<\frac{1}{2}$. One sees that ω_λ and ω are the same states. Let $\gamma=\ln\left(\frac{1-\lambda}{\lambda}\right)$. According to the discussion in § 2, ω_λ is then a KMS (for $\beta=1$) for the gauge transformation $\sigma_t a(f) = a(e^{i\gamma t}f)=e^{i\gamma t}a(f)$.

One can easily interpret the range of values of γ to be those gotten by letting the temperature $(1/\beta)$ change and thus Powers result in this case is

Theorem 1. *Let $\mathscr{A}$ be the CAR algebra. Then there exists a continuous family of KMS states for temperature $0<T<\infty$ giving rise to non-isomorphic type III factor representation.*

Appendix

We reproduce for the reader's reference a theorem of Powers [11]

Theorem. *Let $\mathscr{A}=\mathscr{A}(\mathscr{H})$ be the CAR algebra and suppose $\varrho\in\mathscr{A}^*$. Let $\mathscr{M}_1\subsetneqq\mathscr{M}_2\subsetneqq\cdots\subsetneqq\mathscr{H}$ be an increasing sequence of subspaces with*

$\overline{\cup \mathcal{M}_n} = \mathcal{H}$. *Then given* $\varepsilon > 0$, *there exists* $n > 0$ *such that* $|\varrho(x)| \leqq \varepsilon \|x\|$ *for all* odd $x \in \mathcal{A}(\mathcal{M}_n^\perp)$.

We shall need the following

Lemma 1. *Let* $\{x_i\}$ *be an infinite sequence of* odd *hermitian elements such that* $\{x_i, x_j\} = 0$. *Then* $\left\| \frac{1}{n} \sum_1^n x_i \right\| \to 0$ *as* $n \to \infty$. *So if* ω *is a state on* $\mathcal{A}(\mathcal{H})$ *one has that* $\frac{1}{n} \sum_1^n |\omega(x_i)| \to 0$ *as* $n \to \infty$.

Proof. If $z_n = \frac{1}{n} \sum_1^n x_i$ compute the $\|z_n\|^2 = \|z_n z_n^*\|$ and notice that the off-diagonal terms vanish.

Proof of Theorem. We assume ϱ is hermitian and $\|\varrho\| \leqq 1$. By contradiction we construct a sequence $\{x_i, \|x_i\| = 1, i = 1, 2, \ldots\}$ such that $x_i = x_i^*$, $\{x_i, x_j\} = 0$, $i \neq j$ and $\varrho(x_i) \geqq \delta > 0$ This contradicts Lemma 1.

Then suppose there is $\varepsilon_0 < 1$, such that for each n we can find $x \in \mathcal{A}(\mathcal{M}_n^\perp)$ x odd with $|\varrho(x)| > \varepsilon_0 \|x\|$. Since $\overline{\cup \mathcal{A}(\mathcal{M}_n)} = \mathcal{A}(\mathcal{H})$ there exists $y \in \mathcal{A}(\mathcal{M}_{n_1})$ with y odd and $|\varrho(y)| > \varepsilon_0/2 \|y\|$. We multiply y if necessary by an appropriate scalar obtaining $\varrho(y) > \varepsilon_0/2 \|y\|$. Then let $x_1 = (y + y^*)/\|y + y^*\|$. We see that x_1 is odd, $\varrho(x_1) > \varepsilon_0/2$ and $\|x_1\| = 1$.

By hypothesis there exist $y \in \mathcal{A}(\mathcal{M}_{n_1}^\perp)$, y odd such that $|\varrho(y)| > \varepsilon_0 \|y\|$. Noticing that $\bigcup_{j=n_1+1}^{\infty} (\mathcal{M}_j \cap \mathcal{M}_{n_1}^\perp)$ is dense in $\mathcal{M}_{n_1}^\perp$ we repeat the above procedure obtaining x_2. Continuing we obtain the desired sequence since "orthogonal" odd elements anticommute.

Added in proof: Professor Størmer, in a private discussion with the second author, has pointed out that Jadczyk's method [8] extends to give a simpler proof of the Theorem in § 2.

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