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Exponential Representations of the Canonical Commutation Relations

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Abstract. A class of representations of the canonical commutation relations is investigated. These representations, which are called exponential representations, are given by explicit formulas. Exponential representations are thus comparable to tensor product representations in that one may compute useful criteria concerning various properties. In particular, they are all locally Fock, and non-trivial exponential representations are globally disjoint from the Fock representation. Also, a sufficient condition is obtained for two exponential representations not to be disjoint. An example is furnished by Glimm's model for the $:\Phi^4:$ interaction for boson fields in three space-time dimensions.

I. Introduction

In this paper we investigate a certain class of representations of the canonical commutation relations. Our representations will be called *exponential* Weyl systems. A representation of the canonical commutation relations, or a *Weyl system*, is a map $f \to W(f)$ from a complex inner product space J to unitary operators on a complex Hilbert space H, such that

$$W(f)\, W(g) = e^{i\,\mathrm{Im}(f,g)/2}\, W(f+g)$$

(the Weyl relations), and $t \to (\phi, W(tf)\psi)$ is continuous at $t=0$. If $\{f_j\}$ is an orthonormal basis of J, then

$$W(sf_j) = e^{isQ_j}, \qquad W(tf_k) = e^{itP_k},$$

by Stone's theorem, where Q_j, P_k are self-adjoint. The Weyl relations are

$$e^{isQ_j}\, e^{itP_k} = e^{ist\delta_{jk}}\, e^{itP_k}\, e^{isQ_j},$$

which is an exponentiated version of

$$Q_j P_k - P_k Q_j = i\delta_{jk}\,.$$

A theorem of von Neumann states if $\dim J = n < \infty$, then W is essentially (up to multiplicity) the Schrödinger Weyl system of quantum mechanics:

$$Q_j = \text{multiplication by } x_j, \quad P_k = i^{-1}\frac{\partial}{\partial x_k}, \quad H = L_2(R^n).$$

If $\dim J = \infty$, there are uncountably many inequivalent irreducible Weyl systems. The best known example is the Fock representation. $J = L_2(R^d)$, $d \geqq 1$, is called the single particle space. Let $H = F = \Sigma F_n$ be the direct sum of Hilbert spaces

$$F_n = S L_2(R^{dn}),$$

where S is the symmetrization projection defined on $\Sigma L_2(R^{dn})$ by

$$S\psi_n(k_1, \ldots, k_n) = n!^{-1} \sum_{\sigma \varepsilon S_n} \psi_n(k_{\sigma(1)}, \ldots, k_{\sigma(n)}).$$

Here $k_i \in R^d$, S_n is the permutation group on n letters, and F_0 is the field of complex numbers. Thus $\psi = \Sigma \psi_n \in F$ if $\psi_n \in F_n$ is symmetric, square integrable, and $\Sigma \|\psi_n\|^2 = \|\psi\|^2 < \infty$. F is the Fock space for a neutral scalar boson field. $\psi_n \in F_n$ represents a state of the quantum field in which there are n particles; if $\|\psi_n\| = 1$, then $|\psi_n(k_1, \ldots, k_n)|^2$ is the probability density that their momenta are $k_1, \ldots, k_n$. Since ψ_n is symmetric, the particles are indistinguishable.

$$\Omega = (1, 0, 0, \ldots) \in F$$

is the *Fock vacuum*, and it is the unique (normalized) state in which no particles are present. In general there are an indefinite number of particles in the state $\psi \in F$. If $f \in L_2(R^d)$, the annihilation operator $a(f)$ maps $F_n \to F_{n-1}$, annihilating a particle with wave function f, and the creation operator $a^*(f)$ maps $F_n \to F_{n+1}$, creating a particle with wave function f:

$$(a(f)\psi_n)(k_1, \ldots, k_{n-1}) = n^{1/2} \int f(k_n)\, \psi_n(k_1, \ldots, k_n)\, dk_n$$
$$a^*(f)\psi_n = (n+1)^{1/2}\, S(f \otimes \psi_n).$$

The normalization constants are chosen so that $a^*(f)$ and $a(\bar{f})$ are adjoints; moreover

$$\phi(f) = 2^{-1/2}[a(\bar{f}) + a^*(f)]$$

is self adjoint. $W(f) = e^{i\phi(f)}$ is the Fock representation. The Weyl relations follow from the commutation relations

$$a(f)\, a^*(g) - a^*(g)\, a(f) = \int f(k)\, g(k)\, dk.$$

Exponential representations are constructed as follows. If

$$w = w(k_1, \ldots, k_\nu) \in L_2(R^{d\nu}),$$

the creation operator $a^{*\nu}(w)$ maps $F_n \to F_{n+\nu}$, creating ν particles with wave function w:

$$a^{*\nu}(w)\psi_n = [(n+1)\dots(n+\nu)]^{1/2}\, S(w \otimes \psi_n).$$

Let

$$D = \{\psi = \Sigma\psi_n \in F \colon \psi_n = 0, \text{ large } n; \operatorname{supp}\psi_n \text{ compact}\}$$

be the set of vectors in F with a finite number of particles and bounded momentum. Let $v = v(k_1, \dots, k_\nu)$ be a symmetric measurable function, and let ϱ, σ be lower and upper cutoffs on the magnitude of the largest momentum:

$$\begin{aligned} v_{\varrho\sigma}(k) &= v(k), \ \max_{1 \leqq i \leqq \nu} |k_i| \in [\varrho, \sigma) \\ &= 0 \quad \text{otherwise}. \end{aligned}$$

Note that $v_{\varrho\sigma} = 0$ for $\varrho > \sigma$. Let $v_\sigma = v_{0\sigma}$. Fix $\alpha > 1$, and let

$$\begin{aligned} \alpha(j) &= \alpha^j, \quad & j \geqq 1 \\ &= 0 & j = 0. \end{aligned}$$

Thus α denotes both the constant and the corresponding function of j. Let $v_{j\sigma}$, v_{jk} denote $v_{\alpha(j)\sigma}$, $v_{\alpha(j)\alpha(k)}$ respectively. Suppose $v_\sigma \in L_2(R^{d\nu})$ so that $a^{*\nu}(v_{\varrho\sigma})$ is defined. Observe that for $k < l$, $\alpha(l) \leqq \sigma$,

$$\exp a^{*\nu}(v_{k\sigma}) = \exp a^{*\nu}(v_{kl} + v_{l\sigma}) = \exp a^{*\nu}(v_{kl}) \exp a^{*\nu}(v_{l\sigma})$$

as formal power series.

If v is almost in L_2 in a certain technical sense, we construct a family of cutoff operators $T_{j\sigma}$, $j \geqq 0$, which are modifications of $\exp a^{*\nu}(v_{j\sigma})$. For $k \leqq l$, $\alpha(l) \leqq \sigma$,

$$T_{k\sigma}\psi = T_{l\sigma}\theta,$$

where $\theta \in D$ is a modification of $\exp a^{*\nu}(v_{kl})\psi$. Let $T_j = T_{j\infty}$. If $v \notin L_2$, then T_j maps D out of Fock space. However, $T_j D$ is contained in the algebraic direct sum [11] of F_n, $n \geqq 0$, and has a natural Hilbert space structure after division by an infinite constant. We view $T_k D \subset T_l D$, $k < l$, with the identification $T_k\psi = T_l\theta$. Then the limit

$$\lim_{\sigma \to \infty} (T_{k\sigma}\phi, T_{l\sigma}\psi)\, e^{-\nu!\|v_\sigma\|^2} = (T_k\phi, T_l\psi)_r$$

exists for $\phi, \psi \in D$ and defines a positive definite inner product on the set $\mathscr{D} = \bigcup_{j \geqq 0} T_j D$. The subscript denotes "renormalized". Let $H = F_r$ be the completion of $\mathscr{D}$. Let

$$J = \{f \in L_2(R^d) : \mu(k)^\varepsilon f \in L_2(R^d), \text{ some } \varepsilon > 0\},$$

where $\mu(k)=(\mu_0^2+|k|^2)^{1/2}$ is the energy of a particle with momentum k; $\mu_0>0$ is the rest mass. Then the limit

$$\lim_{\sigma\to\infty}(T_{k\sigma}\phi, W(f)\,T_{l\sigma}\psi)\,e^{-\nu!\|v_\sigma\|^2}=(T_k\phi, W_r(f)\,T_l\psi)_r$$

defines a Weyl system $W_r(f)$, $f\in J$. W_r is called an *exponential* Weyl system.

We now define $T_{j\sigma}$. In general, the power series for $\exp a^{*\nu}(v_{j\sigma})\theta$, $\theta\in F$, does not converge; see [7]. Indeed,

$$\|a^{*\nu}(v_{j\sigma})^n\Omega\|^2\sim K^n n!^\nu,$$

so that convergence cannot be expected for $\nu>2$. Convergence occurs when $\nu=1$, or $\nu=2$ and $v_{j\sigma}$ has Hilbert Schmidt norm less than 1/2 [11, Lemma 4; 12, Lemma 2]. Thus we must omit portions of $\exp a^{*\nu}(v_{j\sigma})$. Suppose $j\leqq l$, $\alpha(l)\leqq\sigma\leqq\alpha(l+1)$. Now

$$\begin{aligned}\exp a^{*\nu}(v_{j\sigma})&=\exp a^{*\nu}\Big(\sum_{j\leqq k\leqq l-1}v_{k,k+1}+v_{l\sigma}\Big)\\&=\prod_{j\leqq k\leqq l}\exp V_{k\sigma}\end{aligned}$$

as formal power series, where

$$\begin{aligned}V_{k\sigma}&=a^{*\nu}(v_{k,k+1}), &&k\leqq l-1\\&=a^{*\nu}(v_{l\sigma}) &&k=l.\end{aligned}$$

Let $\exp_n x=\sum_{l=0}^{n}x^l/l!$. Then for any sequence $n(k)$, $k\geqq 0$,

$$T_{j\sigma}=\prod_{j\leqq k\leqq l}\exp_{n(k)}V_{k\sigma}$$

converges absolutely on D.

Thus F_r and W_r depend on the parameters $v,\alpha,n(k)$. Note that as α decreases and $n(k)$ increases, $T_{j\sigma}$ creates more high energy particles. The choice of $n(k)$ must balance two conflicting objectives. We do not want $T_{j\sigma}$ to create so many high energy particles that $(\cdot,\cdot)_r$ doesn't exist. On the other hand, we want $T_{j\sigma}$ to create enough high energy particles to overcome the effect of the factor $e^{-\nu!\|v_\sigma\|^2}$ so that $\|\cdot\|_r$ is definite. Thus, we require that $n(k)$ be strictly increasing but polynomial bounded in k.

In Ch. II, a basic estimate controlling products of operators of the form $a^{*\nu}(w)$ or $a^{*\nu}(w)^*$ is obtained. In the process, quantized bilinear forms are discussed [6]. Ch. III is devoted to the construction of F_r. We say that an operator B in F_r is the *weak limit* of an operator A in F (written $B=\lim_\sigma A$) if $\mathscr{D}(B)=\mathscr{D}$ and

$$(T_k\phi, B\,T_l\psi)_r=\lim_{\sigma\to\infty}(T_{k\sigma}\phi, A\,T_{l\sigma}\psi)\,e^{-\nu!\|v_\sigma\|^2}.$$

If $\lim_{\sigma} A$ is bounded, then it extends uniquely by continuity to F_r; the extension is also written $\lim_{\sigma} A$. Thus if I_r is the identity in F_r, then $I_r = \lim_{\sigma} I$. Ch. III is also devoted to obtaining operators in F_r as weak limits of quantized operators in F.

Ch. IV contains some results concerning unitary operators and n-parameter unitary groups in F_r which are defined by weak limits. We then conclude (Theorem 2) that $\lim_{\sigma} W(f)$ defines a Weyl system $W_r(f) = e^{i\phi_r(f)}$, $f \in J$. Moreover, $\mathscr{D}$ is a dense set of entire vectors for $\phi_r(f)$, and $\phi_r(f)^j \supset \lim_{\sigma} \phi(f)^j$, $j \geqq 0$. If $\|\mu^\varepsilon(f_n - f)\| \to 0$, some $\varepsilon > 0$, then $\phi_r(f_n) \to \phi_r(f)$ and $W_r(f_n) \to W_r(f)$ strongly on $\mathscr{D}$.

We also show (Theorem 3) that the local systems $W_r(f)$, $|\operatorname{supp} f(k)| \leqq \varrho$, i.e. localized in momentum space, have a non-negative number operator in the sense of Chaiken, and hence are unitarily equivalent to a direct sum of Fock representations. In Theorem 3, we also prove that if the kernel is not in L_2, then the global systems $W_r(f)$, $f \in J$, are disjoint from the Fock representation because every vector in F_r has an infinite number of particles. Finally, in Theorem 4, it is shown that the choice of $n(k)$ is somewhat a matter of technical convenience in the sense that two Weyl systems W_r^1 and W_r^2 with the same *kernel* v and same choice of α are not disjoint. That is, there are invariant subspaces $S_i \subset F$ for W_r^i such that $W_r^1 \mid S_1$ is unitarily equivalent to $W_r^2 \mid S_2$. Moreover, if $n_1(k) = n_2(k)$ for almost all k, then W_r^1 and W_r^2 are unitarily equivalent. If the kernel of W_r is perturbed by a sufficiently small function, then the new Weyl system is not disjoint from the old one. A sufficient condition for unitary equivalence is given. It is hoped that a family of inequivalent exponential representations may be obtained from kernels whose pairwise differences are sufficiently large.

Remarks. 1. $\phi(f)$ is related to the Newton-Wigner field $\phi_{N,W}$ and the relativistic field ϕ_{rel} by the formulas

$$\phi_{N,W}(g) = \phi(\hat{g}), \qquad \phi_{\mathrm{rel}}(g) = \phi\left([(-\Delta + \mu_0^2)^{-1/4} g]^{\wedge}\right),$$

where $\hat{g}(k)$ is the Fourier transform of $g(x)$. Then if we replace ϕ by ϕ_{rel}, the local systems (localized in momentum space) are still unitarily equivalent to a direct sum of Fock representations because $|\operatorname{supp} g| \leqq \varrho$ if and only if

$$|\operatorname{supp} [(-\Delta + \mu_0^2)^{-1/4} g]^{\wedge}| \leqq \varrho .$$

2. Exponential Weyl systems have some of the advantages of tensor product representations: explicit formulas for them are given, and one may compute criteria for various properties of the representations. Tensor product representations have been useful for linear problems in quantum

field theory. M. Reed [15] has even shown that a certain class of renormalized Hamiltonians with nearly diagonal non-linear interactions acts on infinite tensor product spaces. However, tensor product representations cannot be expected to provide solutions to most physical problems. This point of view has been supported by Powers [14] in the case of the representations of the canonical anticommutation relations (CAR) provided by Fermi fields. He proves that a translation invariant vector state of a tensor product representation of the CAR is a generalized free state, i.e. the truncated n point functions vanish for $n > 2$.

Powers' point of view may apply somewhat to exponential representations. Nevertheless, certain renormalized Hamiltonians, with interactions possessing large momentum singularities different from those of Reed's model, are defined on exponential representation spaces. Indeed, exponential representations are suitable for some superrenormalizable interactions which may be far from diagonal. An example is furnished by Glimm's model for the $:\Phi^4:$ interaction for boson fields in three space-time dimensions [5]. This model has an infinite mass renormalization caused by the off-diagonal part of the interaction in the sense that there is fairly strong coupling between low and high energy parts of the interaction. Hepp has demonstrated [8] that the renormalized Hamiltonian for a simplified version of the interaction acts on a closed subspace, the completion of $T_0 D$, of an exponential representation space. The interaction is taken to be

$$a^{*4}(-\Sigma\mu(k_i)v) + a^{*4}(-\Sigma\mu(k_i)v)^* .$$

Here $v = 4$, $d = 2$, $n(k) = k$, $\alpha = 2$ and

$$v(k_1, \ldots, k_4) = -\Pi\mu(k_i)^{-1/2} (\Sigma\mu(k_i))^{-1} \hat{h}(\Sigma k_i),$$

where $\|v_\sigma\|^2$ is logarithmically divergent. v is almost in L_2, but is not close to diagonal. For example, v is not in L_2 on the set

$$\{k: 2|k_1| \leqq |k_2|, |k_3|, |k_4| \leqq 3|k_1|\} .$$

$\hat{h}$ is the Fourier transform of the space cutoff function, which is any non-negative smooth function with compact support. It would be of interest to construct a family of inequivalent representations by choosing different space cutoff functions. Incidentally, Hepp has independently obtained [8, 9] a Weyl system whose properties are similar to those of $W_r = W_r(v, 2, k)$. It is conjectured that the two representations are unitarily equivalent.

Note that "exponential" has been applied to representations in a different context [10].

II. Basic Estimates

In this section we obtain an estimate controlling products of the form $a^{*\nu}(v)^{*p}a^{*\nu}(v')^q$. Products involving v_σ and v'_σ are considered because we want sufficient generality to compare two different exponential Weyl systems in Ch. IV. First we must discuss quantized operators and bilinear forms in Fock space [6]. In order to control the proliferation of constants in the sequel, $K>0$ will denote possibly different constants from one line to the next.

Let B' be a bilinear form, with distribution kernel b, densely defined in $F_m \times F_n$. We associate to B' the quantized bilinear form B defined in $F\times F$ by

$$(\phi, B\psi) = \sum_{t=0}^{\infty} [(t+m)!\,(t+n)!]^{1/2}\, t!^{-1} \cdot \int \bar{\phi}_{m+t}(x,z)\, b(x,y)\, \psi_{n+t}(y,z)\, dx\, dy\, dz\,. \tag{2.1}$$

If $b = f_1 \otimes \cdots \otimes f_{m+n}$, $f_i \in L_2(R^d)$, then one can easily check that $B = a^*(f_1)\dots a^*(f_m)a(f_{m+1})\dots a(f_{m+n})$. Also, if $b = w(k_1,\dots,k_m) \in L_2(R^{dm})$, then $B = a^{*m}(w)$. Thus B creates m particles and annihilates n particles. If δ is the Dirac delta function, then δ is a distribution kernel densely defined in $F_0 \times F_1$. Then $a(k)$, the quantized operator given by $\delta(\cdot - k)$, is defined by

$$(a(k)\phi_n)(k_1,\dots,k_{n-1}) = n^{1/2}\,\phi_n(k,k_1,\dots,k_{n-1})\,. \tag{2.2}$$

Thus $a(k)$ annihilates a particle with momentum k. The adjoint $a^*(k)$ of $a(k)$ is a quantized bilinear form with distribution kernel densely defined in $F_1 \times F_0$, and it is defined by the improper operator

$$(a^*(k)\phi_n)(k_1,\dots,k_{n+1}) = (n+1)^{1/2}\, S\delta(k-k_1)\otimes\phi_n(k_2,\dots,k_{n+1})\,. \tag{2.3}$$

That is,

$$(\psi_{n+1}, a^*(k)\phi_n) = (n+1)^{1/2}\,(\psi_{n+1}(k,k_2,\dots,k_{n+1}), \phi_n(k_2,\dots,k_{n+1}))\,.$$

Thus $a^*(k)$ creates a particle with momentum k. Let $a^\#(k)$ denote $a^*(k)$ or $a(k)$. One may verify that

$$\begin{gathered} B = \int b(k_1,\dots,k_m,k'_1\dots k'_n)\, a^*(k_1)\dots a^*(k_m)\, a(k'_1)\dots a(k'_n)\, dk\, dk' \\ [a(k), a^*(l)] = a(k)\,a^*(l) - a^*(l)\,a(k) = \delta(k-l) \\ [a(k), a(l)] = 0 = [a^*(k), a^*(l)]\,. \end{gathered} \tag{2.4}$$

That is,

$$(\phi, B\psi) = \int (\Pi a(k_i)\phi, b(k,k')\,\Pi a(k'_i)\psi)\,dk\,dk'\,,$$

and

$$\int (\phi, b(k)\, \Pi a^{\#}(k_i)\psi)dk = \int (\phi, b(k)\, \Pi' a^{\#}(k_i)\psi)dk$$
$$= \int \Big(\phi, b(k) \prod_{i \neq j, j+1} a^{\#}(k_i)\, \delta(k_j - k_{j+1})\psi\Big) dk\,,$$
$$+ \int (\phi, b(k)\, \Pi'' a^{\#}(k_i)\psi)dk$$

where Π' denotes a transposition of two factors $a(k_i), a(k_{i+1})$ or $a^*(k_i)$, $a^*(k_{i+1})$, and $''$ denotes a transposition of two factors $a(k_j), a^*(k_{j+1})$.

Note that if B' is a bounded operator from F_n to F_m (for example, if $b \in L_2(R^{d(m+n)})$), then B is a densely defined unbounded operator in F. Indeed, if I is the identity operator on F_1, with distribution kernel $\delta(k-k')$, let N be the quantization of I. Then, by (2.1), $N\psi_n = n\psi_n$, and N is called the number of particles operator. By (2.1),

$$(B\psi)_i = i!^{1/2}(i-m+n)!^{1/2}\,(i-m)!^{-1}\, SB'\psi_{i-m+n}\,,$$

where B' acts on the first n variables of ψ_{i-m+n}, so that $B'\psi_{i-m+n}$ is an unsymmetrized function of i variables. Hence

$$\|B\psi\|^2 \leqq \sum_{i \geqq m} i!\,(i-m+n)!\,(i-m)!^{-2}\, \|S\|^2\, \|B'\|^2\, \|\psi_{i-m+n}\|^2$$
$$\leqq K\|B'\|^2 \sum_{j=0}^{\infty} (j+1)^{m+n}\, \|\psi_j\|^2$$
$$\leqq K\|B'\|^2\, \|(N+1)^{(m+n)/2}\psi\|^2\,.$$

Thus

$$\|B\psi\| \leqq K\|B'\|\, \|(N+1)^{(m+n)/2}\psi\|\,. \tag{2.5}$$

Hence $D \subset \mathscr{D}(B)$, so that B is densely defined.

By (2.4), we may write $N = \int a^*(k)\, a(k)dk$. Similar operators to be considered in the sequel are:

$$N(B_\varrho) = \int_{|k| \leqq \varrho} a^*(k)\, a(k)dk\,,$$
$$N_\tau = \int \mu(k)^\tau\, a^*(k)\, a(k)dk, \qquad \tau < 0\,,$$
$$H_0(B_\varrho) = \int_{|k| \leqq \varrho} \mu(k)\, a^*(k)\, a(k)dk\,,$$

where B_ϱ is the ball of radius $\varrho > 0$ in R^d. $N(B_\varrho)$, N_τ, and $H_0(B_\varrho)$ are quantizations of operators on F_1 given by multiplication by χ_ϱ, $\mu(k)^\tau$, and $\mu(k)\chi_\varrho$ respectively, where χ_ϱ is the characteristic function of B_ϱ. By (2.1), they act on F_n by multiplication by $\sum_{i=1}^{n} \chi_\varrho(k_i)$, $\sum_{i=1}^{n} \mu(k_i)^\tau$, and $\sum_{i=1}^{n} \mu(k_i)\, \chi_\varrho(k_i)$ respectively. $N(B_\varrho)$ is the number of particles operator

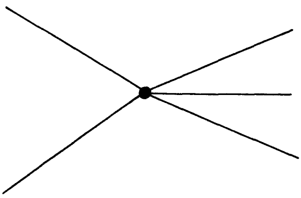

Fig. 1: m=2, n=3.

Fig. 2: j=1.

over $L_2(B_\varrho)$ for the Fock representation and measures the number of particles with momenta bounded by ϱ. $H_0(B_\varrho)$ measures the free energy of these particles.

Suppose B is a quantized bilinear form

$$B = \int b(k, k') \prod_{i=1}^{m} a^*(k_i) \prod_{i=1}^{n} a(k_i') dk\, dk' .$$

To B we associate a graph with m (creating) legs pointing to the left, n (annihilating) legs pointing to the right; all legs issue from a common vertex [4, 5]. (See Fig. 1.) B is also called a *Wick ordered* bilinear form because the $a^*(k_i)$ all appear to the left of the $a(k_j')$. It is determined by its graph and its *kernel b*. The product of two Wick ordered bilinear forms B_1, B_2 is not Wick ordered, but it is a sum of Wick ordered bilinear forms, by repeated application of the commutation relations (2.4). The term with no δ functions has a kernel $b_1 \otimes b_2$ and is denoted $:B_1 B_2:$, the *Wick* product of B_1 and B_2. If a term contains a δ function $\delta(k-l)$, we say that the variables k, l have been *contracted.* The term with $j\delta$ functions, denoted $B_1 \underset{j}{-\!\circ\!-} B_2$, has a kernel obtained from $b_1 \otimes b_2$ by equating contracted variables and summing over all possible contractions with $j\delta$ functions. To $B_1 \underset{j}{-\!\circ\!-} B_2$ is associated a graph obtained by connecting j of the annihilating legs of B_1 with j of the creating legs of B_2. (See Fig. 2.) $B_1 \underset{j}{-\!\circ\!-} B_2$ is determined by its graph and its kernel. Similarly, $B_1 \ldots B_t$ is a sum of Wick ordered bilinear forms B, each of which is determined by its graph and its kernel. Its graph has t ordered vertices and is obtained by specifying the number of annihilating legs of the graph of B_i which are connected to creating legs of B_j, $i<j$. Its kernel b is obtained from $b_1 \otimes \cdots \otimes b_t$ by equating variables and then summing over all possible contractions which produce the same graph. Then $B = \int b(k)\, \Pi a^{\#}(k_i) dk$, where the product extends over the legs of the graph which do not connect two vertices (external legs). Legs which do connect vertices are called internal. Note that there is one variable in b for each leg in the graph, so that one speaks of external and internal variables.

If M is a measurable subset of all variables of b, then

$$\int_M b(k)\,\Pi\, a^{\#}(k_i)\,dk$$

is called a *truncation*. A truncation of a product $B_1 \dots B_t$ is given by a truncation of each of its Wick ordered terms. A truncated power series is given by a truncation of each of its terms.

A subgraph of a graph is a subset of the vertices of the graph, together with all legs issuing from these vertices. Two subgraphs are disjoint if they have no common vertices. (They may have common legs.) Note that a leg may be internal in the full graph but external in a subgraph. In the sequel, let $\mu_t = \mu(k_t)$, let Π_e denote the product over external variables, Π the product over all variables, $I = I(G)$ the internal variables with respect to the graph G, and $\int_I$ the integral over internal variables [5, p. 8].

Let v, v' be symmetric, measurable functions, and $V = a^{*\nu}(v)$, $V' = a^{*\nu}(v')$ the corresponding bilinear forms. We shall be concerned with v, v' which satisfy the following property:

Let b be the kernel of $B = V^* \underset{r}{-\circ-} V'$, $1 \leqq r \leqq \nu - 1$; for $a > 0$ and $1 \leqq t \leqq \nu$, there exists $\varepsilon > 0$ such that

$$\begin{aligned} &\int_I \Pi\, \mu^{\varepsilon} |b| \in L_2(R^{2d(\nu - r)}),\\ &v\Pi\, \mu^{\varepsilon} \mu_t^{-a},\ v'\Pi\, \mu^{\varepsilon} \mu_t^{-a} \in L_2(R^{d\nu}), \qquad (2.6)\\ &\|v_{\varrho\sigma}\|,\ \|v'_{\varrho\sigma}\| \leqq K(\log(\sigma/\varrho) + 1). \end{aligned}$$

Suppose that (2.6) is satisfied for $v = v'$. Then we say that v is *almost in* L_2. In particular, an L_2 function is almost in L_2, and

$$\Lambda(\sigma) = \nu!\, \|v_\sigma\|^2$$

is at worst logarithmically divergent. Choose $\alpha > 1$, $n(k)$ strictly increasing but polynomially bounded, and construct

$$T_{j\sigma} = \prod_{j \leqq k \leqq l} \exp_{n(k)} V_{k\sigma}$$

as described in Ch. I. $T_j = T_j(v, \alpha, n(k))$ is called an exponential dressing transformation. Note that $T_{j\sigma}$ is the truncation of $\exp a^{*\nu}(v_{j\sigma})$ in which $V_{k\sigma}$ appears to at most the power $n(k)$, $k \geqq j$, and does not appear for $k < j$.

Now $V^{*p}(V')^q$ is a sum of Wick ordered bilinear forms; a bilinear form in this sum is called reduced if its graph contains no

$$X = V^* \underset{\nu}{-\circ-} V'$$

components. Each graph is a union of its connected components. If the number of vertices of a connected component is greater than one, then the kernel is in L_2. The improvement of the kernels, as the order of the graph increases, demonstrates the sort of behavior to be expected of superrenormalizable quantum field theories [5, p. 10]. Such estimates have been studied systematically by Eckmann [3]. The following lemma establishes geometric growth of L_2 norms.

Lemma 2.1. *Suppose v, v' satisfy* (2.6). *Let B be a reduced term from $V^{*p}V'^{q}$, b its kernel, and $n=p+q$. Then for $a>0$, there exists $K>0$ such that for $\varepsilon>0$ sufficiently small,*

$$\left\| \Pi_e \mu^{-a} \int_I \Pi \mu^{\varepsilon} |b| \right\| \leqq K^n . \tag{2.7}$$

Here a, ε do not depend on p, q or the reduced term.

Proof. We may suppose that the graph G of B is connected because both sides of (2.7) are products of similar expressions involving the connected components. The cases $n=1,2$ follow from (2.6). Let $n \geqq 3$. G may be written as a disjoint union of subgraphs of v types: a central vertex contracted with $1 \leqq s \leqq v$ vertices [5, p. 8]. This decomposition is obtained by induction on $n \geqq 3$. For $n=3$, use inspection. Suppose we have the decomposition for $3 \leqq n \leqq N$. Let $n=N+1$. Since G is connected, we may choose a subgraph $H \subset G$ of type $s=1$. Then $G-H$ is a disjoint union of connected components H'_j. Let H' be the union of H with all those H'_j which consist of a single vertex. Clearly H' is connected. If $H' \neq G$, then apply induction to H' and the components of $G-H'$. If $H'=G$, then one can show directly that G is decomposable into either one or two subgraphs of the proper type.

Thus, for $n \geqq 3$, we may write $G = \bigcup H_j$, a disjoint union of subgraphs of the proper type. Let $y = \pi y_j$, where y_j is the kernel of H_j. Then the Cauchy-Schwarz inequality in the variables $k \in I(G) - \bigcup_j I(H_j)$ implies that

$$\left\| \Pi_e \mu^{-a} \int_{I(G)} \Pi \mu^{\varepsilon} |y| \right\| \leqq K^n \prod_j \left\| \int_{I(H_j)} \Pi \mu^{\varepsilon} |y_j| \right\| ,$$

where K compensates in the region of small $|k|$ for the factors $\mu(k)^{\varepsilon}$, which appear twice in the right hand side. Since these are a finite number of graphs of the proper type, it suffices to show that $\int_{I(H)} \Pi \mu^{\varepsilon} |y| \in L_2$, where H is a subgraph of type s with kernel y. By assumption, there are no X components, so the case $s=1$ follows from (2.6). For $s \geqq 2$, choose a subgraph $H_0 \subset H$ of type 1. Let H_j be the graphs of the other vertices, and let y_0, y_j be the corresponding kernels. Multiply y_0 by $\prod_{I(H)-I(H_0)} \mu^a$

and y_j by $\prod_{I(H)-I(H_j)} \mu^{-a}$. Then for $a>0$ sufficiently small, we may apply the Cauchy-Schwarz inequality in the variables $I(H)-I(H_0)$ and use the first part of (2.6) for the first factor and the second part of (2.6) for the remaining $s-1$ factors. This completes the proof.

III. Renormalized Hilbert Space

In this chapter we construct F_r, and, in the process, obtain operators in F_r as weak limits of products of Wick ordered operators in F. We first establish a combinatorial lemma which is sufficiently general for the needs of this paper. Suppose Y_i, $1 \leqq i \leqq m$, is a Wick ordered operator in F of the form

$$\int y_i(k_1, \ldots, k_{s_i}) \prod_{j=1}^{s_i} a^{\#}(k_j)\, dk \quad \text{or} \quad \int y(k)\, a^*(k)\, a(k)\, dk ,$$

where y_i is a measurable kernel and y is measurable. Let

$$T_k = T_k(v, \alpha, n(j)), \qquad T'_l = T_l(v', \alpha, n'(j)) ,$$

$$V_\sigma = \sum_j V_{j\sigma}, \qquad V'_\sigma = \sum_j V'_{j\sigma}, \quad \text{and} \quad X(\sigma) = V^*_\sigma \underset{v}{-\circ-} V'_\sigma = v!\,(v_\sigma, v'_\sigma) .$$

Note that we require $\alpha' = \alpha$, so that $V^*_{i\sigma} \underset{v}{-\circ-} V'_{j\sigma} = 0$, $i \neq j$. Let $\phi, \psi \in D$. We consider

$$\left(T_{k\sigma}\phi, \prod_{i=1}^{m} Y_i T'_{l\sigma}\psi\right) e^{-X(\sigma)} = \left(\phi, T^*_{k\sigma} \prod_{i=1}^{m} Y_i T'_{l\sigma}\psi\right) e^{-X(\sigma)} . \tag{3.1}$$

(For $m=0$, $\Pi Y_i = I$.) Now $T^*_{k\sigma} \Pi Y_i T'_{l\sigma}$ is a truncated power series in V^*_σ, Y_i, V'_σ and is an infinite sum of Wick ordered terms with distribution kernels. A term will be called *reduced* if it contains no X components [4]. Let $R_{0\sigma}$ be a reduced term, and let $R_{j\sigma}$ be the sum of all terms whose graphs differ from the graph G of $R_{0\sigma}$ by jX components. (G is called the reduced graph of $R_{j\sigma}$.) Let $e^{X(\sigma)} R_{G\sigma}$ be the bilinear form with graph G and kernel equal to the kernel of $\sum_{j \geqq 0} R_{j\sigma}$, after integration over the variables in the X components. Thus, (3.1) equals $\sum_G (\phi, R_{G\sigma}\psi)$, where the summation extends over all reduced graphs G. $R_{G\sigma}$ does not necessarily have a measurable kernel because of the δ function in $\int y(k)\, \delta(k-l) \cdot a^*(k)\, a(l)\, dk\, dl$. However, by (2.1) and the form of Y_i, $(\phi, R_{G\sigma}\psi)$ is a sum of integrals of the form

$$c_{pq} \int \bar{\phi}_p r_{G\sigma} \psi_q$$

over the variables (some coinciding) of $\bar{\phi}_p$, $r_{G\sigma}$, and ψ_q. Here $r_{G\sigma}$ is a measurable function, ϕ_p and ψ_q are the p and q particle components of ϕ, ψ, and the choices of p, q depend on G.

Let E be the direct sum over all reduced graphs G, and over permissible p, q for each G, of the measure spaces associated with $c_{pq}\int\bar{\phi}_p r_{G\sigma}\psi_q$. Thus (3.1) equals

$$\int_E h_\sigma,$$

where h_σ is a measurable function; we sometimes use the notation $h_\sigma = \phi r_{G\sigma}\psi$. Similarly, $\sum_G(\phi, R_{0\sigma}\psi) = \int_E \bar{\phi} r_{0\sigma}\psi$, where $r_{0\sigma}$ is a measurable function depending on G. Let $|R_{0G}| = |R_0|$ be the bilinear form associated with $|r_0| = |r_{0\infty}|$.

Lemma 3.1. *Let* $X_j(\sigma) = V_{j\sigma}^* \overset{\circ}{\underset{v}{-}} V'_{j\sigma}$. *Suppose* $\bar{v}v'$ *is non-negative. Let* $m(j) = \min(n(j), n'(j))$. *Let* ξ *denote a fixed value of the variables of* $r_{G\sigma}$, $r_{0\sigma}$. *Then*

$$r_{G\sigma}(\xi) = \prod_j e^{-X_j(\sigma)} \exp_{m'(j)} X_j(\sigma)\, r_{0\sigma}(\xi) \tag{3.2}$$

where $m'(j) \leqq m(j)$ *depends on* G *and* ξ, $m'(j)$ *is independent of* σ, $m'(j) = m(j)$ *for almost all* j, *and* $m'(j) = 0$ *for* $j < \max(k, l)$.

Proof. Suppose G has $p\,V_\sigma^*$ and $q\,V'_\sigma$ vertices. It is convenient to regard $T_{k\sigma}^* \Pi Y_i T'_{l\sigma}$ as a power series in $V_{j\sigma}^*$, Y_i, $V'_{j\sigma}$. Let $v_j = v_{j\infty}$. A Wick ordered term will be called *reduced* if it has no $X_j = X_j(\infty) = v!(v_j, v'_j)$ components. Then G is a reduced graph from

$$\prod_{j\geqq k} V_{j\sigma}^{*p(j)}/p(j)!\, \Pi Y_i \prod_{j\geqq l} V_{j\sigma}^{\prime q(j)}/q(j)!,$$

and $r_{0\sigma}(\xi)$ is the value at ξ of the measurable function associated to G. Here $p(j)$, $q(j)$ are determined by G and ξ. Note that

$$\Sigma p(j) = p, \quad \Sigma q(j) = q, \quad p(j) \leqq n(j), \quad q(j) \leqq n'(j).$$

Then $e^{X(\sigma)} R_{G\sigma}$ is the sum of all terms (after integration over the variables in the X_j components) whose reduced graph is G. Consider all such terms with $x(j) X_j$ components, $j \geqq \max(k, l)$, and hence $p(j) + x(j) V_j^*$ and $q(j) + x(j) V'_j$ vertices. By the truncations in $T_{k\sigma}$, $T'_{l\sigma}$,

$$0 \leqq x(j) \leqq \min(n(j) - p(j), n'(j) - q(j)) = m'(j), \quad j \geqq \max(k, l)$$

$$x(j) = 0 = m'(j), \quad j < \max(k, l).$$

There are

$$\binom{p(j) + x(j)}{x(j)} \binom{q(j) + x(j)}{x(j)} x(j)!$$

ways to contract $2x(j)$ vertices into $x(j)X_j$ components. The remaining legs are contracted according to G. Summing over all such sequences $\{x(j)\}$,

$$
\begin{aligned}
r_{G\sigma}(\xi) &= e^{-X(\sigma)} \prod_{j \geqq \max(k,l)} p(j)!\, q(j)!\, [(p(j)+x(j))!\, (q(j)+x(j))!]^{-1} r_{0\sigma}(\xi) \\
&\quad \cdot \sum_{0 \leqq x(j) \leqq m'(j)} \binom{p(j)+x(j)}{x(j)} \binom{q(j)+x(j)}{x(j)} x(j)!\, X_j(\sigma)^{x(j)} \qquad (3.2)\\
&= e^{-X(\sigma)} \sum_{0 \leqq x(j) \leqq m'(j)} \prod_{j \geqq \max(k,l)} X_j(\sigma)^{x(j)}/x(j)!\, r_{0\sigma}(\xi) \\
&= \prod_j e^{-X_j(\sigma)} \exp_{m'(j)} X_j(\sigma)\, r_{0\sigma}(\xi)\,.
\end{aligned}
$$

Clearly, $m'(j)$ has the desired properties.

Lemma 3.2. *Let*

$$c(j_0, \sigma) = c(v, v', j_0, \sigma) = \prod_{j \geqq j_0} e^{-X_j(\sigma)} \exp_{m(j)} X_j(\sigma)\,. \qquad (3.3)$$

Then for all $j_0 \geqq 0$,

$$0 < \lim_{\sigma \to \infty} c(j_0, \sigma) \leqq 1 \qquad (3.4)$$

exists. Moreover

$$\lim_{j_0 \to \infty} c(v, v', j_0, \sigma) = 1 \qquad (3.5)$$

uniformly in σ, v, v', *such that* $X_j = v!(v_j, v'_j) \leqq K$.

Proof. Let

$$a_j(\sigma) = e^{-X_j(\sigma)} \exp_{m(j)} X_j(\sigma)\,.$$

By (2.6), for $\sigma \geqq \eta(j)$,

$$X_j(\sigma) \leqq v!\, \|v_{j,j+1}\|\, \|v'_{j,j+1}\| < K(\log\alpha + 1)^2$$

uniformly in σ. Thus $X_j(\infty)$ and $a_j(\infty)$ exist and $0 \leqq a_j(\sigma)$, $a_j(\infty) \leqq 1$. Thus, by [1, p. 190], it suffices to show that

$$\sum_j (1 - a_j(\sigma)) \leqq K\,,$$

where K is independent of σ.

Observe that, for $x > 0$,

$$1 - e^{-x} \exp_n x \leqq x^{n+1}/(n+1)!\,.$$

Therefore, since $m(j)$ is strictly increasing,

$$\sum_j (1 - a_j(\sigma)) \leqq \sum_j X_j(\sigma)^{m(j)+1}/(m(j)+1)!$$
$$\leqq 2\sum_j K^j/j! \leqq K .$$

The lemma is proved.

Let $|\phi| \in F$ be the vector with n-particle component $|\phi_n|$.

Corollary. (3.1) *is equal to* $\int_E h_\sigma$, *where* $\{h_\sigma\}$ *is a family of measurable functions which converge pointwise as* $\sigma \to \infty$. *Moreover,* $|h_\sigma| \leqq h$, *where* h *is a measurable function on E, and*

$$\int_E h = \sum_G (|\phi|, |R_{0G}|\, |\psi|) . \tag{3.6}$$

Proof. By (3.4), $j_0 = 0$, $r_{G\sigma}(\xi) \to c r_0(\xi)$, where c is some constant. Hence $h_\sigma = \phi r_{G\sigma} \psi$ converges pointwise as $\sigma \to \infty$. Let

$$h = |\phi|\, |r_0|\, |\psi| .$$

Then, by (3.2), $|r_{G\sigma}| \leqq |r_0|$, and $|h_\sigma| \leqq h$. Q.E.D.

Thus, in order to remove the momentum cutoff in (3.1), it suffices to show that h is integrable, i.e. (3.6) is bounded. Then (3.1) possesses a limit as $\sigma \to \infty$ by the bounded convergence theorem. In order to bound (3.6), we strengthen the hypothesis. Let

$$J_m = \{y \in L_2(R^{dm}) : \Pi \mu^\varepsilon y \in L_2(R^{dm}), \quad \text{some } \varepsilon > 0\} .$$

We consider two cases:

a) $v - v' \in J_\nu$, $Y_i = \int y_i \prod_{j=1}^{s_i} a^\#(k_j) dk$, $y_i \in J_{s_i}$, $1 \leqq i \leqq m$,

b) $v = v'$, $Y_i = Y = \int y(k)\, a^*(k)\, a(k) dk$, $1 \leqq i \leqq m$,

where y is a bounded non-negative measurable function such that

$$(v, \Pi \mu^\varepsilon Y^j v) = \int \bar{v} \Pi \mu^\varepsilon \left[\sum_{i=1}^{\nu} y(k_i)\right]^j v \leqq K^j, \quad 1 \leqq j \leqq m . \tag{3.7}$$

Lemma 3.3. *Let* $n = |G|$ *be the number of* V^* *or* V' *vertices in G, and let* $s = \sum_{i=1}^m s_i$. *Then the following estimates are valid:*

a) $(|\phi|, |R_0|\, |\psi|) \leqq K^{s+n} \prod_{i=1}^m \|\Pi \mu^\varepsilon y_i\| \, \alpha^{-\varepsilon c n^{1+\delta}}$,

b) $(|\phi|, |R_0|\, |\psi|) \leqq K^{m+n} \alpha^{-\varepsilon c n^{1+\delta}}$

where $c, \delta > 0$.

Proof. a) First, v, v' satisfy (2.6) as follows. Let $v'' = v - v' \in J_\nu$, and $1 \leqq r \leqq \nu - 1$. Let w, w', and w'' be the kernels of $V^* \underset{r}{-\circ-} V$, $V^* \underset{r}{-\circ-} V'$, and $V^* \underset{r}{-\circ-} V''$ respectively. Then $\int_I \Pi \mu^\varepsilon |w| \in L_2$ by (2.6). Moreover, by the Cauchy-Schwarz inequality,

$$\left\| \int_I \Pi \mu^\varepsilon |w''| \right\| \leqq \|\Pi \mu^{-\varepsilon} v\| \; \|\Pi \mu^{2\varepsilon} v''\| < \infty$$

for ε sufficiently small. Thus,

$$\int_I \Pi \mu^\varepsilon |w'| \leqq \int_I \Pi \mu^\varepsilon |w| + \int_I \Pi \mu^\varepsilon |w''| \in L_2 \,.$$

Observe that in case (a), $|r_0|$ is the kernel of $|R_0|$. Let B be the bilinear form given by the graph of $|R_0|$ and the kernel $\Pi_e \mu^{-a} |r_0|$, where $a > 0$. We may suppose that ϕ_j, ψ_j vanish off a sphere of radius ϱ in R^{dj}, $j \geqq 0$, and ϕ, ψ have n_0 or fewer particles. Then, by (2.5)

$$\begin{aligned} (|\phi|, |R_0| \, |\psi|) &\leqq \sum_{i,j \leqq n_0} (\Pi \mu^a |\phi_i|, B \Pi \mu^a |\psi_j|) \\ &\leqq \varrho^{a n_0} (|\phi|, B |\psi|) \\ &\leqq \varrho^{a n_0} n_0^{(s+\nu n)/2} \|\phi\| \; \|\psi\| \; \|B'\| \\ &\leqq K^{s+n} \left\| \Pi_e \mu^{-a} \int_I |r_0| \right\| . \end{aligned} \tag{3.8}$$

Since $n(j), n'(j)$ are polynomially bounded,

$$\sum_{i=0}^{j-1} (n(i) + n'(i)) < j^\gamma, \qquad j \geqq 0 \,,$$

for some integer γ. Then there are fewer than $j^\gamma = i$ vertices from $T^*_{k\sigma}$ or $T'_{l\sigma}$ with the magnitude of the largest momentum less than $\alpha^j = \alpha^{i^\delta}$, for $\delta = \gamma^{-1} < 1$. Thus

$$\left\| \Pi_e \mu^{-a} \int_I |r_0| \right\| \leqq K^{s+n} \left\| \Pi_e \mu^{-a} \int_I \Pi \mu^\varepsilon |r_0| \right\| \prod_{i=1}^{[(n+1)/2]} \alpha^{-\varepsilon i^\delta} \tag{3.9}$$

because G contains at least $[(n+1)/2]$, the greatest integer less than or equal $(n+1)/2$, vertices from one of T^*_k or T'_l. K compensates in the region of small $|k|$ for factors $\mu(k)^\varepsilon$, and $\alpha^{\varepsilon i^\delta} \leqq \mu(k)^\varepsilon$ for some of the factors.

The first factor in (3.9) is bounded by $K^{s+n} \prod_i \|\Pi \mu^\varepsilon y^i\|$, by Lemma 2.1 and the Cauchy-Schwarz inequality in the contracted variables between $T^*_k T'_l$ and ΠY_i. The exponent of the second factor is bounded by

$$- \sum_{i=1}^{[(n+1)/2]} \varepsilon i^\delta \leqq -\varepsilon \int_0^{n/2} i^\delta di \leqq -\varepsilon c n^{1+\delta}$$

where $c=(1+\delta)2^{-(1+\delta)}$. The lemma follows, in this case, from (3.8) and (3.9).

b) Here $|R_0|$ does not have a measurable kernel since it contains δ functions. Observe that Y conserves the number of particles. Thus, every reduced graph from case (b) is the disjoint union of three subgraphs. The first is the union of all components without any V^*, V' vertices; the second is the union of all components which have exactly one V^* and one V' vertex, at least one Y vertex, and no external legs, for example

$$V^* \underset{\nu-1}{-\circ-} (Y \underset{1}{-\circ-} V') .$$

The third subgraph is a reduced graph from case (a), $s=0$; of course, the corresponding kernel does not arise from case (a) because of the Y's. Some of the legs from a case (a) reduced graph are replaced by legs contracted with one or more Y vertices.

Therefore $|R_0|$ is a product of three factors, corresponding to the three subgraphs. The first factor is estimated by $Y^j|\phi| \leqq K^j|\phi|$, $Y^j|\psi| \leqq K^j|\psi|$. The second factor is estimated by (3.7). The third factor has a measurable kernel $|r'_0|$ which can be estimated by $Y^j|v_\sigma| \leqq K^j|v_\sigma|$, uniformly in σ. Thus, by (3.8) and (3.9)

$$\begin{aligned}(|\phi|, |R_0|\,|\psi|) &\leqq K^{m+n}\left\| \Pi_e \mu^{-a} \int_I |r_0| \right\| \\ &\leqq K^{m+n} \alpha^{-\varepsilon c n^{1+\sigma}} .\end{aligned} \tag{3.10}$$

The lemma is proved.

Lemma 3.4. *There are at most*

a) $K^{s+n}(s/2)!\,(\nu n)!^2$

or b) $K^{m+n} m!\,(\nu n)!^2$

reduced graphs G *such that* $|G|=n$.

Proof. We overestimate by bounding the number of contraction schemes, so that schemes which produce the same graph are counted separately. We use the fact that, for a, b positive integers,

$$\sum_{0 \leqq i \leqq \min(a,b)} \binom{a}{i}\binom{b}{i} i! \leqq \min(a,b)!\, K^{a+b} .$$

G can have pV^* vertices, $0 \leqq p \leqq n$. For fixed p, there are at most

$$\min(\nu p, \nu n - \nu p)!\; K^{\nu p + \nu n - \nu p} \leqq (\nu n)!\, K^n$$

possible contraction schemes between the V^* and V' vertices.

In case (a) there are at most

$$\min(s, \nu n)!\; K^{s+\nu n} \leqq (\nu n)!\, K^{s+n}$$

possible contraction schemes between ΠY_i and the V^*, V' vertices. In case (b), the corresponding bound is $(\nu n)!\,K^{m+n}$.

Suppose ΠY_i has q creating legs in case (a). Then there are at most

$$\min(q, s-q)!\, K^{q+s-q} \leqq (s/2)!\, K^s$$

possible contraction schemes among the Y_i. In case (b) the corresponding bound is $m!\,K^m$.

There are $n+1$ choices for p. The lemma follows from these estimates.

Lemma 3.5. *Let* $T_k = T_k(v, \alpha, n(j))$, $T_l' = T_l(v', \alpha, n'(j))$, Y_i *be as in Lemma 3.2. Then for* $\phi, \psi \in D$

$$\left(T_{k\sigma}\phi, \prod_{i=1}^{m} Y_i T_{l\sigma}' \psi\right) e^{-X(\sigma)} = \int_E h_\sigma \tag{3.11}$$

and both limits

$$\lim_{\sigma\to\infty} \int_E h_\sigma = \int_E \lim_{\sigma\to\infty} h_\sigma \tag{3.12}$$

exist, where E is a measure space, $\{h_\sigma\}$ *is a family of measurable functions dominated by an integrable function h. Moreover*

(a) $\int_E h \leqq K^s (s/2)! \prod_{i=1}^{m} \|\Pi \mu^\varepsilon y_i\|$

or (b) $\int_E h \leqq K^m m!$.

Proof. Let E, $\{h_\sigma\}$, h be given by the Corollary to Lemmas 3.1, 3.2. The lemma then follows from the bounded convergence theorem and a bound on (3.6). By Lemmas 3.3 and 3.4, (3.6) is bounded by

(a) $K^s (s/2)! \prod_{i=1}^{m} \|\Pi \mu^\varepsilon y_i\| \left(\sum_n K^n (\nu n)!^2 \alpha^{-\varepsilon c n^{1+\delta}}\right)$

or (b) $K^m m! \left(\sum_n K^n (\nu n)!^2 \alpha^{-\varepsilon c n^{1+\delta}}\right)$.

But

$$\begin{aligned}\sum_n K^n (\nu n)!^2 \alpha^{-\varepsilon c n^{1+\delta}} &\leqq \sum_n K^n e^{2\nu n \log \nu n} \alpha^{-\varepsilon c n^{1+\delta}} \\ &\leqq \sum_n \alpha^{K n \log n - \varepsilon c n^{1+\delta}} \\ &\leqq K + \sum_{n \geqq n_0} \alpha^{-(\varepsilon c/2) n^{1+\delta}}, \text{ some } n_0 \\ &\leqq K < \infty .\end{aligned}$$

Theorem 1. *Let $T_j = T_j(v, \alpha, n(k))$, Then for $\phi, \psi \in D$, the limit*

$$\lim_{\sigma\to\infty} (T_{k\sigma}\phi, T_{l\sigma}\psi) e^{-\Lambda(\sigma)} = (T_k\phi, T_l\psi)_r \tag{3.13}$$

exists. If $k \geqq l$, $\sigma \geqq \alpha(k)$, then $T_{l\sigma}\psi = T_{k\sigma}\theta$, where

$$\theta = \prod_{j=l}^{k-1} \exp_{n(j)} V_{j\sigma}\psi \in D\,. \tag{3.14}$$

$(\cdot,\cdot)_r$ provides a positive definite inner product for $\mathscr{D} = \bigcup_{j\geqq 0} T_j D$, whose completion is denoted F_r.

Proof. (3.13) is the special case of (3.11) in which $v = v'$, $n(k) = n'(k)$, and $s = 0$. (3.14) follows from the definition of $T_{j\sigma}$. $(\cdot,\cdot)_r$ is clearly an inner product for $\mathscr{D}$. It remains to show that it is positive definite, i.e. if $0 \neq \psi \in D$, then $\|T_j\psi\|_r^2 = (T_j\psi, T_j\psi)_r > 0$.

Let m be the smallest integer such that $\psi_m \neq 0$. Let τ be a lower cutoff on the magnitude of the smallest momentum:

$$\begin{aligned} v^\tau(k) &= v(k), \qquad \min_{1\leqq i\leqq v} |k_i| \geqq \tau \\ &= 0, \qquad \text{otherwise}\,. \end{aligned}$$

Then $T_j^\tau = T_j(v^\tau, \alpha, n(k))$ is an exponential dressing transformation. Suppose that ψ_n vanishes off a sphere of radius τ_0 in R^{dn}, $n \geqq 0$. Let $\sigma \geqq \alpha(k) \geqq \tau_0$ and let $\sum_{n\geqq 0} nP_n(B_{\alpha(k)})$ be the spectral decomposition of $N(B_{\alpha(k)})$. Then

$$\begin{aligned} \|T_{j\sigma}\psi\|^2 \geqq \|P_m(B_{\alpha(k)})\,T_{j\sigma}\psi\|^2 &= \|P_m(B_{\alpha k)})\,T_{j\sigma}\psi_m\|^2 \\ &= \|T_{j\sigma}^{\alpha(k)}\psi_m\|^2 = \|T_{k\sigma}^{\alpha(k)}\psi_m\|^2\,, \end{aligned}$$

because $T_{j\sigma}$ does not annihilate any particles. Let

$$\Lambda^k(\sigma) = v!\,\|v_\sigma^{\alpha(k)}\|^2, \qquad \Lambda_j^k(\sigma) = \|v_{j\sigma}^{\alpha(k)}\|^2\,.$$

Now

$$(\psi_m, T_{k\sigma}^{\alpha(k)*}\,T_{k\sigma}^{\alpha(k)}\psi_m) \tag{3.15}$$

is a sum of terms given by graphs from $T_{k\sigma}^{\alpha(k)*}\,T_{k\sigma}^{\alpha(k)}$. By (3.2), the sum over all graphs with the empty set for reduced graph is given explicitly by

$$\prod_{l\geqq k} \exp_{n(l)} \Lambda_l^k(\sigma)\,\|\psi_m\|^2\,,$$

which is equal to

$$\exp \Lambda^k(\sigma)\,(\|\psi_m\|^2 + \delta(k,\sigma))$$

by (3.5), where $\delta(k,\sigma)\to 0$ as $k\to\infty$ uniformly in σ.

We may bound the other contributions to (3.15) as follows. Since $G \neq \emptyset$ for these terms,

$$\left\| \Pi_e \mu^{-a} \int_I \Pi \mu^{\varepsilon} |r_0| \right\| \leqq K\alpha(k)^{-\varepsilon} \left\| \Pi_e \mu^{-a} \int_I \Pi \mu^{2\varepsilon} |r_0| \right\| .$$

Thus, by (3.8), (3.9), Lemmas 3.1, 3.3, and 3.4, these contributions are bounded by $K\alpha(k)^{-\varepsilon} \exp \Lambda^k(\sigma)$. Hence

$$\begin{aligned} \|T_{j\sigma}\psi\|^2 e^{-\Lambda(\sigma)} &\geqq e^{-\Lambda(\sigma)} \exp \Lambda^k(\sigma) \left(\|\psi_m\|^2 - \delta(k,\sigma) - K\alpha(k)^{-\varepsilon}\right) \\ &\geqq \exp\left(-\nu! \sum_{i=1}^{\nu} \int_{|k_i| \geqq \alpha(k)} |v|^2 \right) \left(\|\psi_m\|^2 - \delta(k,\sigma) - K\alpha(k)^{-\varepsilon}\right) \\ &\geqq \exp(-K \log \alpha(k)) \left(\|\psi_m\|^2 - \delta(k,\sigma) - K\alpha(k)^{-\varepsilon}\right) \\ &= \alpha(k)^{-K} \left(\|\psi_m\|^2 - \delta(k,\sigma) - K\alpha(k)^{-\varepsilon}\right) \\ &> 0 \end{aligned}$$

for k sufficiently large. Therefore $\|T_j\psi\|_r > 0$. The theorem is proved.

Lemma 3.6. *Let Y_i be as in Lemma 3.5. Then $\lim_\sigma \prod_{i=1}^m Y_i$ exists and*

a) $\left\| \lim_\sigma \prod_{i=1}^m Y_i T_k \psi \right\|_r \leqq K^s s!^{1/2} \prod_{i=1}^m \|\Pi \mu^{\varepsilon} y_i\|$

or b) $\|\lim_\sigma Y^m T_k \psi\|_r \leqq K^m m!$.

Proof. By Lemma 3.5, the limits

$$\lim_{\sigma \to \infty} \left(T_{k\sigma}\phi, \prod_{i=1}^m Y_i T_{l\sigma}\psi \right) e^{-\Lambda(\sigma)}, \tag{3.16}$$

$$\lim_{\sigma \to \infty} \left\| \prod_{i=1}^m Y_i T_{l\sigma}\psi \right\|^2 e^{-\Lambda(\sigma)} \tag{3.17}$$

exist. Hence (3.16) is bounded by

$$\|T_k \phi\|_r \lim_{\sigma \to \infty} \left\| \prod_{i=1}^m Y_i T_{l\sigma}\psi \right\| e^{-\Lambda(\sigma)/2},$$

and a bound on (3.17) is obtained in Lemma 3.5. The lemma follows from the Riesz representation theorem.

Corollary. *Let $f_i \in J$. Then $\lim_\sigma \prod_{i=1}^m \phi(f_i)$ exists, and*

$$\left\| \lim_\sigma \prod_{i=1}^m \phi(f_i) T_k \psi \right\|_r \leqq K^m m!^{1/2} \prod_{i=1}^m \|\mu^{\varepsilon} f_i\| . \tag{3.18}$$

Let Y be $N(B_\varrho)$, N_τ, $\tau<0$, or $H_0(B_\varrho)$. Then $\lim_\sigma Y^m$ exists, and

$$\left\|\lim_\sigma Y^m T_k \psi\right\|_r \leqq K^m m!\,. \tag{3.19}$$

Proof. $\prod_{i=1}^m \phi(f_i)$ is the sum of 2^m terms of the form $\prod_{i=1}^m Y_i$, where $Y_i = a^\#(f_i)$, and $a^\#(f)$ denotes $a^*(f)$ or $a(\bar f)$. Thus, the existence of $\lim_\sigma \prod_{i=1}^m \phi(f_i)$ and (3.18) follow from Lemma 3.6. The existence of $\lim_\sigma Y^m$ and (3.19) also follow from Lemma 3.6, provided that (3.7) is established for $Y = N(B_\varrho)$, N_τ, or $H_0(B_\varrho)$. This is verified as follows:

$$(v, \Pi\,\mu^\varepsilon N(B_\varrho)^j v) \leqq K v^j \mu(\varrho)^{2a} \|\Pi\,\mu^\varepsilon \mu_1^{-a} v\|^2 \leqq K^j\,,$$

$$(v, \Pi\,\mu^\varepsilon N_c^j v) \leqq K v^j \|\mu_1^\tau v\|^2 \leqq K^j\,,$$

$$(v, \Pi\,\mu^\varepsilon H_0(B_\varrho)^j v) \leqq K^j v^j \mu(\varrho)^j\, \mu(\varrho)^{2a} \|\Pi\,\mu^\varepsilon \mu_1^{-a} v\|^2 \leqq K^j\,,$$

for $j \geqq 1$, by (2.6).

Let $f\in J$, $\|f\| = 1$, and let $N(f) = a^*(f)\,a(\bar f)$ be the number operator over $\{f\}$ for W [2]. $N(f)$ measures the number of particles with wave function f. Let $T_{kl} = T_{k\alpha(l)}$. We need the following estimates for Theorem 2.

Lemma 3.7. *Let $F_r = F_r(v, \alpha, n(k))$. Let A be $\phi(f)$, $N(f)$, $N(B_\varrho)$, $H_0(B_\varrho)$, or N_τ. Let $\varepsilon>0$. Then for $p, l \geqq k$ sufficiently large and $|t| \leqq t_0$, some t_0:*

$$\sum_{m=p+1}^{\infty} \|(itA)^m/m!\, T_{k\sigma}\psi\|\, e^{-A(\sigma)/2} \tag{3.20}$$

$$+ \sum_{m=1}^{p} \|[(itA)^m/m!, T_{l\sigma}]\, T_{kl}\psi\|\, e^{-A(\sigma)/2} \tag{3.21}$$

$$\leqq \varepsilon\,.$$

Proof. By (3.18) or (3.19), (3.20) is bounded by

$$\sum_{m=p+1}^{\infty} (K|t_0|)^m \leqq \varepsilon/2$$

for p sufficiently large and t_0 sufficiently small. To estimate (3.21), observe that

$$T_{kl}^* [A^m, T_{l\sigma}]^* [A^m, T_{l\sigma}]\, T_{kl}$$

is the sum of all Wick ordered terms from

$$T_{k\sigma}^* A^{2m} T_{k\sigma}$$

in which there is at least one contraction between A and $V_{l\sigma}^*$ or $V_{l\sigma}$. Thus, by Lemma 2.1, the Cauchy-Schwarz inequality in one of the above contracted variables, (3.8)–(3.10), and Lemmas 3.1–3.4, (3.21) is bounded either by

$$K\|\mu^{a+\varepsilon}f\|\;\|\mu_1^{-a}\Pi\mu^{\varepsilon}v_{l\infty}\| \tag{3.22}$$

or

$$K\|A v_{l\infty}\|\,. \tag{3.23}$$

(3.22) holds in case A is $\phi(f)$ or $N(f)$, and (3.23) holds in case A is $N(B_\varrho)$, $H_0(B_\varrho)$, or N_τ. By (2.6) or (3.7), (3.22) and (3.23) are less than $\varepsilon/2$ for $l \geqq k$ sufficiently large.

IV. Exponential Weyl Systems

In this chapter we prove some general results concerning unitary operators and n-parameter unitary groups in F_r which are defined by weak limits. Particular examples are $\lim_\sigma W(tf)$, $f\in J$, $\lim_\sigma e^{itN(B_\varrho)}$, and $\lim e^{ita^*(f)a(f)}$, $f\in J$. Properties of these one-parameter groups and exponential Weyl systems are studied in Theorems 2 and 3. Sufficient conditions for two exponential Weyl systems not to be disjoint (or to be unitarily equivalent) are given in Theorem 4.

For later purposes, we generalize the notion of weak limit. We say that an operator B, which maps a subspace of F_r' into F_r, is the weak limit of an operator A in F written $B = \operatorname{Lim}_\sigma A$ if $\mathscr{D}(B) = \mathscr{D}' = \bigcup_j T_j' D$ and

$$(T_k\phi, BT_l'\psi)_r = \lim_{\sigma\to\infty}(T_{k\sigma}\phi, AT_{l\sigma}'\psi)e^{-\Lambda(\sigma)}\,. \tag{4.1}$$

If $\operatorname{Lim}_\sigma A$ is bounded, then it extends uniquely by continuity to F_r'; the extension is also written $\operatorname{Lim}_\sigma A$. If $F_r = F_r'$, then $\operatorname{Lim}_\sigma A = \lim_\sigma A$.

Lemma 4.1. *Suppose that* $v - v' \in J_v$ *and* $\bar{v}v'$ *is non-negative. Suppose that* U, V *are unitary operators on* F *such that* $\operatorname{Lim}_\sigma U$ *is an operator mapping* F_r' *into* F_r, *and such that* $\lim_\sigma V$, $\lim_\sigma' V$ *are unitary operators on* F_r, F_r' *respectively. Then*

$$\operatorname{Lim}_\sigma U \lim_\sigma{}' V = \operatorname{Lim}_\sigma UV, \qquad \lim_\sigma V \operatorname{Lim}_\sigma U = \operatorname{Lim}_\sigma VU\,. \tag{4.2}$$

Proof. Choose $T_{l(n)}'\theta_n$ such that

$$\|T_{l(n)}'\theta_n - \lim_\sigma{}' VT_l'\psi\|_r' \to 0\,. \tag{4.3}$$

Let $w = v - v'$. By (2.6), (w, v') and (w, v) exist since $w \in J_v$. Note that

$$\begin{aligned}\lim_{\sigma\to\infty}\exp\tfrac{1}{2}(\Lambda'(\sigma)-\Lambda(\sigma)) &= \exp v!(-\|w\|^2/2 - \operatorname{Re}(w, v')) \\ &= c(v, v') > 0\,.\end{aligned} \tag{4.4}$$

Then

$$\left\| \operatorname*{Lim}_{\sigma} U(T'_{l(n)}\theta_n - \operatorname*{lim'}_{\sigma} V T'_l \psi)\right\|_r \to 0$$

and

$$\begin{aligned}
&\limsup_{\sigma\to\infty} \left|(T_k\phi, \operatorname*{Lim}_{\sigma} U \operatorname*{lim'}_{\sigma} V T'_l \psi)_r - (T_{k\sigma}\phi, U V T'_{l\sigma}\psi) e^{-A(\sigma)}\right| \\
&\quad \leqq \lim_{n\to\infty} \limsup_{\sigma\to\infty} |(U^* T_{k\sigma}\phi, T'_{l(n)\sigma}\theta_n - V T'_{l\sigma}\psi)| e^{-A(\sigma)} \\
&\quad \leqq c(v, v') \, \|T_k\phi\|_r \lim_{n\to\infty} \lim_{\sigma\to\infty} \|T'_{l(n)\sigma}\theta_n - V T'_{l\sigma}\psi\| e^{-A'(\sigma)/2} \\
&\quad \leqq K \lim_{n\to\infty} \left(\|T'_{l(n)}\theta_n\|'^2_r + \|T'_l\psi\|'^2_r - 2\operatorname{Re}\left(T'_{l(n)}\theta_n, \operatorname*{lim'}_{\sigma} V T'_l \psi\,'_r\right)^{1/2}\right) \\
&\quad = K\left(\|T'_l\psi\|'^2_r - \|\operatorname*{lim'}_{\sigma} V T'_l\psi\|'^2_r\right) = 0 .
\end{aligned}$$

Thus $\operatorname*{Lim}_{\sigma} UV$ exists and $\operatorname*{Lim}_{\sigma} U \operatorname*{lim'}_{\sigma} V = \operatorname*{Lim}_{\sigma} UV$.

Taking adjoints, we observe that

$$\begin{aligned}
\operatorname*{Lim}_{\sigma} V^* U^* &= \left(\operatorname*{Lim}_{\sigma} U \operatorname*{lim'}_{\sigma} V\right)^* = \left(\operatorname*{lim'}_{\sigma} V\right)^* \left(\operatorname*{Lim}_{\sigma} U\right)^* \\
&= \operatorname*{lim'}_{\sigma} V^* \operatorname*{Lim}_{\sigma} U^* .
\end{aligned}$$

Replacing V^*, U^* by V, U respectively, and interchanging F'_r and F_r, we conclude that the second equality in (4.2) follows from the first. The lemma is proved.

Corollary. *Suppose that U_i, $1 \leqq i \leqq m$, are unitary operators on F and $U_{ir} = \lim_{\sigma} U_i$ are unitary operators on F_r. Then*

$$\prod_{i=1}^{m} U_{ir} = \lim_{\sigma} \prod_{i=1}^{m} U_i . \tag{4.5}$$

Proof. The general case follows immediately from the case $m = 2$, which in turn follows from the lemma (with $F_r = F'_r$). Q.E.D.

It is natural to ask for a sufficient condition that $\operatorname*{Lim}_{\sigma} U$ be a unitary mapping from F'_r onto F_r.

Lemma 4.2. *Suppose U is a unitary operator on F and $\operatorname*{Lim}_{\sigma} U$ is an operator mapping F'_r into F_r. Suppose that for each $k, l \geqq 0$, there exist $\theta_n, \theta'_n \in D$ and $k(n), l(n) \geqq 0$ such that*

$$\lim_{n\to\infty} \lim_{\sigma\to\infty} \sup \|U T'_{l\sigma}\psi - T_{l(n)\sigma}\theta_n\| e^{-A(\sigma)/2} = 0$$

and

$$\lim_{n\to\infty} \limsup_{\sigma\to\infty} \|U^* T_{k\sigma}\psi - T'_{k(n)\sigma}\theta'_n\| e^{-A'(\sigma)/2} = 0 . \tag{4.6}$$

Then $\operatorname*{Lim}_{\sigma} U$ is a unitary operator mapping F'_r onto F_r.

Proof. By (4.1), $\left(\operatorname*{Lim}_{\sigma} U\right)^* = \operatorname*{Lim}_{\sigma} U^*$ is an operator mapping F_r into F_r'. It suffices to show that $\operatorname*{Lim}_{\sigma} U$ is isometric. Then, replacing U by U^* and interchanging F_r' and F_r, we conclude that $\left(\operatorname*{Lim}_{\sigma} U\right)^*$ is isometric, so that $\operatorname*{Lim}_{\sigma} U$ is unitary.

By (4.6),

$$\begin{aligned}
&\lim_{n\to\infty} \left|(T_k \phi, \operatorname*{Lim}_{\sigma} U T_l' \psi - T_{l(n)} \theta_n)_r\right| \\
&= \lim_{n\to\infty} \lim_{\sigma\to\infty} |(T_{k\sigma} \phi, U T_{l\sigma}' \psi - T_{l(n)\sigma} \theta_n)| \, e^{-\Lambda(\sigma)} \\
&\leqq \|T_k \phi\|_r \lim_{n\to\infty} \limsup_{\sigma\to\infty} \|U T_{l\sigma}' \psi - T_{l(n)\sigma} \theta_n\| \, e^{-\Lambda(\sigma)/2} \\
&= 0 .
\end{aligned}$$

Thus $T_{l(n)} \theta_n \to \operatorname*{Lim}_{\sigma} U T_l' \psi$ in F_r, so that

$$\begin{aligned}
\left|\|\operatorname*{Lim}_{\sigma} U T_{l\,l}' \psi\|_r^2 - \|T_l' \psi\|_r'^2\right| &= \lim_{n\to\infty} \left|\|T_{l(n)} \theta_n\|_r^2 - \|T_l' \psi\|_r'^2\right| \\
&\leqq \lim_{n\to\infty} \lim_{\sigma\to\infty} |(T_{l(n)\sigma} \theta_n - U T_{l\sigma}' \psi, U T_{l\sigma}' \psi)| \, e^{-\Lambda(\sigma)} \\
&\quad + \lim_{n\to\infty} \lim_{\sigma\to\infty} |(T_{l(n)\sigma} \theta_n, T_{l(n)\sigma} \theta_n - U T_{l\sigma}' \psi)| \, e^{-\Lambda(\sigma)} \\
&\leqq 2K \limsup_{\sigma\to\infty} \|U T_{l\sigma}' \psi - T_{l(n)\sigma} \theta_n\| \, e^{-\Lambda(\sigma)/2} \\
&= 0 .
\end{aligned}$$

The lemma is proved.

Lemma 4.3. *Suppose that $U(x)$ is an n-parameter unitary group on F and $\lim_{\sigma} U(x) = U_r(x)$ is an n-parameter unitary group on F_r. Let $\int_{R^n} e^{i(x,y)} dP_y$, $\int_{R^n} e^{i(x,y)} dP_{ry}$ be spectral decompositions of $U(x)$ and $U_r(x)$ respectively. If g is a bounded continuous function on R^n, then*

$$\int_{R^n} g(y) dP_{ry} = \lim_{\sigma} \int_{R^n} g(y) dP_y . \tag{4.7}$$

Suppose $n = 1$ and A, A_r are the infinitesimal generators for U, U_r respectively. If $\lim_{\sigma} A^j$ exists for $j \geqq 0$, then $A_r^j \supset \lim_{\sigma} A^j$, $j \geqq 0$, so that $\mathscr{D}$ is a dense set of C^∞ vectors for A_r.

Proof. By definition,

$$\begin{aligned}
\int_{R^n} e^{i(x,y)} d\|P_{ry} T_l \psi\|_r^2 &= (T_l \psi, U_r(x) T_l \psi)_r \\
&= \lim_{\sigma\to\infty} (T_{l\sigma} \psi, U(x) T_{l\sigma} \psi) e^{-\Lambda(\sigma)} \\
&= \lim_{\sigma\to\infty} \int_{R^n} e^{i(x,y)} d(\|P_y T_{l\sigma} \psi\|^2 e^{-\Lambda(\sigma)}) .
\end{aligned}$$

By the Levy continuity theorem for distributions [13, p. 191, p. 205] and the Helly-Bray Theorem [13, p. 182],

$$\int g(y)\,d\,\|P_{ry}T_l\psi\|_r^2=\lim_{\sigma\to\infty}\int g(y)\,d(\|P_yT_{l\sigma}\psi\|^2e^{-A(\sigma)}).$$

(4.7) is then obtained by polarization.

Suppose $n=1$ and $\int_R \lambda\,dP_\lambda$, $\int_R \lambda\,dP_{r\lambda}$ are the spectral decompositions of A, A_r respectively. Then, by the moment convergence theorem for distributions [13, p. 184],

$$\int \lambda^j d\,\|P_{r\lambda}T_l\psi\|_r^2=\lim_{\sigma\to\infty}\int \lambda^j d(\|P_\lambda T_{l\sigma}\psi\|^2e^{-A(\sigma)}),$$

so that $\mathscr{D}(A_r^j)\supset\mathscr{D}$ for all $j\geqq 0$. Again by polarization,

$$\begin{aligned}(T_k\phi, A_r^jT_l\psi)_r&=\int\lambda^j d(T_k\phi, P_{r\lambda}T_l\psi)_r\\&=\lim_{\sigma\to\infty}\int\lambda^j d((T_{k\sigma}\phi, P_\lambda T_{l\sigma}\psi)e^{-A(\sigma)})\\&=\lim_{\sigma\to\infty}(T_{k\sigma}\phi, A^jT_{l\sigma}\psi)e^{-A(\sigma)}.\end{aligned}$$

This completes the proof.

Theorem 2. *Let $F_r=F_r(v,\alpha,n(k))$. Let A be $\phi(f)$, $N(f)$, $N(B_\varrho)$, $H_0(B_\varrho)$, or N_τ. Let $V(t)=e^{itA}$. Then $V_r(t)=\lim_\sigma V(t)$ exists and defines a one-parameter unitary group e^{itA_r} on F_r. $A_r^j\supset\lim_\sigma A^j$, $j\geqq 0$, and $\mathscr{D}$ is a dense set of entire vectors for $\phi_r(f)$ and analytic vectors for the remaining self adjoint generators.*

Corollary. *$W_r(f)=\lim_\sigma W(f)=e^{i\phi_r(f)}$, $f\in J$, is a Weyl system on F_r. If $\|\mu^\varepsilon(f_n-f)\|\to 0$, some $\varepsilon>0$, then $\phi_r(f_n)\to\phi_r(f)$ and $W_r(f_n)\to W_r(f)$ strongly on $\mathscr{D}$.*

Definition. $W_r(f)$, $f\in J$, is called an *exponential Weyl system.*

Let

$$a_r^*(f)=2^{-1/2}[\phi_r(f)+i\phi_r(if)],$$
$$a_r(f)=2^{-1/2}[\phi_r(f)-i\phi_r(if)]$$

be the creation and annihilation operators for the representation. Then $a_r^*(f)\,a_r(\bar f)$ is the self adjoint number operator over $\{f\}$ for W_r [2].

Lemma 4.4. *$N_r(f)=a_r^*(f)\,a_r(\bar f)$. The spectra of $N_r(f)$ and $N_r(B_\varrho)$ are the non-negative integers, and, for $f\in L_2(B_\varrho)$,*

$$e^{itN_r(B_\varrho)}W_r(f)e^{-itN_r(B_\varrho)}=W_r(e^{it}f). \tag{4.8}$$

Theorem 3. *W_r is locally Fock: $W_r(f)$, $f\in L_2(B_\varrho)$, is unitarily equivalent to a direct sum of Fock representations. If $v\notin L_2(R^{d\nu})$, then every vector in*

F_r has an infinite number of particles for the global system $W_r(f)$, $f \in J$. $W_r(f)$, $f \in J$, is disjoint from the Fock representation.

Proof of Theorem 2. By (3.18) or (3.19), and the bounded convergence theorem,

$$V_r(t) = \lim_\sigma \sum_{m=0}^{\infty} (itA)^m/m! = \sum_{m=0}^{\infty} \lim_\sigma (itA)^m/m! \tag{4.9}$$

exists for $|t| \leqq t_0$, some t_0. Since $V(t)$ is unitary, $V_r(t)$ is bounded in norm by one and extends uniquely to F_r. Now $V(t)^* = V(-t)$ so that $V_r(t)^* = V_r(-t)$. We first prove that $V_r(t)$ is unitary. Let $\psi \in D$ and let

$$\theta = \sum_{m=0}^{p} (itA)^m/m! \, T_{kl}\psi \in D .$$

Then by Lemma 4.2, it suffices to show that, given $\varepsilon > 0$,

$$\|V(t) T_{k\sigma}\psi - T_{l\sigma}\theta\| \, e^{-A(\sigma)/2} \leqq \varepsilon \tag{4.10}$$

for $p, l \geqq k$ sufficiently large, $|t| \leqq t_0$. Now (4.10) is bounded by

$$\sum_{m=p+1}^{\infty} \|(itA)^m/m! \, T_{k\sigma}\psi\| \, e^{-A(\sigma)/2}$$
$$+ \sum_{m=1}^{p} \|[(itA)^m/m!, T_{l\sigma}] \, T_{kl}\psi\| \, e^{-A(\sigma)/2} .$$

By Lemma 3.7, this is bounded by ε for $p, l \geqq k$ sufficiently large, $|t| \leqq t_0$.

Thus $V_r(t)$, $|t| \leqq t_0$, is unitary. By (4.5), we may define $V_r(t)$, for all t, by

$$V_r(t) = V_r(t/m)^m = \lim_\sigma V(t/m)^m = \lim_\sigma V(t) , \tag{4.11}$$

where m is chosen such that $|t/m| \leqq t_0$. Thus, by (4.5),

$$V_r(t) \, V_r(t') = \lim_\sigma V(t) \, V(t') = \lim_\sigma V(t+t') = V_r(t+t') .$$

Moreover, by (3.18) or (3.19), and (4.9), $V_r(t)$ is weakly continuous at $t = 0$:

$$|(T_k\phi, (V_r(t) - I_r) \, T_l\psi)_r| \leqq \sum_{m=1}^{\infty} (K|t|)^m$$
$$= K|t| \, (1 - K|t|)^{-1} \to 0$$

as $t \to 0$.

Thus $V_r(t)$ is a one-parameter unitary group. Let $A_r(t)$ be its self adjoint generator. By Lemma 4.3, (3.18), and (3.19), $A_r^j \supset \lim_\sigma A^j$, $j \geqq 0$, and $\mathscr{D}$ is a dense set of entire vectors for $\phi_r(f)$ and analytic vectors for the remaining self adjoint generators.

Proof of Corollary. By (4.5), W_r satisfies the Weyl relations:

$$W_r(f)\,W_r(g) = \lim_\sigma W(f)\,W(g) = \lim_\sigma e^{i\,\mathrm{Im}(f,g)/2}\,W(f+g)$$
$$= e^{i\,\mathrm{Im}(f,g)/2}\,W_r(f+g)\,.$$

We now prove the regularity properties. By the real linearity of and the Weyl relations for W_r, it suffices to show that $\|\mu^\varepsilon f_n\|\to 0$ implies that $\phi_r(f_n)\to 0$ and $W_r(f_n)\to I_r$ strongly on $\mathscr{D}$. This follows directly from (3.18) and (4.9):

$$\|\phi_r(f_n)\,T_k\psi\|_r \leqq K\,\|\mu^\varepsilon f_n\|\to 0$$

and

$$\|(W_r(f_n)-I_r)\,T_k\psi\|_r \leqq \sum_{m=1}^{\infty} (K\,\|\mu^\varepsilon f_n\|)^m m!^{-1/2}$$
$$\leqq K\,\|\mu^\varepsilon f_n\|\to 0$$

as $n\to\infty$.

Proof of Lemma 4.4. For the first conclusion, it suffices to prove that for $f, g\in J$,

$$\phi_r(f)\,\phi_r(g)\supset \lim_\sigma \phi(f)\,\phi(g)\,. \tag{4.12}$$

Then by Theorem 2,

$$\begin{aligned} N_r(f)\supset \lim_\sigma N(f) &= \lim_\sigma a^*(f)\,a(\bar f)\\ &= \lim_\sigma 2^{-1/2}[\phi(f)+i\phi(if)]\,2^{-1/2}[\phi(\bar f)-i\phi(-i\bar f)]\\ &\subset 2^{-1/2}[\phi_r(f)+i\phi_r(if)]\,2^{-1/2}[\phi_r(\bar f)-i\phi_r(-i\bar f)]\\ &= a_r^*(f)\,a_r(\bar f)\,. \end{aligned}$$

Thus $N_r(f)$ and $a_r^*(f)\,a_r(\bar f)$ are self adjoint operators which agree on a dense set of analytic vectors. Hence they are equal.

(4.12) is established as follows.

$$\begin{aligned} \phi_r(f)\,\phi_r(g)\,T_k\psi &= \lim_{s,t\to 0}(W_r(sf)-I_r)\,(is)^{-1}\,(W_r(tg)-I_r)\,(it)^{-1}\,T_k\psi\\ &= \lim_{s,t\to 0}\lim_\sigma (W(sf)-I)\,(is)^{-1}\,(W(tg)-I)\,(it)^{-1}\,T_k\psi\\ &= \lim_{s,t\to 0}\lim_\sigma \sum_{j=1}^{\infty}(is)^{j-1}\,\phi(f)^j \sum_{j=1}^{\infty}(it)^{j-1}\,\phi(g)^j\,T_k\psi\\ &= \lim_\sigma \lim_{s,t\to 0} \sum_{j=1}^{\infty}(is)^{j-1}\,\phi(f)^j \sum_{j=1}^{\infty}(it)^{j-1}\,\phi(g)^j\,T_k\psi\\ &= \lim_\sigma \phi(f)\,\phi(g)\,T_k\psi\,, \end{aligned}$$

by (3.18) and the bounded convergence theorem.

By Lemma 4.3, the spectra of $N(f)$ and $N(B_\varrho)$, both of which are the non-negative integers, coincide with the spectra of $N_r(f)$ and $N_r(B_\varrho)$ respectively.

By (4.5) and [2], for $f \in L_2(B_\varrho)$,

$$\begin{aligned} e^{itN_r(B_\varrho)} W_r(f) e^{-itN_r(B_\varrho)} &= \lim_\sigma e^{itN(B_\varrho)} W(f) e^{-itN(B_\varrho)} \\ &= \lim_\sigma W(e^{it} f) = W_r(e^{it} f). \end{aligned} \tag{4.8}$$

Proof of Theorem 3. By (4.8), $N_r(B_\varrho)$ is a number operator for $W_r(f)$, $f \in L_2(B_\varrho)$, which is therefore unitarily equivalent to a direct sum of Fock representations [2, Theorem 2].

Suppose $v \notin L_2(R^{dv})$. Let $\{f_i\}_{i=1}^\infty$, $f_i \in J$, be an orthonormal basis of J, and let $\sum_{m \geqq 0} m P_{rm}(f_i)$, $\sum_{m \geqq 0} m P_m(f_i)$ be the spectral decompositions of $N_r(f_i)$, $N(f_i)$ respectively. Let

$$\Gamma = \{\gamma = \{\gamma_i\} : \gamma_i \text{ non-negative integers, } \gamma_i = 0 \text{ almost always}\},$$

and let Γ be assigned the measure for which each point has measure one. For $\gamma \in \Gamma$, let

$$P_{r\gamma} = \prod_i P_{r\gamma_i}(f_i) \qquad P_\gamma = \prod_i P_{\gamma_i}(f_i).$$

Then [2, p. 31 and Prop. 3.1] implies that $\{P_{r\gamma}\}_{\gamma \in \Gamma}$, $\{P_\gamma\}_{\gamma \in \Gamma}$ are both mutually orthogonal families of projections. Fix $k, p \geqq 0$. Let g_σ, g be functions mapping Γ into R and defined by

$$\begin{aligned} g_\sigma(\gamma) &= \|P_\gamma T_{k\sigma} \psi\|^2 e^{-A(\sigma)} && \Sigma\gamma_i \leqq p \\ &= 0 && \Sigma\gamma_i > p \\ g(\gamma) &= \|P_{r\gamma} T_k \psi\|_r^2 && \Sigma\gamma_i \leqq p \\ &= 0, && \Sigma\gamma_i > p. \end{aligned}$$

By Lemma 4.3, with $U(x) = e^{i\Sigma x_j N(f_j)}$, $g_\sigma \to g$ pointwise as $\sigma \to \infty$.

For $\mathcal{M} \subset J$, $\dim \mathcal{M} < \infty$, let $\sum_{m \geqq 0} m P_{rm}(\mathcal{M})$ $\sum_{m \geqq 0} m P_m(\mathcal{M})$ be the spectral decompositions of $N_r(\mathcal{M})$, $N(\mathcal{M})$ respectively. Let

$$Q_{rp}(\mathcal{M}) = \sum_{m=0}^{p} P_{rm}(\mathcal{M}), \qquad Q_p(\mathcal{M}) = \sum_{m=0}^{p} P_m(\mathcal{M}),$$

$$Q_{rp} = \lim_{\mathcal{M} \to J} Q_{rp}(\mathcal{M}), \qquad Q_p = \lim_{\mathcal{M} \to J} Q_p(\mathcal{M}).$$

Here convergence of finite dimensional subspaces is net convergence, and Q_p is the projection onto $\sum_{m \leqq p} F_m$ [2].

We assert that $Q_{rp} \leqq \lim_{\sigma} Q_p = 0$. Hence the projection

$$Q_r = \text{strong}\lim_{p\to\infty} Q_{rp}$$

onto the finite particle subspace is zero. Thus every vector in F_r has an infinite number of particles, so that no subrepresentation is unitarily equivalent to the Fock representation [2, Theorem 2].

Let $\mathscr{M}_n = \{f_i\}_{i=1}^n$. Noting that $Q_{rp}(\mathscr{M})$ decreases as $\mathscr{M}$ increases, we prove that $Q_{rp} \leqq \lim_{\sigma} Q_p = 0$:

$$\begin{aligned}
\|Q_{rp} T_k \psi\|_r^2 &= \lim_{\mathscr{M}\to J} \|Q_{rp}(\mathscr{M}) T_k \psi\|_r^2 \\
&\leqq \lim_{n\to\infty} \|Q_{rp}(\mathscr{M}_n) T_k \psi\|_r^2 \\
&\lim_{n\to\infty} \int_{\{\gamma:\gamma_i=0,\, i>n\}} g(\gamma) \\
&= \int_\Gamma g(\gamma) \leqq \liminf_{\sigma\to\infty} \int_\Gamma g_\sigma(\gamma) \quad \text{(Fatou's Lemma)} \\
&= \liminf_{\sigma\to\infty} \lim_{n\to\infty} \int_{\{\gamma:\gamma_i=0,\, i>n\}} g_\sigma(\gamma) \\
&= \liminf_{\sigma\to\infty} \lim_{n\to\infty} \|Q_p(\mathscr{M}_n) T_{k\sigma} \psi\|^2 e^{-\Lambda(\sigma)} \\
&= \liminf_{\sigma\to\infty} \|Q_p T_{k\sigma} \psi\|^2 e^{-\Lambda(\sigma)} \\
&\leqq \liminf_{\sigma\to\infty} K \exp_p \Lambda(\sigma) e^{-\Lambda(\sigma)} = 0
\end{aligned}$$

because $\Lambda(\sigma) \to \infty$. The theorem is proved.

We now compare two exponential Weyl systems.

Theorem 4. *$W_r = W_r(v, \alpha, n(k))$, $W_r' = W_r(v', \alpha, n'(k))$ are not disjoint if $v - v' \in J_v$ and $\bar{v}v'$ is non-negative. If, moreover,* (4.6) *holds, for $U = I$, then W_r and W_r' are unitarily equivalent.*

Corollary. *If $v = v'$ and $n(k) = n'(k)$ for almost all k, then W_r and W_r' are unitarily equivalent.*

Proof of Corollary. Let $\psi \in D$ and let $\theta = T_{jk}' \psi \in D$. Then for $k \geqq j$ sufficiently large,

$$T_{j\sigma}' \psi = T_{k\sigma} \theta .$$

The same statement holds with T and T' interchanged. Thus (4.6) holds trivially.

Proof of Theorem. By Theorem 2, $\lim_{\sigma} W(f)$ and $\lim_{\sigma}' W(f)$ are unitary operators on F_r, F_r' respectively. By Lemma 3.6 and (4.4), $V = \operatorname{Lim}_{\sigma} I$ is an operator mapping F_r' into F_r and bounded in norm by $c(v, v')$. U is non-zero because

$$(T_0\Omega, VT_0'\Omega)_r = \lim_{\sigma\to\infty} \prod_j e^{-X_j(\sigma)} \exp_{n(j)} X_j(\sigma)\, e^{X(\sigma)-\Lambda(\sigma)} > 0$$

by (3.2), (3.4), and $\lim_{\sigma\to\infty}(X(\sigma)-\Lambda(\sigma)) = -v!(v,w)$. Finally, by (4.2),

$$VW_r'(f) = \operatorname*{Lim}_{\sigma} I\,W(f) = \operatorname*{Lim}_{\sigma} W(f)\,I = W_r(f)\,V.$$

Thus V intertwines $W_r(f)$ and $W_r'(f)$, which are therefore not disjoint.

If (4.6) holds for $U = I$, then, by Lemma 4.2, V is unitary. The theorem is proved.

Bibliography

1. Ahlfors, L.: Complex Analysis. New York: McGraw Hill 1966.
2. Chaiken, J.: Ann. Phys. **42**, 23 (1967).
3. Eckmann, J.: A theorem on kernels of superrenormalizable theories. To appear.
4. Friedrichs, K.: Perturbation of Spectra in Hilbert Space. Providence: Amer. Math. Soc. 1965.
5. Glimm, J.: Commun. Math. Phys. **10**, 1 (1968).
6. — Advan. Math. **3**, 101 (1969).
7. Jaffe, A.: Commun. Math. Phys. **1**, 127 (1965).
8. Hepp, K.: Lecture Notes. Ecole Polytechnique (1968—1969).
9. — Proceedings of CNRS conference on systems with an infinite number of degrees of freedom. Paris 1969.
10. Klauder, J.: Exponential Hilbert space: Fock space revisited. To appear.
11. Kristensen, P., Mejlbo, L., Poulsen, E.: Math. Scand. **14**, 129 (1964).
12. — — — Commun. Math. Phys. **6**, 29 (1967).
13. Loeve, M.: Probability Theory. Princeton: Van Nostrand 1963.
14. Powers, R.: Princeton thesis (1967).
15. Reed, M.: Commun. Math. Phys. **11**, 346 (1969).

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