Commun. math. Phys. 18, 204—226 (1970)

On Parastatistics

K. DRÜHL, R. HAAG, and J. E. ROBERTS
II. Institut für Theoretische Physik der Universität, Hamburg

Received March 2, 1970

Abstract. The physical content of a para-Fermi field theory is analysed from the point of view of its local observables. The parafield theory leads to parastatistics only for special choices of the observable algebra, and only then does it give a complete description of the relevant physical states. On the other hand there is always a physically equivalent description in terms of a certain number of ordinary Fermi fields from which the observables are selected by a gauge group (in general non-Abelian). Thus one can always achieve a reduction to Fermi statistics by considering a system with different particle types which are distinguished by hidden (unobservable) quantum numbers.

I. Introduction

Do all particles obey either Bose or Fermi statistics? Specific theoretical models for other alternatives (parastatistics) have been proposed by H. S. Green [1] and studied by various authors [1]. In terms of physical observations the essential feature of such models is the following. Let $\Phi_1, \ldots, \Phi_N$ be different states such that Φ_i is well localized at time $t=0$ within a space region $\mathscr{V}_i$, the different $\mathscr{V}_i$ being far apart. In the case of ordinary statistics there is exactly one "product state" which corresponds to specifying that observations at $t=0$ in $\mathscr{V}_i$ find the state Φ_i and observations outside of any $\mathscr{V}_i$ find the vacuum state. In the case of parastatistics, on the other hand, there are several states answering to these specifications. The distinction between these states can be found either by means of a very large measuring apparatus at time $t=0$ (covering several of the regions $\mathscr{V}_i$) or, more realistically, by measurements at a much later time.

A situation of this sort also arises in another context. Consider the idealized theory of strong interactions in which the electromagnetic and weak interactions are strictly neglected. In such a theory the basic fields have ordinary commutation relations but only those quantities which are invariant under the transformations of a certain symmetry group (the isospin group) can be observed [2]. In particular, the distinction between a single neutron and a single proton becomes impossible; yet

[1] See e.g. [2–4]. There are also various discussions of general statistics which are not directly based on Green's models. Compare for instance [5] and the literature quoted there.

[2] This is so if all possible measuring devices are governed by the laws of the theory. In other words, if the theory does not contain any electromagnetic interactions, then we may exclude the possibility of measurements which rely on electromagnetic effects.

for two far separated nucleons with prescribed spatial and spin wave functions there are two distinguishable 2-particle states in the theory, namely the isospin singlet and the isospin triplet. One may therefore ask whether a parafield theory is just an alternative description of a theory with normal commutation relations but equipped with a non-Abelian gauge group (non-Abelian superselection rules). This would mean that parastatistics is reduced to ordinary statistics if one introduces a larger number of particle types than those which are distinguished by the observed quantum numbers.

A strong indication in support of this conjecture is already given by H. S. Green's decomposition of a parafield. Thus, for instance, a para-Fermi field of order n may be written as the sum of n mutually commuting Fermi fields. Still, for the analysis of the contents of such a model one must know what the observable quantities are supposed to be. The observables should be functions of the parafield and the theoretical possibilities for their selection are restricted by the principle of locality. There remain, however, several distinct choices some of which have been mentioned in the literature [4].

We study here in some detail the example of a para-Fermi field of order 2 and come to the following conclusions. If the algebra of observables is large then the Hilbert space generated from the vacuum state by the parafield algebra does not contain all the relevant states of the observable algebra which are needed for a complete particle description. In such cases the parafield is not a convenient tool for describing the physical content of the theory. It does not generate the appropriate "field algebra" in the sense of Ref. [6]. There are, on the other hand, choices of the observable algebra for which the Hilbert space generated by the parafield from the vacuum contains all relevant states. Then the parafield description is appropriate and perhaps even convenient. In such cases one finds, however, that the model may be alternatively described by a field algebra with normal commutation relations from which the observable algebra is selected by a non-Abelian gauge group.

In Section IV we give a brief discussion of the case of para-Fermi fields of higher order showing that the above conclusions remain valid.

II. Notation. Description of Possible Observable Algebras

As indicated we shall study the example of a para-Fermi field of order 2. We need consider only quantities referring to the time $t = 0$. The form of the dynamical laws will play no role in the discussion[3]. We shall

[3] In the next section we shall assume some qualitative properties of the ground state (vacuum state) and these are, of course, implicit dynamical assumptions. They allow a more detailed illustration of the physical features of the models but are not necessary for our main conclusion, namely the equivalence between a parafield theory and a normal field theory with a suitable gauge group.

not require relativistic invariance nor assume any specific transformation character of the parafield ψ under rotations. Therefore it is irrelevant for our purpose whether ψ has any spinor indices. We shall omit indicating them.

If f is a square-integrable function in 3-dimensional space, then we shall write

$$\psi(f) = \int \psi(x) f(x) d^3x$$

with the adjoint

$$\psi(f)^* = \int \psi^*(x) \bar{f}(x) d^3x .$$

It will simplify the subsequent formulae considerably if we combine ψ and ψ^* into one symbol $\boldsymbol{\psi}$ and correspondingly introduce 2-component test functions $\boldsymbol{f} = (f', f'')$ such that

$$\boldsymbol{\psi}(\boldsymbol{f}) = \psi(f') + \psi(f'')^* . \tag{2.1}$$

The adjoint is then

$$\boldsymbol{\psi}(\boldsymbol{f})^* = \boldsymbol{\psi}(\hat{\boldsymbol{f}}) \quad \text{with} \quad \hat{\boldsymbol{f}} = (\bar{f}'', \bar{f}') . \tag{2.2}$$

The commutation relations of the parafield $\boldsymbol{\psi}$ are most conveniently described indirectly, using the decomposition of $\boldsymbol{\psi}$ into its Green components $\boldsymbol{\psi}^{(1)}$ and $\boldsymbol{\psi}^{(2)}$ [1]. We have

$$\boldsymbol{\psi} = \boldsymbol{\psi}^{(1)} + \boldsymbol{\psi}^{(2)} \tag{2.3}$$

with the commutation relations [4]

$$[\boldsymbol{\psi}^{(1)}(\boldsymbol{f}), \boldsymbol{\psi}^{(2)}(\boldsymbol{g})] = 0 , \tag{2.4}$$

$$\{\boldsymbol{\psi}^{(k)}(\boldsymbol{f}), \boldsymbol{\psi}^{(k)}(\boldsymbol{g})\} = (\hat{\boldsymbol{f}}, \boldsymbol{g}) \quad \text{for} \quad k = 1, 2 \tag{2.5}$$

where

$$(\boldsymbol{f}, \boldsymbol{g}) = \int (\bar{f}'(x) g'(x) + \bar{f}''(x) g''(x)) d^3x . \tag{2.6}$$

We now have to consider various algebras which are associated in some way with the parafield. Actually we shall not only be interested in these algebras themselves; each such algebra has a set of distinguished subalgebras corresponding to regions in (3-dimensional) space. To emphasize this structure we shall speak of a "net" $\mathfrak{B}$ when we have an assignment of a C^*-algebra $\mathfrak{B}(\mathscr{V})$ to each finite space region $\mathscr{V}$. To deserve the name "net" the correspondence $\mathscr{V} \to \mathfrak{B}(\mathscr{V})$ should satisfy a few general requirements which we shall not spell out here in detail since they are trivially satisfied in the examples appearing in our context. We only mention here two features for notational purposes.

1. The net $\mathfrak{B}$ defines a total algebra also denoted by $\mathfrak{B}$ which contains all $\mathfrak{B}(\mathscr{V})$. $\mathfrak{B}$ is the smallest C^*-algebra containing the $\mathfrak{B}(\mathscr{V})$.

[4] $[A, B] = AB - BA$; $\{A, B\} = AB + BA$.

2. We assume covariance under spatial translations. Thus a translation by $\boldsymbol{x} \in R^3$ shall be represented by an automorphism $\alpha_{\boldsymbol{x}}$ of the algebra $\mathfrak{B}$ such that

$$\alpha_{\boldsymbol{x}}(\mathfrak{B}(\mathscr{V})) = \mathfrak{B}(\mathscr{V} + \boldsymbol{x})$$

where $\mathscr{V} + \boldsymbol{x}$ denotes the translated region.

Let $\mathfrak{F}^{(1)}$ be the net generated by the Green component $\psi^{(1)}$. In other words, $\mathfrak{F}^{(1)}(\mathscr{V})$ is the *Clifford algebra*[5] generated by all the $\psi^{(1)}(f)$ for which the support of f is contained in $\mathscr{V}$. Replacing $\psi^{(1)}$ by $\psi^{(2)}$ we get the (isomorphic) net $\mathfrak{F}^{(2)}$. The net $\mathfrak{F}_G$ is defined as the tensor product of the two[6]

$$\mathfrak{F}_G(\mathscr{V}) = \mathfrak{F}^{(1)}(\mathscr{V}) \otimes \mathfrak{F}^{(2)}(\mathscr{V}) \tag{2.7}$$

and $\mathfrak{F}_P$ is the subnet of $\mathfrak{F}_G$ generated by the parafield quantities ψ (see Eq. (2.3)). In $\mathfrak{F}_P$ we consider the automorphism γ inducing the transformation $\psi(f) \to -\psi(f)$. The subnet of $\mathfrak{F}_P$ containing all those elements which are invariant under γ will be denoted by $\mathfrak{A}_0$. In other words, $\mathfrak{A}_0$ is the even part of $\mathfrak{F}_P$.

Now we want to consider the observables. As indicated in the introduction we shall demand that the net $\mathfrak{A}$ generated by the observables satisfy the two requirements

i) $\mathfrak{A}(\mathscr{V}) \subset \mathfrak{F}_P(\mathscr{V})$,

ii) local commutativity:

$$[\mathfrak{A}(\mathscr{V}), \mathfrak{A}(\mathscr{V}')] = 0 \quad \text{when } \mathscr{V} \cap \mathscr{V}' \text{ is empty}.$$

We observe

2.1 Lemma. *The net* $\mathfrak{A}_0$ *satisfies properties* (i) *and* (ii). *If* $\mathfrak{A}$ *satisfies* (i) *and* (ii), *then*

$$\mathfrak{A}(\mathscr{V}) \subset \mathfrak{A}_0(\mathscr{V}). \tag{2.8}$$

Proof. Elements of $\mathfrak{F}_G$ may be classified according to their Bose or Fermi character in the component fields.

For $F \in \mathfrak{F}_G(\mathscr{V})$ we have the unique decomposition

$$F = F_{++} + F_{+-} + F_{-+} + F_{--} \quad \text{with} \quad F_{ab} \in \mathfrak{F}_G(\mathscr{V}). \tag{2.9}$$

The subscripts refer to the transformation properties under the automorphisms $\gamma^{(i)}$ defined by

$$\gamma^{(i)}(\psi^{(i)}) = -\psi^{(i)}; \quad \gamma^{(i)}(\psi^{(k)}) = \psi^{(k)} \quad \text{for} \quad i \neq k\,^7. \tag{2.10}$$

[5] See e.g. [7].

[6] The C^*-tensor product of two Clifford algebras is unique [8].

[7] Thus the subscript + means Bose type, − Fermi type.

The first subscript relates to $\gamma^{(1)}$, the second to $\gamma^{(2)}$. The locality condition (ii) demands that if $F \in \mathfrak{A}(\mathscr{V})$

$$[F, \alpha_{\boldsymbol{x}}(F)] = 0 \quad \text{for sufficiently large } |\boldsymbol{x}| . \tag{2.11}$$

We first show that (2.11) implies

$$F_{+-} = F_{-+} = 0 . \tag{2.12}$$

If $\mathscr{V} \cap \mathscr{V} + \boldsymbol{x}$ is empty then

$$G_{\boldsymbol{x}} \equiv [F, \alpha_{\boldsymbol{x}}(F)]_{++} = 2F_{+-}\alpha_{\boldsymbol{x}}(F_{+-}) + 2F_{-+}\alpha_{\boldsymbol{x}}(F_{-+}) . \tag{2.13}$$

Therefore (2.11) demands that

$$\lim_{|\boldsymbol{x}| \to \infty} \omega(G_{\boldsymbol{x}}^* G_{\boldsymbol{x}}) = 0 \tag{2.14}$$

for any state ω[8].

We shall test (2.14) by choosing

$$\omega = \omega^{(1)} \otimes \omega^{(2)}$$

where $\omega^{(i)}$ is the trace state[9] over the Clifford algebra $\mathfrak{F}^{(i)}$. The state ω has the following convenient properties which follow immediately from the corresponding properties of the trace states $\omega^{(i)}$:

a) ω is translationally invariant and satisfies the cluster relation

$$\omega(F\alpha_{\boldsymbol{x}}(G)) \to \omega(F)\,\omega(G) \quad \text{as} \quad |\boldsymbol{x}| \to \infty , \tag{2.15}$$

b) $\omega(\gamma^i(F)) = \omega(F)$ and hence

$$\omega(F) = \omega(F_{++}) , \tag{2.16}$$

c) $\omega(F^* F) = 0$ implies $F = 0$.

Using (2.15) and (2.16) we get for $|\boldsymbol{x}| \to \infty$

$$\omega(G_{\boldsymbol{x}}^* G_{\boldsymbol{x}}) \to \{\omega(F_{+-}^* F_{+-})\}^2 + \{\omega(F_{-+}^* F_{-+})\}^2 .$$

Thus we get (2.12) from (2.14) and property c). Condition (2.12) is synonymous with the invariance of F under the automorphism $\gamma = \gamma^{(1)}\gamma^{(2)}$ which transforms ψ into $-\psi$. We have seen therefore that only the subnet of γ-invariant quantities in $\mathfrak{F}_G$ (the "even part") can satisfy the local

[8] A "state" shall be understood as an expectation functional, i.e. a state is described by a normalized positive linear form over the algebra under consideration.

[9] The trace state over the Clifford algebra $\mathfrak{F}^{(i)}$ is the quasifree state with the 2-point function

$$\omega^{(i)}(\psi^{(i)}(f)\,\psi^{(i)}(g)) = \tfrac{1}{2}(\hat{f}, g) .$$

It is the only state over the Clifford algebra with the property

$$\omega^{(i)}(AB) = \omega^{(i)}(BA) .$$

commutativity condition. Conversely, a trivial computation shows that the even part of $\mathfrak{F}_G$ indeed satisfies local commutativity. Restricting our attention to the subnet $\mathfrak{F}_P$ we obtain Lemma 2.1.

We shall now consider three possible choices for the nets of observable algebras. The study of these three examples should suffice to illustrate the relevant phenomena. The first example is the net $\mathfrak{A}_0$ defined above. Clearly, $\mathfrak{A}_0(\mathscr{V})$ is generated by elements of the form $\psi(f)\,\psi(g)$ with the support of both f and g restricted to $\mathscr{V}$. We can write

$$\psi(f)\,\psi(g) = r(f,g) + s(f,g) \tag{2.17}$$

with

$$r(f,g) = \tfrac{1}{2}[\psi(f),\psi(g)] + (\hat{f},g) = \psi^{(1)}(f)\,\psi^{(1)}(g) + \psi^{(2)}(f)\,\psi^{(2)}(g)\,, \tag{2.18}$$

$$s(f,g) = \tfrac{1}{2}\{\psi(f),\psi(g)\} - (\hat{f},g) = \psi^{(1)}(f)\,\psi^{(2)}(g) + \psi^{(2)}(f)\,\psi^{(1)}(g)\,. \tag{2.19}$$

Now one observes that $r(f,g)$ can actually be expressed as a polynomial in the $s(\cdot,\cdot)$. To see this one computes the commutator $[s(e,f), s(g,h)]$ which yields $r(f,g)$ if the test functions are chosen so that $(\hat{f},g)=(\hat{e},g)=(\hat{f},h)=0$ and $(\hat{e},h)\neq 0$. In this way we obtain a subset of the $r(\cdot,\cdot)$ namely those for which $(\hat{f},g)=0$. By taking commutators again between pairs of elements from this subset we can obtain any $r(\cdot,\cdot)$. Therefore $\mathfrak{A}_0$ is already generated by the $s(\cdot,\cdot)$.

The second example, denoted by $\mathfrak{A}_1$, is the net generated by the $r(\cdot,\cdot)$.

The third example results when we add a principle of charge conservation. We abandon the equal treatment of the components (ψ,ψ^*) of ψ and allow as observables only quantities with an equal number of starred and unstarred factors. If we start from the generators $r(\cdot,\cdot)$ of the net $\mathfrak{A}_1$ then the subset of such ("chargeless") quantities is obtained by restricting the components of f and g by

$$f' = g'' = 0\,. \tag{2.20}$$

This leads to terms of the form

$$\varrho(f,g) = \psi^{(1)}(f)^*\,\psi^{(1)}(g) + \psi^{(2)}(f)^*\,\psi^{(2)}(g) \tag{2.21}$$

where f and g are now 1-component functions. The net generated by the $\varrho(\cdot,\cdot)$ will be denoted by $\mathfrak{A}_2$. Clearly we have the following inclusions for the nets which have been mentioned so far:

$$\mathfrak{F}_G \supset \mathfrak{F}_P \supset \mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2\,. \tag{2.22}$$

We shall show that the nets $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ can be characterized by invariance properties. For this purpose we should consider them not as subnets of $\mathfrak{F}_G$ or $\mathfrak{F}_P$ but as subnets of a net $\mathfrak{F}$ which is generated by two

Fermi fields with normal commutation relations. Using the same notation as in Eq. (2.1) we take two fields $\phi^{(i)}$ with the commutation relations

$$\{\phi^{(i)}(f), \phi^{(k)}(g)\} = \delta_{ik}(\hat{f}, g) \tag{2.23}$$

and define adjoints by

$$\phi^{(i)}(f)^* = \phi^{(i)}(\hat{f}). \tag{2.24}$$

Then $\mathfrak{F}(\mathscr{V})$ denotes the C^*-algebra generated by the $\phi^{(1)}(f)$, $\phi^{(2)}(f)$ with support of f contained in $\mathscr{V}$. If we represent $\mathfrak{F}_G$ in Fock space then $\mathfrak{F}$ will be the Klein transform of $\mathfrak{F}_G$. Since the Klein transformation is macroscopically nonlocal, the algebra $\mathfrak{F}(\mathscr{V})$ is not contained in $\mathfrak{F}_G$. But we can consider the Klein operator, denoted by K_2, as an additional algebraic element satisfying the relations:

$$\{\phi^{(2)}(f), K_2\} = 0; \quad [\phi^{(1)}(f), K_2] = 0, \tag{2.25}$$

$$K_2 = K_2^{-1} = K_2^*. \tag{2.26}$$

Defining then $\tilde{\mathfrak{F}}(\mathscr{V})$ as the algebra whose general element is of the form $F + F' K_2$ with $F, F' \in \mathfrak{F}(\mathscr{V})$ we obtain a net $\tilde{\mathfrak{F}}$ in which both $\mathfrak{F}$ and $\mathfrak{F}_G$ are embedded[10]. Within $\tilde{\mathfrak{F}}$ we can identify the generators of $\mathfrak{F}_G$ by

$$\psi^{(1)}(f) = \phi^{(1)}(f) K_2; \quad \psi^{(2)}(f) = i\phi^{(2)}(f) K_2. \tag{2.27}$$

One easily checks that this identification gives the right commutation relations and definition of the adjoint for the $\psi^{(i)}$. We see then that the even parts of $\mathfrak{F}_G$ and $\mathfrak{F}$ coincide since the factor K_2 drops out in all even polynomials in $\psi^{(1)}, \psi^{(2)}$. The parafield becomes

$$\psi = (\phi^{(1)} + i\phi^{(2)}) K_2. \tag{2.28}$$

The generators of the nets $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2$ are respectively

$$s(f, g) = -i(\phi^{(1)}(f)\,\phi^{(2)}(g) - \phi^{(2)}(f)\,\phi^{(1)}(g)), \tag{2.29}$$

$$r(f, g) = \phi^{(1)}(f)\,\phi^{(1)}(g) + \phi^{(2)}(f)\,\phi^{(2)}(g), \tag{2.30}$$

$$\varrho(f, g) = \phi^{(1)}(f)^*\,\phi^{(1)}(g) + \phi^{(2)}(f)^*\,\phi^{(2)}(g). \tag{2.31}$$

One recognizes in the expressions (2.29), (2.30), (2.31) the fundamental invariants of certain groups. Consider automorphisms of $\mathfrak{F}$ which are induced by transformations of the form

$$\begin{aligned} \alpha_g(\phi^{(i)}(x)) &= \sum_k g_{ki}\phi^{(k)}(x), \\ \alpha_g(\phi^{(i)*}(x)) &= \sum \bar{g}_{ki}\phi^{(k)*}(x) \end{aligned} \tag{2.32}$$

[10] Actually $\tilde{\mathfrak{F}}(\mathscr{V})$ may be regarded as the covariance algebra [9] of $\mathfrak{F}(\mathscr{V})$ with respect to the cyclic group of order 2 whose generator acts on $\mathfrak{F}$ by changing the sign of all elements which are of Fermi character in the second index.

where g is a unitary 2×2 matrix. The group of automorphisms resulting when g in (2.32) runs through $SO(2)$ (orthogonal matrices with determinant 1) will be called $\mathscr{G}_0$. Correspondingly we define $\mathscr{G}_1$ and $\mathscr{G}_2$ as groups of automorphisms of the form (2.32), $\mathscr{G}_1$ being isomorphic to the full orthogonal group $O(2)$ and $\mathscr{G}_2$ isomorphic to the unitary group $U(2)$. It is clear that both $r(\cdot,\cdot)$ and $s(\cdot,\cdot)$ are invariant under $\mathscr{G}_0$; $r(\cdot,\cdot)$ is invariant under $\mathscr{G}_1$ and $\varrho(\cdot,\cdot)$ is invariant under $\mathscr{G}_2$. One now has

2.2 Theorem. *The nets $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ are the subnets of $\mathfrak{F}$ consisting of precisely those elements of $\mathfrak{F}$ which are invariant under the automorphism groups $\mathscr{G}_0$, $\mathscr{G}_1$, $\mathscr{G}_2$ respectively.*

Proof. We have already seen that $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ are subnets of $\mathfrak{F}$ and that the generating elements (2.29), (2.30), (2.31) are respectively invariant under the automorphism groups $\mathscr{G}_0$, $\mathscr{G}_1$, $\mathscr{G}_2$. Polynomials in $\phi^{(i)}(\cdot)$ and $\phi^{(i)}(\cdot)^*$ are norm dense in $\mathfrak{F}$. Averaging elements of $\mathfrak{F}$ over one of these groups is a norm-continuous operation and the mean of a polynomial in $\phi^{(i)}(\cdot)^*$ is again such a polynomial. Hence every invariant element of $\mathfrak{F}$ can be approximated in norm by an invariant polynomial and it suffices to show that every invariant polynomial is a polynomial in the respective fundamental invariants $s(\cdot,\cdot)$, $r(\cdot,\cdot)$ or $\varrho(\cdot,\cdot)$. If the $\phi^{(i)}(\cdot)$, $\phi^{(i)}(\cdot)^*$ were a commutative set then the classical theorems on invariant polynomials would be applicable. In that case the $s(\cdot,\cdot)$ and $r(\cdot,\cdot)$ together would be a generating set of invariants for the group $SO(2)$; similarly the $r(\cdot,\cdot)$ are the fundamental invariants of $O(2)$, and the $\varrho(\cdot,\cdot)$ are those of $U(2)$. We show in the appendix that a generating set of invariants for the classical polynomial algebra remains a generating set for the Clifford algebra. The only change is that in the Clifford algebra the $r(\cdot,\cdot)$ are also expressible as polynomials in the $s(\cdot,\cdot)$ so that we do not need the $r(\cdot,\cdot)$ as generators in the case of $\mathfrak{A}_0$.

III. Physical Content

The preceding discussion was of a purely algebraic nature. Now we want to consider the relevant states of the observable algebra. For elementary particle physics the interesting states are those which (asymptotically) coincide with the vacuum state[11] ω_0 for observations in far away regions of finite size. In the traditional treatment of Quantum Field Theory these states appear as the normal[12] states in a Hilbert space

[11] By "vacuum state" we mean the dynamically determined ground state not the Fock vacuum.

[12] These are states which can be described by vectors or density matrices in the Hilbert space in question.

representation of the field algebra generated from the vacuum. In fact one may say that the basic reason for embedding the observable algebra in a field algebra is precisely the wish to unite and connect all relevant states in one Hilbert space representation [10, 6, 11]. Therefore, while the field algebra does not have an intrinsic physical significance and is not uniquely determined by the physical content of the theory the arbitrariness which remains in the construction of a field algebra is severely limited. In our context we have the algebras $\mathfrak{F}_P$, $\mathfrak{F}$, $\tilde{\mathfrak{F}}$ as possible candidates. It will turn out that $\mathfrak{F}$ is always an appropriate field algebra (under some simplifying assumptions concerning the vacuum state), whereas $\mathfrak{F}_P$ is appropriate only in one of the three examples described in the last section, namely when $\mathfrak{A}_2$ is the observable algebra. The model with $\mathfrak{A}_0$ describes ordinary Fermi statistics in disguise.

The Hilbert space representation referred to above is obtained by the Gelfand-Naimark-Segal construction once the vacuum state, originally defined over the observable algebra, is extended to the field algebra. We first note that there is a unique extension of the state ω_0 defined over $\mathfrak{A}_i$ to a $\mathscr{G}_i$-invariant state of $\mathfrak{F}$. We proceed as follows: since the automorphisms $\alpha_g \in \mathscr{G}_i$ of $\mathfrak{F}$ are induced by a unitary group of operators on the test function space we have a strongly continuous group of automorphisms of the Clifford algebra when $\mathscr{G}_i$ is given its natural topology as a matrix group, i.e.

$$g \to \alpha_g(F) \quad \text{is norm continuous for each} \quad F \in \mathfrak{F}.$$

Let

$$m_i(F) = \int_{\mathscr{G}_i} \alpha_g(F)\, d\mu(g) \tag{3.1}$$

where μ denotes normalized Haar measure. Then we extend the vacuum state to $\mathfrak{F}$ by defining

$$\omega_0(F) = \omega_0(m_i(F)) \quad \text{for all} \quad F \in \mathfrak{F}. \tag{3.2}$$

We now make three simplifying assumptions. First the extension of ω_0 defined by (3.2) shall be a pure state on $\mathfrak{F}$. This means that we do not discuss here the possibility of a "spontaneous breakdown of the gauge symmetry" or in the terminology of Yang [12] the possible occurrence of an "off-diagonal long range order". This first assumption implies, that (3.2) is the only possible extension of ω_0 from $\mathfrak{A}_i$ to $\mathfrak{F}$.

Secondly (and related to this) we assume that, in the case when $\mathfrak{A}_0$ is the observable algebra, ω_0 as a state over $\mathfrak{A}_0$ is invariant under the automorphisms $\gamma^{(i)}$. If this were not the case then there would be another translationally invariant state ω_0' arising from ω_0 by the transformation $\gamma^{(2)}$. This state might be another ground state, degenerate with ω_0, corresponding to the spontaneous breakdown of an observable symmetry; we do not wish to discuss this possibility here. If ω_0' is not another ground state it is not an interesting state for elementary particle physics but is typically a many-body state with finite mean particle density. The parafield, which connects ω_0 and ω_0' is then not a useful object in elementary particle physics. The third assumption is that the representation of the observable algebra resulting from ω_0 by the GNS-construction is faithful. This assumption ensures that we have a situation as envisaged in [6] and typical of local relativistic theories (e.g. existence of antiparticles). This assumption would not hold if ω_0 were the Fock vacuum annihilated by all the $\psi(f)$.

In the representation of $\mathfrak{F}$ resulting by the GNS construction from the extended vacuum state (3.2) we denote the operator representing $F \in \mathfrak{F}$ by $\pi(F)$ and the vector corresponding to ω_0 by Ω.

Since ω_0 is invariant we also have a continuous unitary representation of the "gauge group" $\mathscr{G}_i$, the representative $U(g)$ of $\alpha_g \in \mathscr{G}_i$ being defined by

$$U(g)\,\pi(F)\Omega = \pi(\alpha_g(F))\,\Omega \quad \text{for all} \quad F \in \mathfrak{F}. \tag{3.3}$$

As shown in [6] the representation space $\mathscr{H}$ may be decomposed as

$$\mathscr{H} = \bigoplus_\sigma \mathscr{H}_\sigma \otimes \mathscr{H}_\sigma' \tag{3.4}$$

where the index σ characterizes the superselection quantum numbers which in turn correspond to (equivalence classes of) irreducible representations of the gauge group in question. $\mathscr{H}_\sigma$ is the finite dimensional space carrying the irreducible representation U_σ of $\mathscr{G}_i$. In other words, in the decomposition (3.2) we have

$$U(g) = \bigoplus_\sigma U_\sigma(g) \otimes I_\sigma'. \tag{3.5}$$

On the other hand, $\mathscr{H}_\sigma'$ is a representation space for the observable algebra, carrying an irreducible representation π_σ of $\mathfrak{A}$, i.e.

$$\pi(A) = \bigoplus_\sigma I_\sigma \otimes \pi_\sigma(A). \tag{3.6}$$

We now have

3.1 Theorem. *The superselection quantum numbers are in one-to-one correspondence with the equivalence classes of irreducible representations of the "gauge group" $\mathscr{G}_i$*[13].

Proof. We need only show that all representations of the gauge group occur. Given any equivalence class σ of irreducible representations of

[13] Compare [6; Theorem 3.6] where this result is derived using the Reeh-Schlieder property.

$\mathscr{G}_i$ we can find a non-zero tensor $F \in \mathfrak{F}$ of character σ and it suffices to show that

$$\pi(F)E_0 \neq 0 \tag{3.7}$$

where E_0 is the projection onto the vacuum sector. For any vector $\pi(F)E_0\Phi$ transforms according to σ. Suppose $\pi(F)E_0 = 0$ then

$$E_0\pi(F^*F)E_0 = \pi\left(m(F^*F)\right)E_0 = 0\,.$$

However, we have assumed that the vacuum representation of the observables is faithful, hence $m(F^*F) = 0$. Thus $\omega\left(m(F^*F)\right) = 0$ where ω is the unique trace state of $\mathfrak{F}$. However ω is invariant under all automorphisms of $\mathfrak{F}$, so $\omega\left(m(F^*F)\right) = \omega(F^*F) = 0$. Thus $F = 0$ contrary to hypothesis and the proof is complete.

Let us now look at the superselection quantum numbers in the three different cases mentioned. This is the purely group theoretical problem of classifying the irreducible representations of $SO(2)$, $O(2)$ and $U(2)$.

Case 1) $\mathfrak{A}_0$; $\{SO(2)\}$

Here σ runs through all integers n (positive, negative, and zero). The rotation by an angle ϕ is represented in U_n by the number $e^{in\phi}$. Thus $\mathscr{H}_n$ is 1-dimensional.

Case 2) $\mathfrak{A}_1$; $\{O(2)\}$

For each positive integer $|n|$ we have a 2-dimensional representation of $O(2)$, tying together the two representations of $SO(2)$ with $n = \pm|n|$. In addition there are two 1-dimensional representations of $O(2)$ denoted by 0^+ and 0^-.

Case 3) $\mathfrak{A}_2$; $\{U(2)\}$

Here σ is characterized by two quantum numbers which may be interpreted as isospin I and baryon number B. Of course, B runs through all integers (positive, negative, zero) and $2I$ is a non-negative integer. The two numbers are related by

$$2I + B = \text{even}\,. \tag{3.8}$$

The dimensionality of $\mathscr{H}_{I,B}$ is $2I+1$.

The position of the subgroup $\mathscr{G}_0$ within $\mathscr{G}_2$ is such that

$$n = 2I_2 \tag{3.9}$$

where I_2 is the second component of the isospin vector $I = (I_1, I_2, I_3)$.

We now wish to compare this classification of states by the above charge quantum numbers with the one resulting from the parafield description. This comparison is simplified by observing that from (2.25), (2.26)

$$K_2 F K_2^{-1} = \gamma^{(2)}(F)$$

and furthermore $\gamma^{(2)} \in \mathscr{G}_1$. As ω_0 is invariant under $\gamma^{(2)}$ we can represent K_2 by an operator $\pi(K_2)$ in $\mathscr{H}$ defined by

$$\pi(K_2)\,\pi(F)\Omega = \pi(\gamma^{(2)}(F))\Omega \quad \text{for all} \quad F \in \mathfrak{F}\,. \tag{3.10}$$

In particular $\pi(K_2)\Omega = \Omega$. Thus we have a representation of $\tilde{\mathfrak{F}}$ and a fortiori of $\mathfrak{F}_P$ in $\mathscr{H}$. The state ω_0 has thus been extended to $\mathfrak{F}_P$ and this extension is in fact unique. Using $\mathfrak{F}_P$ as a field algebra instead of $\mathfrak{F}$ means restricting attention to the subspace $\mathscr{H}_P$ generated by applying the parafield algebra to Ω. Writing now

$$\begin{aligned} \chi_+(f) &= \phi^{(1)}(f) + i\phi^{(2)}(f)\,, \\ \chi_-(f) &= \phi^{(1)}(f) - i\phi^{(2)}(f)\,, \end{aligned} \tag{3.11}$$

we see that χ_+ raises the quantum number n (defined with respect to the observable algebra $\mathfrak{A}_0$) by one unit whereas χ_- lowers n by one unit. On the other hand, by (2.28) and (3.10) we have

$$\psi(f)\Omega = \chi_+(f)\Omega \qquad (n=1)\,, \tag{3.12}$$

$$\psi(f)\,\psi(g)\Omega = \chi_+(f)\,\chi_-(g)\Omega \qquad (n=0)\,, \tag{3.13}$$

$$\psi(f)\,\psi(g)\,\psi(h)\Omega = \chi_+(f)\,\chi_-(g)\,\chi_+(h)\Omega \qquad (n=1)\,. \tag{3.14}$$

Hence the subspace $\mathscr{H}_P \subset \mathscr{H}$ which is generated by applying the parafield algebra to the vacuum contains only two sectors, namely those states with "charge" $n=0$ or $n=1$.

This paradoxical situation is best illustrated by looking at (3.13) with functions f and g, having supports in two far separated regions $\mathscr{V}_1$ and $\mathscr{V}_2$ respectively. Then the state (3.13) means, for observations from the algebra $\mathfrak{A}_0$, that we have charge $+1$ in $\mathscr{V}_2$ and charge -1 in $\mathscr{V}_1$, whereas if we look at the left hand side of (3.13) and at (3.12) we might be led to believe that in both regions we have charge $+1$. The conclusion is therefore: if $\mathfrak{A}_0$ is the observable algebra then the parafield description is very unsuitable because the product of parafields does not correspond to the physical notion of the product of states well localized in far separated regions. Furthermore only a small part of the physically relevant states are described by the vectors in $\mathscr{H}_P$. We have for example no states with total charge -1 in $\mathscr{H}_P$, although we do have states in which the charge of a certain region is -1. The true physical content of the theory in this case is completely equivalent to that of a theory with Fermi statistics; the parastatistics are here only simulated by an artificial and physically inadmissible restriction on the manifold of states which are considered.

Let us now turn to the case where $\mathfrak{A}_2$ is the observable algebra. First we observe that the subspace $\mathscr{H}_P^{(0)}$ got by applying even polynomials in

ψ to Ω contains all vectors in $\mathscr{H}$ with "charge" $n=0$. Similarly $\mathscr{H}_P^{(1)}$, spanned by the vectors $\psi(f)\mathscr{H}_P^{(0)}$, contains all vectors in $\mathscr{H}$ with $n=1$. Further, the vectors in $\mathscr{H}_P^{(0)}$ have even baryon number and hence by (3.5) integer isospin; those in $\mathscr{H}_P^{(1)}$ have odd baryon number, hence half integer isospin. In every isospin multiplet with integer I there is a vector with $I_2=0$ and for half integer I we have a vector with $I_2=\frac{1}{2}$. Since all the vectors belonging to one isospin multiplet give the same expectation values for the algebra $\mathfrak{A}_2$ i.e. they correspond to one and the same state over $\mathfrak{A}_2$ it suffices to pick one state vector from each multiplet and we can choose in particular the one with $I_2=0$ (or $\frac{1}{2}$) for I integer (or half integer). Therefore $\mathscr{H}_P$ contains indeed all the relevant states of the observable algebra $\mathfrak{A}_2$. Each such state is represented by exactly one state vector in $\mathscr{H}_P$, whereas it is redundantly represented in $\mathscr{H}$ (for a state with isospin I we have a $(2I+1)$-dimensional subspace of observationally equivalent vectors in $\mathscr{H}$). The parafield description is consistent and perhaps even convenient when $\mathfrak{A}_2$ is the observable algebra. The alternative description by means of the field algebra $\mathfrak{F}$ (normal commutation relations and the non-Abelian gauge group $\mathscr{G}_2$) shows however, that the parastatistics are reduced to ordinary statistics when "hidden variables" are introduced i.e. when certain quantities outside of $\mathfrak{A}_2$ are added as possible observables.

The case where $\mathfrak{A}_1$ is the observable algebra illustrates both the phenomena encountered in discussing $\mathfrak{A}_0$ and $\mathfrak{A}_2$. On the one hand, the group $O(2)$ is non-Abelian. Therefore the model is not equivalent to Fermi statistics (at least without introducing hidden variables). On the other hand the parafield description is again inadequate in this case because $\mathscr{H}_P$ does not contain all the relevant states over $\mathfrak{A}_1$.

IV. Para-Fermi Fields of Higher Order

The main results of Section II can be generalized to parafields of order $p>2$. We shall restate these results in a form appropriate to the case $p>2$ and outline the proofs, without however giving details.

For a para-Fermi field of order p we have Green's decomposition

$$\psi=\sum_{i=1}^{p}\psi^{(i)} \tag{4.1}$$

with commutation relations

$$[\psi^{(i)}(f),\psi^{(j)}(g)]=0 \quad \text{for} \quad i\neq j\,, \tag{4.2}$$

$$\{\psi^{(i)}(f),\psi^{(i)}(g)\}=(\hat{f},g) \text{ for } \quad i=1,\ldots,p\,. \tag{4.3}$$

The net $\mathfrak{F}_G$ is now defined by

$$\mathfrak{F}_G(\mathscr{V}) = \bigotimes_{i=1}^{p} \mathfrak{F}^{(i)}(\mathscr{V})$$

where $\mathfrak{F}^{(i)}$ is the net generated by the Green component $\psi^{(i)}$. $\mathfrak{F}_P$ is the subnet of $\mathfrak{F}_G$ generated by the parafield quantities ψ.

The elements of $\mathfrak{F}_G$ may be classified according to their transformation properties under the automorphisms $\gamma^{(i)}$ defined by

$$\gamma^{(i)}(\psi^{(j)}(f)) = (-1)^{\delta_{ij}} \psi^{(j)}(f) . \tag{4.4}$$

For $F \in \mathfrak{F}_G(\mathscr{V})$ we have the unique decomposition

$$F = \sum_{N \in \mathscr{N}} F_N \quad \text{with} \quad F_N \in \mathfrak{F}_G(\mathscr{V}) \tag{4.5}$$

where

$$\mathscr{N} = \overbrace{Z_2 \times \cdots \times Z_2}^{p\text{-times}} = \{(N_1, \ldots, N_p) \mid N_i = 0, 1\} \tag{4.6}$$

and

$$\gamma^{(i)}(F_N) = (-1)^{N_i} F_N . \tag{4.7}$$

Actually the $\gamma^{(i)}$ generate a representation of the group $\mathscr{N}$ by automorphisms of $\mathfrak{F}_G$. Since $\mathscr{N}$ may be identified with its own dual, (4.5) corresponds to the decomposition of this representation into irreducible components. Let $\mathscr{V}$ and $\mathscr{V}'$ be two disjoint regions and $F \in \mathfrak{F}_G(\mathscr{V})$, $F' \in \mathfrak{F}_G(\mathscr{V}')$. One has

$$F_N F'_M = (-1)^{\sum_{i=1}^{p} N_i M_i} F'_M F_N . \tag{4.8}$$

We now define subnets $\mathfrak{A}_0$ and $\mathfrak{A}_1$ of $\mathfrak{F}_P$ by:

$$\begin{aligned} \mathfrak{A}_0(\mathscr{V}) &= \{F_{(0,\ldots,0)}, F_{(1,\ldots,1)} \mid F \in \mathfrak{F}_P(\mathscr{V})\} , \\ \mathfrak{A}_1(\mathscr{V}) &= \{F_{(0,\ldots,0)} \mid F \in \mathfrak{F}_P(\mathscr{V})\} . \end{aligned} \tag{4.9}$$

Consider the following conditions for a net $\mathfrak{A}$ of observables:
(i) $\mathfrak{A}(\mathscr{V}) \subset \mathfrak{F}_P(\mathscr{V})$
(ii) $[\mathfrak{A}(\mathscr{V}), \mathfrak{A}(\mathscr{V}')] = 0$ if $\mathscr{V} \cap \mathscr{V}'$ is empty.

4.1 Lemma. *A net* $\mathfrak{A}$ *satisfies* (i) *and* (ii), *if and only if*

$$\begin{aligned} \mathfrak{A}(\mathscr{V}) &\subset \mathfrak{A}_0(\mathscr{V}), \quad p \text{ even} \\ \mathfrak{A}(\mathscr{V}) &\subset \mathfrak{A}_1(\mathscr{V}), \quad p \text{ odd}. \end{aligned} \tag{4.10}$$

Proof. Let $\mathfrak{A}$ be a net satisfying (i) and (ii), and let $F \in \mathfrak{A}(\mathscr{V})$. Then for all x such that $\mathscr{V} \cap \mathscr{V} + x$ is empty we have

$$\begin{aligned}[F, \alpha_x(F)] &= \sum_{N, N'} [F_N, \alpha_x(F_{N'})] \\ &= \sum_{N, N'} [1 - (-1)^{\sum_i N_i N_i'}] F_N \alpha_x(F_{N'}) \\ &= 0 .\end{aligned} \tag{4.11}$$

Let $\mathscr{N}_F = (N \in \mathscr{N} \,|\, F_N \neq 0)$.

Applying the same technique as in the proof of Lemma 2.1 one may show that

$$\sum_i N_i N_i' \quad \text{is even for all } N, N' \in \mathscr{N}_F . \tag{4.12}$$

Furthermore since F is invariant under any permutation of the Green indices, so is $\mathscr{N}_F$. Using (4.12) one may thus derive the following condition for all $N \in \mathscr{N}_F$

$$N_i = N_j \quad \text{for all } i, j . \tag{4.13}$$

Combining (4.12) and (4.13) we get

$$\begin{aligned} &\mathscr{N}_F \subset \{(0, \ldots, 0)\} && \text{for } p \text{ odd,} \\ &\mathscr{N}_F \subset \{(0, \ldots, 0), (1, \ldots, 1)\} && \text{for } p \text{ even} \end{aligned} \tag{4.14}$$

and hence the result.

Condition (4.14) is equivalent to the invariance of F under each of the following automorphisms

$$\gamma = \prod_{i=1}^{p} \gamma^{(i)}; \qquad \gamma^{(ij)} = \gamma^{(i)} \gamma^{(j)} \quad \text{for all} \quad i \neq j .$$

For $p > 2$ these automorphisms do not induce automorphisms on $\mathfrak{F}_P$ so that we cannot, in general, characterize $\mathfrak{A}_0$ by invariance properties of $\mathfrak{F}_P$.

A generating set for $\mathfrak{A}_0$ can now be obtained by using decomposition theorems [13] for polynomials in the parafields and taking norm limits. These generators are of the form

$$\begin{aligned} r(f, g) &= \sum_{i=1}^{p} \psi^{(i)}(f) \, \psi^{(i)}(g) , \\ s(f_1, \ldots f_p) &= \sum_{\pi} \psi^{(\pi_1)}(f_1) \ldots \psi^{(\pi_p)}(f_p) , \end{aligned} \tag{4.15}$$

where the sum is to be taken over all permutations π of $(1, \ldots, p)$. The quantities $r(f, g)$ alone form a generating set for $\mathfrak{A}_1$. We again consider the net $\mathfrak{A}_2$ constructed by using the "neutral" generators $\varrho(\cdot, \cdot)$, where:

$$\varrho(f, g) = \sum_{i=1}^{p} \psi^{(i)}(f)^* \psi^{(i)}(g) . \tag{4.16}$$

The following inclusions hold:

$$\mathfrak{F}_G \supset \mathfrak{F}_P \supset \mathfrak{A}_0 \supset \mathfrak{A}_1 \supset \mathfrak{A}_2 . \tag{4.17}$$

The nets $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ may now be characterized in terms of invariance properties by considering them as subnets of a net $\mathfrak{F}$ generated by p Fermi fields with normal commutation relations:

$$\{\phi^{(i)}(f), \phi^{(j)}(g)\} = \delta_{ij}(\hat{f}, g) \quad \text{for} \quad i = 1, \ldots, p .$$

We can embed both $\mathfrak{F}$ and $\mathfrak{F}_G$ in a net $\tilde{\mathfrak{F}}$ where $\tilde{\mathfrak{F}}(\mathscr{V})$ is generated by elements of the form

$$\sum_{H \in \mathscr{K}} F_H H \quad \text{with} \quad F_H \in \mathfrak{F}(\mathscr{V})$$

where the group $\mathscr{K}$ of Klein operators is generated by elements H_k defined by the relations:

$$\begin{aligned} &\{H_k, \phi^{(i)}(f)\} = 0 \quad \text{for all } i \geqq k , \\ &[H_k, \phi^{(j)}(f)] = 0 \quad \text{for all } j < k , \\ &H_k = H_k^* = H_k^{-1} \quad \text{for } k \text{ even } \leqq p+1 . \end{aligned} \tag{4.18}$$

Within $\tilde{\mathfrak{F}}$ the generators of $\mathfrak{F}_G$ can be identified by

$$\begin{aligned} \psi^{(j)}(f) &= \phi^{(j)}(f) H_{j+1} \quad \text{for } j \text{ odd} \\ &= i\phi^{(j)}(f)\, H_j \quad \text{for } j \text{ even} . \end{aligned} \tag{4.19}$$

The parafield becomes:

$$\begin{aligned} \psi(f) &= \sum_{j=1}^{\frac{1}{2}p} \{\phi^{(2j-1)}(f) + i\phi^{(2j)}(f)\} H_{2j} \quad \text{for } p \text{ even} , \\ \psi(f) &= \sum_{j=1}^{\frac{1}{2}(p-1)} \{\phi^{(2j-1)}(f) + i\phi^{(2j)}(f)\} H_{2j} + \phi^{(p)}(f) \quad \text{for } p \text{ odd} . \end{aligned} \tag{4.20}$$

Again this identification gives the correct commutation relations and definition of the adjoint for the $\psi^{(i)}$. The generators of $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ may now be written in terms of the fields $\phi^{(i)}$ as:

$$r(f,g) = \sum_{i=1}^{p} \phi^{(i)}(f)\,\phi^{(i)}(g)\,, \tag{4.21}$$

$$s(f_1,\ldots,f_p) = (-i)^{[\frac{1}{2}p]} \sum_{\pi} \operatorname{sign}(\pi)\,\phi^{(\pi_1)}(f_1)\ldots\phi^{(\pi_p)}(f_p) \tag{4.22}$$

where $[\frac{1}{2}p]$ is the largest integer $\leqq \frac{1}{2}p$ and $\operatorname{sign}(\pi)$ is the signature of the permutation π.

$$\varrho(f,g) = \sum_{i=1}^{p} \phi^{(i)}(f)^*\,\phi^{(i)}(g)\,. \tag{4.23}$$

Consider automorphisms α_g on $\mathfrak{F}$ defined by (2.32), where g is a unitary $p\times p$ matrix. We get automorphism groups $\mathscr{G}_0$, $\mathscr{G}_1$ and $\mathscr{G}_2$ by letting g run through the matrix groups $SO(p)$, $O(p)$ and $U(p)$ respectively.

4.2 Theorem. *The nets $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ are precisely those subnets of $\mathfrak{F}$ which are invariant under the automorphism groups $\mathscr{G}_0$, $\mathscr{G}_1$, $\mathscr{G}_2$ respectively.*

The proof of this theorem is analogous to the proof of Theorem 2.2 and completes the extension of the results of Section II to para-Fermi fields of order $p>2$.

The assumptions and discussion in Section III leading to Theorem 3.1 apply without modification to order p. The representation π of $\mathfrak{F}$ can be extended to $\tilde{\mathfrak{F}}$ by requiring

$$\pi(H)\Omega = \Omega, \qquad H \in \mathscr{H}. \tag{4.24}$$

It remains to determine which states are obtained from the subspace $\mathscr{H}_P$ generated by applying the parafield algebra to Ω. To this end we study symmetry operations on certain subspaces of $\mathfrak{F}$ and $\mathfrak{F}_P$.

Let $f_1, f_2, , f_n$ be linearly independent test functions and consider the linear spaces $\mathfrak{F}^n \subset \mathfrak{F}$ and $\mathfrak{F}^n_P \subset \mathfrak{F}_P$ spanned by

$$\{\varphi^{(i_1)}(f_1)\ldots\varphi^{(i_n)}(f_n)\,|\,i_k = 1, 2, \ldots p\}$$

and $\{\psi(f_{\tau(1)})\ldots\psi(f_{\tau(n)})\,|\,\tau\in S_n\}$ respectively. Here S_n denotes the permutation group on n objects. On both spaces we get a representation of S_n by permuting test functions

$$\begin{aligned}\tilde{\tau}(\varphi^{(i_1)}(f_1)\ldots\varphi^{(i_n)}(f_n)) &= \varphi^{(i_1)}(f_{\tau(1)})\ldots\varphi^{(i_n)}(f_{\tau(n)})\\ &= \operatorname{sign}(\tau)\,\varphi^{(i_{\tau^{-1}(1)})}(f_1)\ldots\varphi^{(i_{\tau^{-1}(n)})}(f_n)\,,\end{aligned} \tag{4.25}$$

$$\tilde{\tau}(\psi(f_{\pi(1)})\ldots\psi(f_{\pi(n)})) = \psi(f_{\tau\pi(1)})\ldots\psi(f_{\tau\pi(n)})\,. \tag{4.26}$$

Considering $\mathfrak{F}^n$ and $\mathfrak{F}_P^n$ as subspaces of $\mathfrak{F}$, we get

$$\tilde{\tau}\left(\sum_{H\in\mathscr{K}} F_H H\right) = \sum_{H\in\mathscr{K}} \tilde{\tau}(F_H)H \quad \text{for} \quad \sum_{H\in\mathscr{K}} F_H H \in \mathfrak{F}_P^n. \tag{4.27}$$

We now show that $\mathscr{H}_P$ contains all relevant states of the observable algebra $\mathfrak{A}_2$.

4.3 Theorem. *Given an equivalence class σ of irreducible representations of $U(p)$ there is a non-zero vector in $\mathscr{H}_P$ transforming according to σ.*

Proof. Arguing as in the proof of Theorem 3.1, it suffices to show that we can find a non-zero tensor $F\in\mathfrak{F}$ of character σ, such that

$$\pi(F)E_0\mathscr{H}\subset\mathscr{H}_P. \tag{4.28}$$

Now $\mathfrak{F}^n$ carries a representation of $U(p)$ equivalent to the n-fold tensor product of the defining representation. This may be decomposed into irreducible components using Young symmetrizers from the group algebra of S_n where the action of S_n on $\mathfrak{F}^n$ is given by

$$\tau(\varphi^{(i_1)}(f_1)\dots\varphi^{(i_n)}(f_n)) = \varphi^{(i_{\tau^{-1}(1)})}(f_1)\dots\varphi^{(i_{\tau^{-1}(n)})}(f_n). \tag{4.29}$$

Consider a Young symmetrizer

$$\varepsilon = \sum_{\tau,\pi} \operatorname{sign}(\tau)\tau\pi \tag{4.30}$$

corresponding to a Young tableau of n squares and at most p rows, where π permutes the rows and τ the columns.

Comparing (4.25) and (4.29) we have $\tilde{\tau} = \operatorname{sign}(\tau)\tau$ and hence

$$\varepsilon = \sum_{\tau,\pi} \operatorname{sign}(\pi)\tilde{\tau}\tilde{\pi} \equiv \tilde{\varepsilon}'. \tag{4.31}$$

Now Y. Ohnuki and S. Kamefuchi [14] show that the representation (4.26) of S_n contains irreducible components corresponding to all Young tableaux with at most p columns and hence that for some $\sum_{H\in\mathscr{K}} F_H H\in\mathfrak{F}_P^n$

$$\tilde{\varepsilon}'\left(\sum_{H\in\mathscr{K}} F_H H\right) \neq 0. \tag{4.32}$$

But the F_H here are linearly independent since they have different transformation properties under the automorphisms (4.4) and hence by (4.27)

$$\varepsilon\left(\sum_{H\in\mathscr{K}} F_H\right) = \tilde{\varepsilon}'\left(\sum_{H\in\mathscr{K}} F_H\right) \neq 0. \tag{4.33}$$

Further

$$\pi\left(\varepsilon\sum_{H\in\mathscr{K}} F_H\right)E_0\mathscr{H} = \pi\left(\tilde{\varepsilon}'\sum_{H\in\mathscr{K}} F_H H\right)E_0\mathscr{H}\subset\mathscr{H}_P. \tag{4.34}$$

To show that all representations occur it suffices to remark that

$$\alpha_g(s(f_1, \ldots, f_p)) = (\det g)^{-1} s(f_1, \ldots, f_p), \qquad g \in U(p), \tag{4.35}$$

if the test functions are chosen so that $f_i' = 0$, $i = 1, 2, \ldots, p$. Hence multiplying an $\varepsilon\left(\sum_{H \in \mathscr{K}} F_H\right)$ by a suitable product of the $s(\cdot, \ldots, \cdot)$ we obtain a non-zero tensor of arbitrary character satisfying (4.28).

In conclusion we remark that the decomposition theorem [13] for polynomials in the parafield when combined with (4.20) shows at once that the only representations of $O(p)$ contained in $\mathscr{H}_P$ are those corresponding to Young tableaux with a single column. Hence when either $\mathfrak{A}_0$ or $\mathfrak{A}_1$ is the observable algebra not all physically relevant states are described by vectors from $\mathscr{H}_P$.

Appendix

For certain groups of linear transformations it is possible to prove what is known as the first main theorem in the theory of invariants [15], namely that every invariant polynomial in any number of vector variables is a polynomial in a finite number of fundamental invariants. At the end of Section II we wish to apply this result to the groups $U(2)$, $O(2)$ and $SO(2)$ but in a context which is not the classical one because the algebra of polynomials has been replaced by the Clifford algebra. In this appendix we justify this step by showing that the analogous result holds in the Clifford algebra for any group of linear transformations for which it can be proved in the polynomial algebra.

Let G be a group of linear transformations on a vector space E_0 over a field K which we may take to be the real or complex numbers. Let E be the direct sum of any infinite number of copies of E_0.

$$E = \bigoplus_{\alpha \in A} E_\alpha, \qquad E_\alpha \equiv E_0. \tag{A.1}$$

Let $\{\phi^i\}_{i \in I}$ be a basis for E_0 then we have a corresponding natural basis $\{\phi^i(\alpha)\}_{i \in I, \alpha \in A}$ for E. We now consider the polynomials in the variables $\{\phi^i(\alpha)\}_{i \in I, \alpha \in A}$[14]. The algebra of such polynomials is the symmetric algebra of E and we denote it by $\mathscr{S}(E)$. The algebra $\mathscr{S}(E)$ is independent of the choice of basis in E.

We also consider the algebra $\mathscr{A}(E)$, the antisymmetric algebra of E (or Grassmann algebra), where the variables $\phi^i(\alpha)$ are taken to anti-

[14] The notation for the basis has been chosen to help the reader apply the result to Section 2. The vector $\phi^i(\alpha)$ should be thought of as $\phi^{(i)}(f_\alpha)$ where f_α is a test function. The index i labels the component while $\{f_\alpha\}_{\alpha \in A}$ is chosen as a basis in test function space.

commute rather than commute

$$\phi^i(\alpha)\,\phi^j(\beta) + \phi^j(\beta)\,\phi^i(\alpha) = 0 \quad \text{all} \quad i, j \in I,\ \alpha, \beta \in A\,. \tag{A.2}$$

The group G acts in a natural way on $\mathscr{S}(E)$ and $\mathscr{A}(E)$ and we may speak of invariant elements of $\mathscr{S}(E)$ or $\mathscr{A}(E)$. In what follows we need to deal with mappings between spaces on which G acts. If such mappings commute with the action of G they will be called G-morphisms, linear G-morphisms, G-isomorphisms etc.

An element $\phi \in \mathscr{S}(E)$ (or $\mathscr{A}(E)$) will be called *generic* if it is a sum of monomials in each of which no index $\alpha \in A$ occurs more than once. In other words ϕ is generic if its partial degree in each vector variable $\phi(\alpha)$ is at most one. The first step in proving the first main theorem is to show that it suffices to restrict oneself to generic elements. The technique involved is called complete polarization [15].

Let $B \subset A$ be any finite set of indices and consider the subalgebra $\mathscr{A}(B)$ generated by $\varphi^i(\beta)$ with $\beta \in B$, $i \in I$. Let $\mathscr{A}_s(B)$ denote the G-invariant linear subspace of $\mathscr{A}(B)$ consisting of elements homogeneous of degree s. Pick distinct indices $(\beta, r) \in A$, $\beta \in B$, $r = 1, 2, \ldots, s$ and let $B_s = \{(\beta, r) \in A\colon \beta \in B,\ r = 1, 2 \ldots s\}$. Define

$$F(\phi^{i_1}(\beta_1)\ \ldots\ \phi^{i_s}(\beta_s)) = \frac{1}{s!}\sum_{\pi} \phi^{i_1}(\beta_1, \pi(1))\ \ldots\ \phi^{i_s}(\beta_s, \pi(s))\,, \tag{A.3}$$

where the sum is taken over all permutations π of $1, 2, \ldots, s$. To show that F can be extended to a linear G-isomorphism of $\mathscr{A}_s(B)$ onto a subspace of $\mathscr{A}(B_s)$, it suffices to check that F is well-defined. However an elementary calculation shows that

$$\begin{aligned} &F(\phi^{i_{\pi(1)}}(\beta_{\pi(1)})\ \ldots\ \phi^{i_{\pi(s)}}(\beta_{\pi(s)})) \\ &\qquad = \operatorname{sign}(\pi)\, F(\phi^{i_1}(\beta_1)\ \ldots\ \phi^{i_s}(\beta_s)), \quad \text{for all } \pi\,, \end{aligned} \tag{A.4}$$

where $\operatorname{sign}(\pi)$ is the signature of the permutation π. Hence F is well-defined. Further we may define a G-homomorphism F' of $\mathscr{A}(B_s)$ onto $\mathscr{A}_s(B)$ by substituting the variables $\phi(\beta)$ for $\phi(\beta, r)$, $r = 1, 2, \ldots, s$. In other words we define F' by setting

$$F'(\phi^i(\beta, r)) = \phi^i(\beta), \quad \text{all} \quad \beta \in B,\ i \in I,\ r = 1, 2, \ldots, s\,. \tag{A.5}$$

From (A.3) and (A.5) the composed mapping $F'F$ acts as the identity on $\mathscr{A}_s(B)$. Now given an invariant $\phi \in \mathscr{A}_s(B)$, $F(\phi)$ is a generic invariant in $\mathscr{A}(B_s)$. If $F(\phi)$ is a polynomial in the fundamental invariants in $\mathscr{A}(B_s)$

then using F' we deduce that ϕ itself is a polynomial in the fundamental invariants in $\mathscr{A}(B)$. The same argument holds, mutatis mutandis, for the symmetric algebra.

Hence in proving the first main theorem of invariants in $\mathscr{S}(E)$ or $\mathscr{A}(E)$ it suffices to consider the generic invariants of maximal degree in vector variables with indices from an arbitrary finite subset $B \subset A$. Order the elements of B as $\beta_1, \beta_2, \ldots, \beta_n$ say, and let $\mathscr{S}\langle B\rangle$ and $\mathscr{A}\langle B\rangle$ denote the G-invariant linear subspaces of generic elements homogeneous of degree n in $\mathscr{S}(B)$ and $\mathscr{A}(B)$ respectively. Since $\{\phi^{i_1}(\beta_1), \ldots, \phi^{i_n}(\beta_n)$: $i_1, i_2 \ldots i_n \in I\}$ can be considered as a basis for either $\mathscr{S}\langle B\rangle$ or $\mathscr{A}\langle B\rangle$ we have a linear G-isomorphism of these two linear spaces which we use to identify corresponding elements.

Suppose $\phi \in \mathscr{S}\langle B\rangle$ is invariant and may be expressed as a polynomial p in the fundamental invariants $f_1, f_2, \ldots, f_k$ of G.

$$\varphi = p(f_1, f_2, \ldots, f_k) \text{ in } \mathscr{S}(B). \tag{A.6}$$

Expand the polynomial p as a sum of monomials; in each monomial the indices $\beta_1, \beta_2, \ldots, \beta_n$ will require a certain permutation to bring them into their natural order. Let p' be the polynomial obtained by multiplying each monomial of p by the sign of the corresponding permutation, then

$$\phi = p'(f_1, f_2, \ldots, f_k) \text{ in } \mathscr{A}(B). \tag{A.7}$$

Conversely (A.7) implies (A.6) and we have proved

Theorem A.1. *$f_1, f_2, \ldots, f_k$ is a fundamental set of invariants for G in $\mathscr{A}(E)$ if and only if it is a fundamental set in $\mathscr{S}(E)$.*

We have still to extend the results to cover the Clifford algebra. Suppose E has a G-invariant symmetric bilinear form $(\,,\,)$, then the Clifford algebra $\mathscr{C}(E)$ is generated by $\{\varphi^i(\alpha)\}$, $i \in I$, $\alpha \in A$ with the commutation relations

$$\phi^i(\alpha)\,\phi^j(\beta) + \phi^j(\beta)\,\phi^i(\alpha) = (\phi^i(\alpha), \phi^j(\beta)). \tag{A.8}$$

Now $\mathscr{A}(E)$ and $\mathscr{C}(E)$ are quotient algebras of the free algebra generated by $\{\varphi^i(\alpha)\}_{i \in I, \alpha \in A}$ with respect to the ideals $\mathscr{I}_1$ and $\mathscr{I}_2$ generated by the relations (A.2) and (A.8) respectively. We note the following two facts:

a) If $\phi_2 \in \mathscr{I}_2$ has degree n there exists a $\phi_1 \in \mathscr{I}_1$ such that $\phi_1 - \phi_2$ is of degree at most $n-2$.

b) $\mathscr{I}_1$ is a direct sum of linear subspaces of elements homogeneous of degree n, $n = 2, 3, \ldots$.

Theorem A.2. *If $f_1, f_2, \ldots, f_k$ is a fundament set of invariants for G in $\mathscr{A}(E)$, it is a fundamental set in the Clifford algebra $\mathscr{C}(E)$.*

Proof. We proceed by induction on the degree of the invariant. Suppose all invariants in $\mathscr{C}(E)$ of degree $<n$ are polynomials in $f_1, f_2, \ldots, f_k$. Given an invariant $\phi \in \mathscr{C}(E)$ of degree n then by writing it in a totally antisymmetric fashion we may consider it as an invariant of degree n in $\mathscr{A}(E)$. Then by hypothesis $\phi = p(f_1, f_2, \ldots, f_k)$ in $\mathscr{A}(E)$, i.e.

$$\phi - p(f_1, f_2, \ldots, f_k) \in \mathscr{I}_1 .$$

Furthermore by b) we may suppose that $p(f_1, f_2, \ldots, f_k)$ also has degree n. Hence by a) $\phi - p(f_1, f_2, \ldots, f_k)$ has degree at most $n-2$ in $\mathscr{C}(E)$. Thus ϕ must also be a polynomial in $f_1, f_2, \ldots, f_k$ in $\mathscr{C}(E)$. However for invariants of degree 0, and 1 the induction hypothesis is trivially satisfied and the proof is complete.

As we have seen in Section II, the converse of this theorem need not hold. We may well require fewer fundamental invariants in $\mathscr{C}(E)$ than in $\mathscr{A}(E)$ and hence than in $\mathscr{S}(E)$. Naturally we can prove the analogue of Theorem A.2 for the algebra of the canonical commutation relations

$$\phi^i(\alpha)\, \phi^j(\beta) - \phi^j(\beta)\, \phi^i(\alpha) = (\phi^i(\alpha), \phi^j(\beta))$$

where (,) is now a G-invariant skew-symmetric bilinear form.

References

1. Green, H. S.: A generalized method of field quantization. Phys. Rev. **90**, 270 (1953).
2. Volkov, D.: S-matrix in the generalized quantization method. Soviet Phys. JETP **11**, 375 (1960).
3. Greenberg, O. W., Messiah, A. M. L.: Selection rules for parafields and the absence of para particles in nature. Phys. Rev. B **138**, 1155 (1965).
4. — Parafield theory. In: Proceedings of Conference on the Mathematical Theory of Elementary Particles. Edited by R. Goodman and I. Segal. London-New York: MIT-Press 1966.
5. Landshoff, P. V., Stapp, Henry P.: Parastatistics and a unified theory of identical particles. Ann. Phys. **45**, 72 (1967).
6. Doplicher, S., Haag, R., Roberts, J. E.: Fields, observables and gauge transformations I. Commun. Math. Phys. **13**, 1 (1969).
7. Segal, I. E.: Tensor algebras over Hilbert spaces II. Ann. Math. **63**, 160 (1956).
8. Guichardet, A.: Produits tensoriels infinis et répresentations des relations d'anticommutation. Ann. Sci. Ecole Norm. Super. **83**, 1 (1966).
9. Doplicher, S., Kastler, D., Robinson, D. W.: Covariance algebras in field theories and statistical mechanics. Commun. Math. Phys. **3**, 1 (1966).
10. Borchers, H. J.: Local rings and the connection of spin and statistics. Commun. Math. Phys. **1**, 291 (1965).

11. Doplicher, S., Haag, R., Roberts, J. E.: Fields, observables and gauge transformations II. Commun. Math. Phys. **15**, 173 (1969).
12. Yang, C. N.: Concept of off-diagonal long-range order and the quantum phases of liquid He and of superconductors. Rev. Mod. Phys. **34**, 694 (1962).
13. Ohnuki, Y., Kamefuchi, S.: Some general properties of para-Fermi field theory. Phys. Rev. **170**, 1279 (1968).
14. — — Wavefunctions of identical particles. Ann. Phys. **51**, 337 (1969).
15. Weyl, H.: The classical groups. Princeton University Press, Princeton 1946.

K. Drühl
R. Haag
J. E. Roberts
II. Institut für Theoretische Physik
der Universität
D-2000 Hamburg 50, Luruper Chaussee 149