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The Operational Approach to Algebraic Quantum Theory I

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Abstract. Recent work of Davies and Lewis has suggested a mathematical framework in which the notion of repeated measurements on statistical physical systems can be examined. This paper is concerned with an examination of their formulation in the abstract and its application to the C^*-algebra model for quantum mechanics. In particular, a study is made of the notion of the restriction of a physical system and a definition, which coincides with the usual definition in the C^*-algebra model, is formulated.

§ 1. Introduction

In the conventional approach to quantum mechanics, the observables are represented by measures defined on the Borel sets of the real line $\mathbb{R}$ and taking values in the set of projection operators on some separable Hilbert space X, and hence correspond in general to self-adjoint operators on X. The set of states is represented by the set of infinite convex combinations of vector states of the Von Neumann algebra $\mathfrak{L}(X)$ of bounded linear operators on X and hence corresponds to the set of positive trace class operators on X of trace unity. In this approach, the bounded observables may be identified with those projection-valued measures concentrated on compact subsets of $\mathbb{R}$, or equivalently, as self-adjoint elements of $\mathfrak{L}(X)$. There are strong physical reasons for considering only the bounded observables of a physical system. In general, the very nature of the experiments used to determine the properties of many physical observables automatically imposes boundedness conditions on the observables. In addition, the observables usually measured are those which can take only two possible values, the so-called questions. In the conventional approach, the questions correspond to projection operators and are, of course, always bounded. For this reason it has been suggested that a possible model for the set $\mathcal{O}$ of bounded observables of a quantum mechanical system should be some subset $\mathcal{D}$ of the set of self-adjoint elements of $\mathfrak{L}(X)$. In this sense, $\mathcal{D}$ is supposed to represent a subset of the set of all observables and for this reason it was supposed that the states would continue to be represented by infinite convex combinations of vector states of $\mathfrak{L}(X)$, though some would be expected to coalesce on $\mathscr{D}$. Regarded as positive normalised linear functionals on $\mathfrak{L}(X)$, this set of states is precisely the set of ultraweakly continuous positive linear functionals on $\mathfrak{L}(X)$ of norm unity, when regarded as elements of the dual space $\mathfrak{L}(X)^*$ of $\mathfrak{L}(X)$. Moreover, the set of ultraweakly continuous positive linear functionals on any Von Neumann algebra $\mathfrak{B}$ coincides with the set of positive normal linear functionals on $\mathfrak{B}$.

Various candidates for the set $\mathscr{D}$ have been proposed. One of the principle properties which the set $\mathscr{D}$ must possess, if an approach along the lines described above is to be attempted, is that the spectral projections corresponding to an element of $\mathscr{D}$ must also lie in $\mathscr{D}$. That is to say, the question "does a measurement of the observable $\mathscr{A}$ lead to a value in the Borel subset M of $\mathbb{R}$" must be observable if $\mathscr{A}$ is observable. The strongest contender for the role of $\mathscr{D}$ is the set of self-adjoint elements of a Von Neumann algebra $\mathfrak{B}$. This not only possesses the spectral property described above, but also the set of positive normal linear functionals of norm unity on $\mathfrak{B}$ is the set of infinite convex combinations of vector states. These remarks are valid whether or not X is separable and so a modification of the conventional model for quantum theory leads to the following model.

(a) Observables — Self-adjoint elements in a Von Neumann algebra $\mathfrak{B}$ acting on a Hilbert space X.
States — Positive normal linear functionals of norm unity on $\mathfrak{B}$-the normal states of $\mathfrak{B}$.

Another possible contender for $\mathscr{D}$ was considered by Davies [2] and Plymen [24]. By considering the spectral properties of operators Davies arrived at a definition for a Σ^*-algebra $\mathfrak{B}$, which may be represented concretely as a C^*-sub-algebra of $\mathfrak{L}(X)$ closed in the weak sequential topology. The σ-states of $\mathfrak{B}$ are defined to be these elements f of $\mathfrak{B}^*$ of norm unity such that if $\{T_n\}$ is a sequence of elements of $\mathfrak{B}$ converging weakly to T in $\mathfrak{B}$, then $f(T_n)$ converges to $f(T)$. When X is separable, every Σ^*-algebra $\mathfrak{B}$ is a Von Neumann algebra and the set of σ-states of $\mathfrak{B}$ coincides with its set of normal states. Plymen [24] showed that the following model satisfies the essential properties of a physical system as described by Mackey [23].

(b) Observables — Self-adjoint elements in a Σ^*-algebra $\mathfrak{B}$ acting on a Hilbert space X.
States — σ-states of $\mathfrak{B}$.

In a sense, models (a) and (b) coincide when X is separable.

A further approach, at first sight completely divorced from either models (a) or (b), was suggested by Segal [26].

(c) Observables — Self-adjoint elements of an abstract C^*-algebra $\mathfrak{A}$.
States — A full family of positive elements of $\mathfrak{A}^*$ of norm unity-a full family of states of $\mathfrak{A}$.

Here a full family B_0 of states of $\mathfrak{A}$ is defined to be a convex subset of the set $S(\mathfrak{A})$ of states of $\mathfrak{A}$ such that, if $f(a) \geqq 0$ for all f in B_0 and some a in $\mathfrak{A}$, then $a \geqq 0$. Kadison [20] showed that B_0 is a full family if and only if it is weak* dense in $S(\mathfrak{A})$. $S(\mathfrak{A})$ is itself an example of a full family. When examined from the conventional viewpoint there appears little physical motivation for this model. The spectral projections of an arbitrary observable are no longer well defined and even if a faithful representation ϕ of $\mathfrak{A}$ is taken on some Hilbert space X, there is no guarantee that the spectral projections of a self-adjoint element of $\phi(\mathfrak{A})$ lie in $\phi(\mathfrak{A})$. These spectral projections do however lie in the Von Neumann algebra $\phi(\mathfrak{A})''$ generated by $\phi(\mathfrak{A})$ and indeed also in the smallest Σ^*-algebra $\phi(\mathfrak{A})^\sigma$ containing $\phi(\mathfrak{A})$. In order to connect the model (c) with models (a) and (b), it seems to be necessary to take some representation ϕ of $\mathfrak{A}$ on Hilbert space. Different representations will of course produce different models (a), (b) for the same model (c). In addition, in order that the states of the model (c) shall have any meaning at all, it is clearly necessary that for each normal state ω of $\phi(\mathfrak{A})''$ or σ-state ω of $\phi(\mathfrak{A})^\sigma$, ϕ must be chosen in such a way that $\omega \circ \phi$ lies in B_0. Such representations were said to be complete by Kadison [20]. Haag and Kastler [16] studied the problems involved in taking the C^*-algebra model and its complete representations and suggested reasons for the existence of an underlying model (c). Using a notion of physical equivalence for models (a) obtained from a particular model (c) by taking complete representations, they showed that the physical information obtained by taking faithful representations was essentially the same in each case.

Recently, Davies and Lewis [4] have considered an axiomatic approach to statistical physical theories which differs from the conventional approach of Von Neumann [29] and Mackey [23]. This theory depends upon the set of states being regarded as the basic physical object, whilst the observables are defined by means of operations on the states. Haag and Kastler [16] first suggested this approach in the context of model (c).

In this and later papers the assumptions and results of Lewis and Davies will be considered in some detail in the abstract and applied to the special case in which the set of states is represented by some subset either of the set of positive linear functionals on a C^*-algebra $\mathfrak{A}$ or of the set of positive normal linear functionals on a Von Neumann algebra $\mathfrak{B}$. Without loss of generality but with considerable saving in technical detail, $\mathfrak{A}$ will be supposed to have an identity. The principle results which emerge are the following. The sets of states described above are sufficient to describe all the models so far proposed, not only for quantum but also for classical probability theories. An abstract definition of the restriction of a physical system may be formulated which when applied to the C^*-algebra model explains the necessity of taking representations of $\mathfrak{A}$. In fact, it is shown that restrictions, in the sense adopted here, are in one-one correspondence with quasi-equivalent classes of representations. These results rest heavily on the order structure properties of C^*-algebras and Von Neumann algebras (see [1, 11, 25, 27]).

§ 2. Partially Ordered Vector Spaces

This section is devoted to a description of the results required to discuss the abstract formulation of statistical physical systems. The reader is referred to [1, 10, 12, 13, 27] for proofs.

A non-empty subset K of a real vector space V is said to be a *cone* for V providing that $K+K\subset K$, $\alpha K\subset K$, $\alpha \geqq 0$, and $K\cap(-K)=\{0\}$. The cone K defines a partial ordering on elements of V, if $f\geqq g$ is defined to mean that $f-g\in K$. If $K-K=V$, V is said to be *positively generated* or alternatively, generated by K. An *order ideal* L of V is a vector subspace of V such that $0\leqq f\leqq g\in L$ implies that $f\in L$. If L is an order ideal in V, K/L is a cone in V/L and if K generates V, K/L generates V/L. L is said to be *positively generated* if $L=L\cap K-L\cap K$. A non-empty convex subset B of K is said to be a *base* for K if for each $f\in K$, $f\neq 0$, there exist uniquely $g\in B, \alpha>0$ such that $f=\alpha g$. A linear functional T on V is said to be *positive* if $T(K)\geqq 0$ and *strictly positive* if $T(K-\{0\})>0$. K has a base B if and only if there exists a strictly positive linear functional e on V and in this case B may be written as $\{f: f\in K, e(f)=1\}$. If for $f\in V$,

$$\|f\|_B=\inf\{\lambda>0: f\in\lambda\,\mathrm{conv}(B\cup(-B))\} \tag{2.1}$$

where $\mathrm{conv}(B\cup(-B))$ is the convex hull of $B\cup(-B)$, then, providing that K generates V, $\|\,.\,\|_B$ is a semi-norm on V and

$$\|f\|_B=\inf\{e(f_1)+e(f_2): f_1, f_2\in K, f_1-f_2=f\} \tag{2.2}$$

where e is the strictly positive linear functional corresponding to B. $\|\,.\,\|_B$ is a norm on V if and only if $\mathrm{conv}(B\cup(-B))$ is linearly bounded and under these conditions (V, B) is said to be a *base norm space*. $\mathrm{conv}(B\cup(-B))$ contains the open unit ball and is contained in the closed unit ball for the base norm $\|\,.\,\|_B$. When $\mathrm{conv}(B\cup(-B))$ is linearly compact it coincides with the closed unit ball in (V, B).

Let (V, B) be a base norm space with generating cone K. A non-empty convex subset F of B is said to be a *face* providing that for $f\in F$ such that

$f = tf_1 + (1-t) f_2$ with $f_1, f_2 \in B$, $t \in (0,1)$ implies that $f_1, f_2 \in F$. A non-empty subset H of K such that $H + H \subset H$, $\alpha H \subset H$, $\alpha \geqq 0$ and such that for $f \in H$, $f = tf_1 + (1-t) f_2, f_1, f_2 \in K, t \in (0,1)$ implies $f_1, f_2 \in H$, is said to be an *extremal set*. H is an extremal set of K if and only if $H \cap B$ is a face of B and every extremal set arises in this way. An order ideal L in V intersects B in a face and K in an extremal set providing $L \cap K \neq \{0\}$. Conversely, for each extremal set H, $L = H - H$ is an order ideal.

Let U be a real vector space with generating cone C. The partial ordering defined by C is said to be *almost Archimedean* if $-\lambda b \leqq a \leqq \lambda b$ for some $b \in C$ and all $\lambda > 0$ implies $a = 0$ and is said to be *Archimedean* if $a \leqq \lambda b$ for some $b \in C$ and all $\lambda > 0$ implies $a \leqq 0$. An element e of C is said to be an *order unit* for C if for each $a \in U$, there exists $\lambda > 0$ such that $-\lambda e \leqq a \leqq \lambda e$. Let

$$\|a\|_e = \inf\{\lambda > 0 : -\lambda e \leqq a \leqq \lambda e\}. \tag{2.3}$$

Then $\|.\|_e$ is a norm on U if and only if U is almost Archimedean ordered in which case (U, e) is said to be an *order unit space*. $[-e, e]$ is contained in the closed unit ball and contains the open unit ball for the order unit norm $\|.\|_e$. If U is Archimedean ordered $[-e, e]$ is the closed unit ball for the norm $\|.\|_e$.

Let U be a locally convex Hausdorff space with dual U^* and let C, C' be cones in U, U^* respectively. Let

$$C^* = \{f : f \in U^*, f(a) \geqq 0, \forall a \in C\},$$
$$C'_* = \{a : a \in U, f(a) \geqq 0, \forall a \in C'\}.$$

Then C^* is a cone in U^* if and only if $C - C$ is dense in U and if C is closed in U, $C^* - C^*$ is $\sigma(U^*, U)$ dense in U^*. C, C' are said to be *compatible* cones if $C' = C^*, C = C'_*$. This is the case if and only if, either $C' = C^*$ and C is closed or $C = C'_*$ and C' is $\sigma(U^*, U)$ closed.

Let (V, B) be a base norm space with closed generating cone K. Then its dual space V^* possesses an order unit e such that $B = \{f : f \in K, e(f) = 1\}$ and the norm on V^* as a dual space coincides with its order unit norm. Moreover, K, K^* are compatible cones and the ordering of V^* is Archimedean. Hence (V^*, e) is a complete order unit space with closed unit ball $[-e, e]$. V^* may be identified with the space of bounded affine functionals on B in the supremum norm. Conversely, let U be a real vector space with generating cone C and order unit e and possessing a locally convex Hausdorff topology τ such that $[0, e]$ is τ-compact. Then, (U, e) is a complete order unit space and there exists a complete base norm space (V, B) with generating cone K, norm closed in V, such that (U, e) is the dual of (V, B) and K, C are compatible cones. V may be identified with the space of linear functionals on U, τ-continuous on norm bounded subsets of U. The topologies τ and $\sigma(U, V)$ coincide on norm bounded subsets of U and on C. Moreover, $B=\{f : f\in K, e(f)=1\}$. For the dual of this result the reader is referred to [10].

Let U be a complete normed real vector space and let U^* be its dual. Let C, C^* be compatible cones in U, U^* respectively. For each subset J of U, let $J^0=\{f : f\in U^*, f(a)=0, \forall a\in J\}$ and for each subset L of U^*, let $L_0=\{a : a\in U, f(a)=0, \forall f\in L\}$. Then, J^0 is a $\sigma(U^*, U)$ closed subspace of U^*, $(J^0)_0$ is the closed subspace of U generated by J, L_0 is a norm closed subspace of U and $(L_0)^0$ is the $\sigma(U^*, U)$ closed subspace of U^* generated by L. If $J\subset J\cap C - J\cap C$, in which case J is said to be positively generated, J^0 is an order ideal and if L is positively generated L_0 is an order ideal.

A positively generated, norm closed order ideal J in the complete, Archimedean ordered order unit space (U, e) is said to be *Archimedean* if when $\phi : U\to U/J$ is the canonical mapping, $(U/J, \phi(e))$ is an Archimedean ordered order unit space in the quotient ordering. A linear mapping ψ from the order unit space (U, e) with cone C onto the order unit space (U', e') with cone C' is said to be an *order homomorphism* if $\psi(C)=C'$, $\psi(e)=e'$ and the kernel J of ψ is positively generated. An order ideal J in (U, e) is Archimedean if and only if it is the kernel of some order homomorphism. The following conditions on a Archimedean ideal J in (U, e) are equivalent.

(i) It is the kernel of an order homomorphism which is an open mapping for the order unit norm topologies,

(ii) $(U/J, \phi(e))$ is complete,

(iii) The quotient norm on U/J is equivalent to the order unit norm in $(U/J, \phi(e))$,

(iv) J^0 is positively generated.

In this case J is said to be *strongly Archimedean*. In particular when the quotient and order unit norms in U/J coincide, J is said to be strongly Archimedean and of *characteristic unity*. There exists a one-one correspondence between Archimedean ideals in (U, e) and $\sigma(U^*, U)$ closed faces F of the set $B=\{f : f\in C^*, e(f)=1\}$ which satisfy the condition that for each $a\in U$, non-negative on F, there exists $b\in C$, $b\geqq a$ and $b(f)=a(f)\ \forall f\in F$. A weaker dual of this result is proved in § 4.

Let (V, B) be a complete base norm space with norm closed generating cone K and let $\mathfrak{L}(V)$ denote the set of bounded linear mappings from V to itself. A mapping j from V to itself such that $j(K)\subset K$ is said to be positive and the set of all such mappings forms a cone $\mathfrak{K}$ in $\mathfrak{L}(V)$, closed in the strong topology.

A *Borel space* $(\mathscr{S}, \mathscr{B})$ is a space $\mathscr{S}$ together with a set $\mathscr{B}$ of subsets of $\mathscr{S}$ which is closed under the formation of countable unions, countable intersections and complements and contains $\mathcal{S}$ as an element. Let V be a real topological vector space with a generating cone K. A mapping $\mu:\mathcal{B}\to K$ such that for any set $\{M_n\}$ of mutually disjoint elements of B,

$$\mu\left(\bigcup_{n=1}^{\infty} M_n\right) = \sum_{n=1}^{\infty} \mu(M_n)$$

where the sum converges in the topology of V is said to be a positive V-valued measure on $(\mathcal{S}, \mathcal{B})$.

§ 3. Physical Formulation

This section is devoted to a discussion of the mathematical formulation of the operational approach to statistical physical systems suggested by Davies and Lewis [4]. The *states* of the physical system consist of ensembles of identical copies of the system and it is supposed that it is possible to perform filtering operations on the states. For each state f, $e(f)$ denotes a positive real number proportional to the number of copies in the ensemble. Two states f, g may be combined to form a further state $f+g$ and then clearly $e(f+g)=e(f)+e(g)$. When the number of copies in the state f is increased in some proportion $\alpha>0$, a new state αf is formed and clearly $e(\alpha f)=\alpha e(f)$. The empty state 0 of the system clearly satisfies $e(0)=0$ and in addition, if $f\neq 0$, $e(f)\neq 0$. Hence, the set of states forms an abstract cone K and e is a strictly positive affine functional on K. e is said to be the *strength functional.* K can be identified with a generating cone for a real vector space V and e can be extended to a strictly positive linear functional on V in the obvious manner. Hence $B=\{f : f\in K, e(f)=1\}$ is a base for K. Elements of B are said to be normalised states and these clearly determine all the states of the system. Hence, V possesses a semi-norm $\|.\|_B$ defined by (2.2) though, in general this need not be a norm on V.

In most statistical approaches to physical theories (see, for example [23]) it is supposed that the set of states is closed not only under the formation of finite mixtures, but also under the formation of countable mixtures. This assumption is equivalent to the postulate that for each monotone increasing sequence $\{f_n\}$ in K (i.e. $m\leqq n$ implies $f_m\leqq f_n$) such that $\{e(f_n)\}$ is bounded above, there exists a unique element f in K such that $f_n\leqq f$ and $e(f_n)\to e(f)$. Under these conditions it may be shown (see [9]) that (V, B) is a complete base norm space. It is not in general true that K is closed for the base norm though a result of Ellis [13] shows that the closure $\bar{B}$ of B is a base for the cone $\bar{K}$, the closure of K, and that $(V, \bar{B})$ is a complete base norm space with norm identical to that in (V, B). Hence, there is little loss of generality in supposing that K is closed in the base norm. Closing the cone is equivalent to adding certain possibly "non-physical" states. It was assumed without any discussion by Davies and Lewis [4] that (V, B) is a complete base norm space with norm closed cone K. The formation of countable mixtures of normalised states has been studied by Gerzon [15].

Postulate 1. *The set of states of a physical system is represented by the set of elements of a closed generating cone K for a complete base norm space (V, B).*

An *operation* j on a state f is a filtering process which changes f into a new state $j(f)$. Then clearly, $e(j(f)) \leqq e(f)$ and the number $e(j(f))/e(f)$ may be interpreted as the transmission probability of the state f under the operation j. It follows from the definitions of addition and scalar multiplication of states that for $f, g \in K$, $\alpha \geqq 0$, $j(f+g) = j(f)+j(g)$, $j(\alpha f) = \alpha j(f)$. Hence j can be extended to a positive linear mapping from V to itself in the obvious manner. Moreover, a simple calculation using (2.2) shows that for $f \in V$, $\|j(f)\|_B \leqq \|f\|_B$ and hence that j is an element of the unit ball $\mathfrak{L}(V)_1$ of the Banach space $\mathfrak{L}(V)$ of bounded linear operators from V to itself.

Postulate 2. *The set $\mathcal{P}$ of operations on the physical system is represented by the set of positive elements in the unit ball of $\mathfrak{L}(V)$.*

Since (V, B) is a base norm space, its dual (V^*, e) is an order unit space with $\sigma(V^*, V)$ closed generating cone K^*. Moreover, K, K^* are compatible cones. For each element j of $\mathcal{P}$, the mapping $f \mapsto e(j(f))$ is a positive bounded affine functional on B such that $e(j(f)) \leqq e(f) = 1$ for f in B and hence, there exists $T(j)$ in V^* such that

$$0 \leqq T(j) \leqq e \tag{3.1}$$

and

$$T(j)(f) = e(j(f)) \tag{3.2}$$

for all f in V. It follows that $j \mapsto T(j)$ is a mapping from $\mathcal{P}$ to $[0, e]$. In addition, for $T \in [0, e]$, let $j: V \to V$ be defined for $f \in V$ by

$$j(f) = T(f)\, g \tag{3.3}$$

for some $g \in B$. Then, clearly $j \in \mathcal{P}$ and $T(j) = T$. It follows that the mapping $j \mapsto T(j)$ sends $\mathcal{P}$ onto $[0, e]$. The set $\mathcal{Q} = [0, e]$ is said to be the set of *simple observables* of the system. Two elements $j, k \in \mathcal{P}$ are said to be *isotonic* when $T(j) = T(k)$. There is clearly a one-one correspondence between simple observables and isotony classes of operations.

Postulate 3. *The set $\mathcal{Q}$ of simple observables of the physical system is represented by the set $[0, e]$ in the dual space (V^*, e) of (V, B).*

A discussion of the many physically interesting subsets of $\mathcal{P}$ and $\mathcal{Q}$ will appear elsewhere.

According to [4], an *instrument* $\mathscr{E}$ may be regarded as a movable slit parametrised by the set $\mathscr{B}$ of Borel subsets of some Borel space $(\mathscr{S}, \mathscr{B})$. Hence, for $M \in \mathscr{B}, \mathscr{E}(M) \in \mathscr{P}$ and for each set $\{M_n\}$ of mutually disjoint elements of $\mathscr{B}$, and each state f,

$$\mathscr{E}\left(\bigcup_{n=1}^{\infty} M_n\right)(f) = \sum_{n=1}^{\infty} \mathscr{E}(M_n)(f) \tag{3.4}$$

where the sum is supposed to converge in the base norm topology of (V, B). In addition, for each state f,

$$e(\mathscr{E}(\mathscr{S})(f)) = e(f). \tag{3.5}$$

Postulate 4. *The set of instruments on the physical system is represented by the set of $\mathscr{P}$-valued measures $\mathscr{E}$ on Borel spaces $(\mathscr{S}, \mathscr{B})$ such that $e(\mathscr{E}(\mathscr{S})(f)) = e(f)$, $\forall f \in K$.*

In the Mackey approach [23], $\mathscr{S}$ is always supposed to be some subset of $\mathbb{R}$. In general, for $f \in K$, $M \in \mathscr{B}$,

$$e(\mathscr{E}(M)\, f)/e(f)$$

is the probability that a measurement with the instrument $\mathscr{E}$ on the state f yields a value in M.

Two instruments $\mathscr{E}, \mathscr{E}'$ both based on $(\mathscr{S}, \mathscr{B})$ are said to be *isotonic* if for all $M \in B$,

$$T(\mathscr{E}(M)) = T(\mathscr{E}'(M)). \tag{3.6}$$

Let $\mathscr{E}$ be an instrument based on $(\mathscr{S}, \mathscr{B})$ and for $M \in \mathscr{B}$, define $\mathscr{A}(M) \in \mathscr{Q}$ by

$$\mathscr{A}(M) = T(\mathscr{E}(M)). \tag{3.7}$$

Then, clearly $M \mapsto \mathscr{A}(M)$ is a $\mathscr{Q}$-valued measure based on $(\mathscr{S}, \mathscr{B})$ satisfying $\mathscr{A}(\mathscr{S}) = e$. Such a mapping is said to be an *observable.* It follows from (3.6) that there is a one-one correspondence between isotony classes of instruments and observables.

Postulate 5. *The set of observables of the physical system is represented by the set of $\mathscr{Q}$-valued measures $\mathscr{A}$ on Borel spaces $(\mathscr{S}, \mathscr{B})$ such that $\mathscr{A}(\mathscr{S}) = e$.*

Let $(\mathscr{S}_1, \mathscr{B}_1), (\mathscr{S}_2, \mathscr{B}_2)$ be standard Borel spaces (see [22]) and let $\mathscr{E}_1, \mathscr{E}_2$ be instruments based on $(\mathscr{S}_1, \mathscr{B}_1), (\mathscr{S}_2, \mathscr{B}_2)$ respectively. Theorem 1 of [4] shows that there exists a unique instrument $\mathscr{E}$ on the product Borel space $(\mathscr{S}_1 \times \mathscr{S}_2, \mathscr{B})$ such that for $M_1 \in \mathscr{B}_1, M_2 \in \mathscr{B}_2$,

$$\mathscr{E}(M_1 \times M_2) = \mathscr{E}_1(M_1)\, \mathscr{E}_2(M_2). \tag{3.8}$$

$\mathscr{E}$ is said to be the *composition* of $\mathscr{E}_1$ and $\mathscr{E}_2$ and corresponds to a physical process in which $\mathscr{E}_1$ follows $\mathscr{E}_2$. An instrument $\mathscr{E}$ on $(\mathscr{S}, \mathscr{B})$ is said to be *repeatable* if for $M_1, M_2 \in \mathscr{B}$, $f \in K$,

$$e(\mathscr{E}(M_1)\,\mathscr{E}(M_2)\,f) = e(\mathscr{E}(M_1 \times M_2)\,f)\,. \tag{3.9}$$

When $\mathscr{S}$ is a discrete space, $\mathscr{E}$ is said to be a *discrete instrument*. Such an instrument $\mathscr{E}$ on $\mathscr{S}$ is said to be *strongly repeatable* if,

(a) For $r, s \in \mathscr{S}$, $f \in K$,

$$\mathscr{E}(r)\,\mathscr{E}(s)\,f = \delta_{rs}\mathscr{E}(r)\,f\,, \tag{3.10}$$

(b) $e(\mathscr{E}(s)\,f) = e(f)$ implies $\mathscr{E}(s)\,f = f$. (3.11)

(c) If $T \in \mathscr{Q}$ satisfies $T(\mathscr{E}(s)f) = 0\ \forall s \in \mathscr{S}, f \in K$, then $T = 0$. (3.12)

The full significance of this definition is explained in [4], but repeatable instruments are those which satisfy Von Neumann's repeatability hypothesis.

§ 4. Restrictions

The main concern of this section is the study of the restrictions of the physical system described in § 3. Physically, a restriction is supposed to be some device which decreases the number of allowed states in a particular way. Hence a restriction defines a subset H of K. The restricted system must still be a physical system in the sense of § 3, which implies that H must also be a cone. Moreover, if $f \in H$ is a mixture of two other states $f_1, f_2 \in K$, then clearly it is required that $f_1, f_2 \in H$. Hence, H is an extremal set in K. Let $L = H - H$ and let $F = H \cap B$. Then, in order to maintain the requirement that H is the set of states of a physical system, it is clearly required that (L, F) be a complete base norm space with norm closed cone H. Further, it is clear that for $f \in L$,

$$\|f\|_B \leqq \|f\|_F \tag{4.1}$$

and that the strictly positive linear functional e_F on (L, F) such that $F = \{f : f \in H, e_F(f) = 1\}$ is the restriction of e to L. Hence, for $f \in H$,

$$e_F(f) = e(f)\,. \tag{4.2}$$

Let (L^*, e_F) be the dual space of (L, F). Then, (L^*, e_F) is a complete order unit space with $\sigma(L^*, L)$ closed generating cone H^*, $[-e_F, e_F]$ is the unit ball in (L^*, e_F) and H, H^* are compatible cones. The set of simple observables of the restricted system is represented by the set $[0, e_F]$ in H^*. However, simple observables of a restriction have an alternative characterization as simple observables of the unrestricted system where two are identified when they give identical probabilities on the set H of states. The annihilator $H^0 (= L^0 = F^0)$ of H is a $\sigma(V^*, V)$ closed order ideal in V^*. Let $\phi: V^* \to V^*/H^0$ be the canonical mapping. Then $\phi(K^*) = K^*/H^0$ is a generating cone for V^*/H^0 and $\phi(e)$ is clearly an order unit for this quotient ordering. The alternative characterization of the simple observables of the restricted system is the set $\phi([0, e])$ in K^*/H^0. However, it follows from (4.1) that $V^*/H^0 \subset L^*$ and clearly, $K^*/H^0 \subset H^*$, $[0, \phi(e)] \subset [0, e_F]$, $\phi(e) = e_F$. Hence, identifying the alternative characterizations of the restricted simple observables gives $\phi([0, e]) = [0, e_F]$. Clearly, $\phi([0, e]) \subset [0, \phi(e)]$ and hence,

$$\phi([0, e]) = [0, \phi(e)] = [0, e_F]. \tag{4.3}$$

This condition is clearly equivalent to the following: For each affine functional $\tilde{T}$ on H such that $\tilde{T}(0) = 0$ and $0 \leqq \tilde{T}(f) \leqq e(f) \, \forall f \in H$, there exists an affine functional T on B such that $T(f) = \tilde{T}(f) \, \forall f \in H$ and $0 \leqq T(f) \leqq e(f) \, \forall f \in K$.

Since $\phi(e)$, e_F are order units, it follows from (4.3) that $K^*/H^0 = H^*$, $V^*/H^0 = L^*$ and hence that $(V^*/H^0, \phi(e))$ is a complete order unit space which may be identified with the dual of (L, F). A result of Ellis [12] may be utilised to show that $(V^*/H^0, \phi(e))$ is also the dual of a further complete base norm space $(\bar{L}, \bar{L} \cap B)$ with norm closed generating cone $\bar{L} \cap K$, where $\bar{L}$ denotes the closure of L in (V, B). Notice that $\bar{L} = (H^0)_0$. In addition the order unit norm and quotient norm of V^*/H^0 are identical and the base norm of $\bar{L}$ coincides with its norm as a closed subspace of (V, B). It follows that $\bar{L} \cap K$ is the set of states of a further restriction which has the same simple observables as that described by H. Essentially, there should be a duality between sets of simple observables of restrictions and sets of states of restrictions. This difficulty is resolved by choosing H to be closed in (V, B) for in this case it may be shown that $\bar{L} \cap K = (H^0)_0 \cap K = H$ and $\bar{L} = L$.

Hence the two conditions so far placed on H are that it is a norm closed extremal set in K and that it possesses the extension property for bounded positive affine functionals described above. The further condition, that H^0 is positively generated, will be imposed. Physically, it might be hoped that the set $H^0 \cap [0, e]$ of simple observables vanishing in the restricted system forms a set of simple observables of a complementary restriction. In this case H^0 would be required to be a complete order unit space in some way. No such restrictive condition will be imposed. However, the far weaker requirement, that H^0 is positively generated, is adopted.

Postulate 6. *The set of states of a restriction of the physical system is represented by a norm closed extremal set H in K such that, if $L = H - H$, $F = H \cap B$,*

(i) *(L, F) is a complete base norm space with norm closed generating cone H,*

(ii) *For each affine functional $\tilde{T}$ on H such that $\tilde{T}(0)=0$ and $0 \leqq \tilde{T}(f) \leqq e(f)\, \forall f \in H$, there exists an affine functional T on K such that $0 \leqq T(f) \leqq e(f)\, \forall f \in K$ and $T(f)=\tilde{T}(f)\, \forall f \in H$,*

(iii) *H^0 is positively generated.*

An alternative characterization of restrictions in terms of simple observables follows from the following result.

Theorem 4.1. *Let (V, B) be a complete base norm space with norm closed generating cone K and let (V^*, e) be its dual space with $\sigma(V^*, V)$ closed generating cone K^*. Then, there exists a one-one correspondence between norm closed extremal sets H in K satisfying the conditions of Postulate 6 and $\sigma(V^*, V)$ closed strongly Archimedean ideals J in V^* of characteristic unity satisfying the condition that if $\phi: V^* \to V^*/J$ is the canonical mapping, $\phi([0, e]) = [0, \phi(e)]$. This correspondence is defined by $J = H^0, H = J_0 \cap K$. Further, if $L = H - H, F = H \cap B, L$ is a norm closed subspace of (V, B) and the norm on (L, F) as a complete base norm space coincides with its norm as a closed subspace of (V, B).*

Proof. Let H be a norm closed extremal set in K satisfying the conditions of Postulate 6. Then H^0 is a $\sigma(V^*, V)$ closed positively generated order ideal in (V^*, e). Let $\phi: V^* \to V^*/H^0$ be the canonical mapping and let K^*/H^0 be the generating cone for V^*/H^0 in the quotient ordering. Since ϕ is positive, $\phi(e)$ is an order unit for V^*/H^0 and $\phi([0, e]) \subset [0, \phi(e)]$. It follows from 16.11 of [15] that V^*/H^0 with the quotient norm and the quotient $\sigma(V^*, V)$ topology may be identified with the dual of $(H^0)_0 = (L^0)_0 = \bar{L}$ the closure of L in (V, B).

Let (L^*, e_F) be the dual of (L, F) and let H^* be the dual cone. Then (see § 2), (L^*, e_F) is a complete order unit space with unit ball $[-e_F, e_F]$. From (4.1), (4.2) it follows that $V^*/H^0 \subset L^*, K^*/H^0 \subset H^*, [0, \phi(e)] \subset [0, e_F]$ and $\phi(e) = e_F$. It follows from Postulate 6 (ii) that for $\tilde{T} \in [0, e_F]$ there exists $T \in [0, e]$ such that $\phi(T) = \tilde{T}$. Therefore,

$$\phi([0, e]) = [0, \phi(e)] = [0, e_F]. \tag{4.3}$$

Hence, as was remarked above, since $\phi(e), e_F$ are order units it follows that $K^*/H^0 = H^*$ and hence that $V^*/H^0 = L^*$. In addition, since the ordering in L^* is Archimedean, it follows that the ordering in V^*/H^0 defined by K^*/H^0 is Archimedean. Hence, H^0 is an Archimedean ideal in V^* and $(V^*/H^0, \phi(e))$ is a complete Archimedean ordered order unit space with unit ball $[-\phi(e), \phi(e)]$. It follows from the second equality in (4.3) that $[-\phi(e), \phi(e)] = [-e_F, e_F]$ and hence that $(V^*/H^0, \phi(e))$ and (L^*, e_F) are not only isomorphic as partially ordered vector spaces but are also isometrically isomorphic as Banach spaces. In the following V^*/H^0 and L^* will be identified. It follows from the first equality in (4.3) that $\phi([-e, e]) = [-\phi(e), \phi(e)]$ and hence ϕ maps the closed unit ball in (V^*, e) onto the closed unit ball in V^*/H^0 for the order unit norm. However, a standard property of quotient mappings (see II, 4.14 of [8]) shows that ϕ maps the closed unit ball in (V^*, e) onto a subset of the closed unit ball, containing the open unit ball, for the quotient norm of V^*/H^0. It follows that the order unit and quotient norms of V^*/H^0 coincide and hence that H^0 is strongly Archimedean and of characteristic unity.

Further, ϕ is continuous and open for the $\sigma(V^*, V)$ topology of V^* and the $\sigma(V^*/H^0, \bar{L})$ topology of V^*/H^0 and since $[0, e]$ is $\sigma(V^*, V)$ compact, being the intersection of the unit ball $[-e, e]$ in the dual space (V^*, e) and the $\sigma(V^*, V)$ closed cone K^*, (4.3) shows that $[0, \phi(e)]$ is $\sigma(V^*/H^0, \bar{L})$ compact. It follows from Theorem 6 of [12] that $(V^*/H^0, \phi(e))$ is the dual of a complete base norm space (U, A) having a norm closed generating cone C such that $C, K^*/H^0$ are compatible cones. Moreover U is the space of linear functionals on $(V^*/H^0, \phi(e))$ which are $\sigma(V^*/H^0, \bar{L})$ continuous on order unit norm bounded sets, C is the set of positive elements of U and A is the set of elements f of C such that $e(f) = 1$. Since the order unit and quotient norms on V^*/H^0 coincide, it follows that $U = \bar{L}, C = \bar{L} \cap K, A = \bar{L} \cap B$. Clearly, $H \subset (H^0)_0 \cap K = \bar{L} \cap K$. Let $f \in \bar{L}, f \notin H$. Then, since H is a closed cone in V, there exists $T \in V^*$, such that $T(f) < 0, T(g) \geqq 0 \ \forall g \in H$ (see Theorem 6, 1.6 of [5]). Hence $\phi(T) \in K^*/H^0$ and since K^*/H^0 and $(H^0)_0 \cap K$ are compatible cones and $\phi(T)(f) = T(f) < 0$, $f \notin (H^0)_0 \cap K$. Hence, $H = (H^0)_0 \cap K$ and $L = H - H = (H^0)_0 \cap K - (H^0)_0 \cap K = \bar{L}$. It follows that L is a base norm closed subspace of (V, B) and, since the quotient and order unit norms of V^*/H^0 coincide, that the base norm of (L, F) and the norm of L, as a closed subspace of (V, B), coincide.

Conversely, let J be a $\sigma(V^*, V)$ closed strongly Archimedean ideal in (V^*, e) of characteristic unity such that if $\phi: V^* \to V^*/J$ is the canonical mapping, $\phi([0, e]) = [0, \phi(e)]$. Then, in the quotient ordering $(V^*/J, \phi(e))$ is an Archimedean ordered complete order unit space whose order unit norm and quotient norm coincide. Further, ϕ is continuous and open for the $\sigma(V^*, V)$ topology of V^* and the $\sigma(V^*/J, J_0)$ topology of V^*/J. Hence, as above, $[0, \phi(e)]$ is $\sigma(V^*/J, J_0)$ compact and $(J_0, J_0 \cap B)$ is a complete base norm space with norm closed generating cone $J_0 \cap K$ and dual $(V^*/J, \phi(e))$. Hence, Postulate 6, (i) and (ii) hold for $J_0 \cap K$ and since, $(J_0 \cap K)^0 = (J_0)^0 = J$, Postulate 6 (iii) also holds. This completes the proof of the theorem.

It follows that the set of restrictions of the physical system has an alternative characterization in terms of the simple observables of the system.

Corollary 4.2. *There exists a one-one correspondence between restrictions of the physical system and $\sigma(V^*, V)$ closed strongly Archimedean* *ideals J in (V^*, e) of characteristic unity which satisfy the condition $\phi([0, e]) = [0, \phi(e)]$ where $\phi: V^* \to V^*/J$ is the canonical mapping. The set of simple observables of the restriction corresponding to J is $[0, \phi(e)]$.*

§ 5. Preliminary Results on C^*-Algebras

In this section the basic results required for the discussion of the C^*-algebra model for statistical physical theories are given. For the details in the case of C^*-algebras and Von Neumann algebras the reader is referred to [6, 7] and in the case of Σ^*-algebras to [2, 3].

Let $\mathfrak{B}$ be a Von Neumann algebra acting on the Hilbert space X and let $e_{\mathfrak{B}}$ be its identity, supposed also to be the identity in the Von Neumann algebra $\mathfrak{L}(X)$ of all bounded linear operators on X. Let $\mathfrak{B}_*$ be the pre-dual of $\mathfrak{B}$, the space of ultraweakly continuous linear functionals on $\mathfrak{B}$. For $\omega \in \mathfrak{B}_*$, there exist sequences $\{x_n\}$, $\{y_n\}$ in X such that

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty, \qquad \sum_{n=1}^{\infty} \|y_n\|^2 < \infty,$$

and,

$$\omega = \sum_{n=1}^{\infty} \omega_{x_n y_n} \tag{5.1}$$

where, for $x, y \in X$, $\omega_{xy}(T) = \langle Tx, y \rangle$, $\forall T \in \mathfrak{B}$ and convergence is in the norm topology of $\mathfrak{B}_*$. Let $V(\mathfrak{B})$ be the subspace of $\mathfrak{B}_*$ consisting of hermitean linear functionals and let $K(\mathfrak{B})$, $B(\mathfrak{B})$ be the subsets of $V(\mathfrak{B})$ consisting of positive elements and positive elements ω such that $\omega(e_{\mathfrak{B}}) = 1$ respectively. $K(\mathfrak{B})$ is said to be the set of positive normal linear functionals on $\mathfrak{B}$ and $B(\mathfrak{B})$ is said to be the set of normal states of $\mathfrak{B}$. For $\omega \in K(\mathfrak{B})$, there exists a sequence $\{x_n\}$ in X such that

$$\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$$

and

$$\omega = \sum_{n=1}^{\infty} \omega_{x_n} \tag{5.2}$$

where for $x \in X$, $\omega_x(T) = \langle Tx, x \rangle$, $\forall T \in \mathfrak{B}$. Moreover,

$$\|\omega\| = \omega(e_{\mathfrak{B}}) = \sum_{n=1}^{\infty} \|x_n\|^2 . \tag{5.3}$$

The following result is a restatement of a well-known theorem for Von Neumann algebras (see [6]).

Proposition 5.1. *$(V(\mathfrak{B}), B(\mathfrak{B}))$ is a complete base norm space with norm closed generating cone $K(\mathfrak{B})$ and the base norm coincides with its norm as a subspace of $\mathfrak{B}_*$. The dual of $(V(\mathfrak{B}), B(\mathfrak{B}))$ is the complete order unit space $(V^*(\mathfrak{B}), e_{\mathfrak{B}})$ having the dual cone $K^*(\mathfrak{B})$ where $V^*(\mathfrak{B})$ is the space of self-adjoint elements of $\mathfrak{B}$, $K^*(\mathfrak{B})$ is the set of positive elements of $\mathfrak{B}$, the order unit and operator norms coincide and the $\sigma(V^*(\mathfrak{B}), V(\mathfrak{B}))$ topology is the ultraweak topology.*

Let $\mathfrak{A}$ be a C^*-algebra with identity $e_{\mathfrak{A}}$ and let $U(\mathfrak{A})$, $C(\mathfrak{A})$ be the sets of self-adjoint and positive elements of $\mathfrak{A}$ respectively. Let $\mathfrak{A}^*$ be the Banach space dual of $\mathfrak{A}$. The following results from 1.1.6 and 12.3.4 of [7].

Proposition 5.2. *$(U(\mathfrak{A}), e_{\mathfrak{A}})$ is a complete Archimedean ordered order unit space with norm closed generating cone $C(\mathfrak{A})$ and with the order unit and C^*-algebra norms coinciding. The dual of $(U(\mathfrak{A}), e_{\mathfrak{A}})$ is the complete base norm space $(U^*(\mathfrak{A}), S(\mathfrak{A}))$ having dual cone $C^*(\mathfrak{A})$ where $U^*(\mathfrak{A})$ is the space of hermitean bounded linear functionals on $\mathfrak{A}$, $C^*(\mathfrak{A})$ is the set of positive linear functionals on $\mathfrak{A}$ and $S(\mathfrak{A})$ is the set of states of $\mathfrak{A}$. The base norm in $(U^*(\mathfrak{A}), S(\mathfrak{A}))$ coincides with its norm as a subspace of $\mathfrak{A}^*$.*

To each element f of $C^*(\mathfrak{A})$ there corresponds a cyclic representation π_f of $\mathfrak{A}$ on a Hilbert space X_f with cyclic vector x_f such that for $a \in \mathfrak{A}$,

$$f(a) = \langle \pi_f(a)\, x_f, x_f \rangle \tag{5.4}$$

and $\|f\| = f(e_{\mathfrak{A}}) = \langle x_f, x_f \rangle$. Any two such representations are unitarily equivalent. Moreover, if $g = \alpha f$, $\alpha > 0$, it is possible to choose $x_g = \alpha^{\frac{1}{2}} x_f$ and $\pi_f = \pi_g$. The representation

$$\pi = \bigoplus_{f \in S(\mathfrak{A})} \pi_f \tag{5.5}$$

acting on the Hilbert space

$$X = \bigoplus_{f \in S(\mathfrak{A})} X_f$$

is said to be the universal representation of $\mathfrak{A}$. Since every essential representation of $\mathfrak{A}$ is unitarily equivalent to a direct sum of cyclic representations, up to unitary equivalence, every essential representation is a sub-representation of some multiple of π. π is a faithful representation of $\mathfrak{A}$ and hence is an isometric $*$-isomorphism between the C^*-algebras $\mathfrak{A}$ and $\pi(\mathfrak{A})$ and in addition $\pi(e_{\mathfrak{A}})$ is the identity in $\mathfrak{L}(X)$. Let $\mathfrak{B} = \pi(\mathfrak{A})^-$, the weak closure of $\pi(\mathfrak{A})$. Then, denoting the commutant of a subset $\mathfrak{B}$ of $\mathfrak{L}(X)$ by $\mathfrak{B}'$, $\pi(\mathfrak{A})^- = \pi(\mathfrak{A})''$, a Von Neumann algebra. $\mathfrak{B}$ is said to be the Von Neumann envelope of $\mathfrak{A}$. For $f \in C^*(\mathfrak{A})$, ω_{x_f} is an element of $K(\mathfrak{B})$ and, indeed, the mapping $f \mapsto \omega_{x_f}$ extends to a mapping $f \mapsto \omega_f$ from $\mathfrak{A}^*$ onto $\mathfrak{B}_*$ mapping $U^*(\mathfrak{A})$ onto $V(\mathfrak{B})$, $C^*(\mathfrak{A})$ onto $K(\mathfrak{B})$ and

$S(\mathfrak{A})$ onto $B(\mathfrak{B})$. In particular, this implies that, for $\omega \in K(\mathfrak{B})$, there exists $f \in C^*(\mathfrak{A})$ such that $\omega = \omega_{x_f}$. Hence, Corollary 12.1.3 of [7] may otherwise be stated as follows.

Proposition 5.3. *There exists an isometric isomorphism $f \mapsto \omega_f$ between the complete base norm spaces $(U^*(\mathfrak{A}), S(\mathfrak{A}))$ and $(V(\mathfrak{B}), B(\mathfrak{B}))$ which maps $C^*(\mathfrak{A})$ onto $K(\mathfrak{B})$ and $S(\mathfrak{A})$ onto $B(\mathfrak{B})$ and satisfies*

$$\omega_f(\pi(a)) = f(a) \tag{5.6}$$

$\forall\, a \in \mathfrak{A}$. *Hence, there exists an isometric isomorphism $T \mapsto \hat{T}$ between $(V^*(\mathfrak{B}), e_{\mathfrak{B}})$ and the second dual $(U^{**}(\mathfrak{A}), e_{\mathfrak{A}})$ of $(U(\mathfrak{A}), e_{\mathfrak{A}})$, which maps $K^*(\mathfrak{B})$ onto $C^{**}(\mathfrak{A})$ and $e_{\mathfrak{B}}$ onto $e_{\mathfrak{A}}$. This mapping is a homeomorphism for the ultraweak operator topology and the $\sigma(U^{**}(\mathfrak{A}), U^*(\mathfrak{A}))$ topology.*

Notice that it is the properties of the mapping $f \mapsto \omega_f$ which determines the representation theory of $\mathfrak{A}$ (see [19]). In the following, $\mathfrak{A}$ and $\pi(\mathfrak{A})$ will be identified, as will $\mathfrak{A}^*$ and $\mathfrak{B}_*$ by means of Proposition 5.3.

Let $\mathfrak{A}^\sigma$ denote the smallest Σ^*-algebra containing $\mathfrak{A}$. Then, $\mathfrak{A}^\sigma$ is said to be the σ-envelope of $\mathfrak{A}$. Let $V^\sigma(\mathfrak{A}^\sigma)$ be the space of Hermitean linear functionals on $\mathfrak{A}^\sigma$, continuous for the weak sequential topology and let $K^\sigma(\mathfrak{A}^\sigma)$, $B^\sigma(\mathfrak{A}^\sigma)$ be the subsets of $V^\sigma(\mathfrak{A}^\sigma)$ consisting of linear functionals v which are, respectively positive, and positive such that $v(e_{\mathfrak{A}}) = 1$. Then, as above, $(V^\sigma(\mathfrak{A}^\sigma), B^\sigma(\mathfrak{A}^\sigma))$ is a complete base norm space with norm closed generating cone $K^\sigma(\mathfrak{A}^\sigma)$. Moreover, as in Proposition 5.3, there exists an isometric isomorphism $f \mapsto v_f$ from $(U^*(\mathfrak{A}), S(\mathfrak{A}))$ onto $(V^\sigma(\mathfrak{A}^\sigma), B^\sigma(\mathfrak{A}^\sigma))$ mapping $C^*(\mathfrak{A})$ onto $K^\sigma(\mathfrak{A}^\sigma)$, $S(\mathfrak{A})$ onto $B^\sigma(\mathfrak{A}^\sigma)$ and satisfying

$$v_f(a) = f(a) \tag{5.7}$$

$\forall\, a \in \mathfrak{A}$.

To each cyclic representation π_f of $\mathfrak{A}$, there corresponds a unique projection E_f in $\mathfrak{A}'$ such that π_f is unitarily equivalent to the representation $a \mapsto aE_f = E_f a$ on $E_f X$. Hence π_f extends to a normal representation $T \mapsto TE_f = E_f T$ of $\mathfrak{B}$ and similarly to a σ-representation of $\mathfrak{A}^\sigma$. Hence, every essential representation ψ of $\mathfrak{A}$ has a unique extension ψ^n to a normal representation of $\mathfrak{B}$ and a unique extension ψ^σ to a σ-representation of $\mathfrak{A}^\sigma$. Two representations ψ_1, ψ_2 of $\mathfrak{A}$ are said to be quasi-equivalent if ψ_1^n, ψ_2^n have the same kernel $\mathfrak{J}$ in $\mathfrak{B}$ (see [14]). $\mathfrak{J}$ is an ultraweakly closed two-sided ideal in $\mathfrak{B}$. Such ideals are weakly closed and there exists a projection E in the centre of $\mathfrak{B}$ such that $\mathfrak{J} = E\mathfrak{B}E = E\mathfrak{B} = \mathfrak{B}E$. If ψ_1, ψ_2 are subrepresentations of π, there exist unique projections E_1, E_2 in $\mathfrak{A}' = \mathfrak{B}'$, such that, ψ_1, ψ_2 are unitarily equivalent to the representations $a \mapsto aE_1$, $a \mapsto aE_2$ respectively and ψ_1, ψ_2 are quasi-equivalent if and only if E_1, E_2 have the same central support E in $\mathfrak{B}'$. Hence, there exists a one-one correspondence between quasi-equivalence classes of representations of $\mathfrak{A}$ and projections in the centre of $\mathfrak{B}$. Let ψ be an essential representation of A and let

$$H_\psi = \{\omega_x \circ \psi : x \in X_\psi\} .$$

Then, H_ψ is a subset of $K(\mathfrak{B})$. Let $\mathrm{conv}(H_\psi)$ be the convex hull of H_ψ and let $\overline{\mathrm{conv}(H_\psi)}$ be its norm closure. Then $\overline{\mathrm{conv}(H_\psi)}$ is a norm closed extremal set in $K(\mathfrak{B})$ which can clearly be identified with $K(\psi(\mathfrak{A})^-)$ $= K(\psi''(\mathfrak{B}))$. ψ_1, ψ_2 are quasi-equivalent if and only if $\overline{\mathrm{conv}(H_{\psi_1})} = \overline{\mathrm{conv}(H_{\psi_2})}$ (see [28]).

A subset $\mathfrak{L}$ of $\mathfrak{B}_*$ is said to be $\mathfrak{A}$-invariant if and only if for $\omega \in \mathfrak{L}$, $a \in \mathfrak{A}$, the linear functional $a^*\omega a$ defined for $b \in \mathfrak{A}$ by $(a^*\omega a)(b) = \omega(a^*ba)$ is an element of $\mathfrak{L}$, and is said to be $\mathfrak{B}$-invariant if the same result holds for $a, b \in \mathfrak{B}$. Notice that for $\omega \in K(\mathfrak{B})$ there exists $x \in X$ such that $\omega = \omega_x$. Hence, $\mathfrak{L} \cap K(\mathfrak{B})$ is $\mathfrak{A}$-invariant if and only if for $x \in X$ such that $\omega_x \in \mathfrak{L} \cap K(\mathfrak{B})$ and $a \in A$, $\omega_{ax} \in \mathfrak{L} \cap K(\mathfrak{B})$. If $\mathfrak{L} \cap K(\mathfrak{B})$ is norm-closed, it follows that $\mathfrak{L} \cap K(\mathfrak{B})$ is $\mathfrak{A}$-invariant if and only if it is $\mathfrak{B}$-invariant since $\mathfrak{A}$ is weakly dense in $\mathfrak{B}$. Hence if $\mathfrak{L} \subset \mathfrak{B}_*$ is norm closed and positively generated it is $\mathfrak{A}$-invariant if and only if it is $\mathfrak{B}$-invariant.

Let $\mathfrak{J}$ be an ultraweakly closed two-sided ideal in $\mathfrak{B}$ and let E be the projection in the centre of $\mathfrak{B}$ such that $\mathfrak{J} = E\mathfrak{B}E$. Then clearly $\mathfrak{B} = E\mathfrak{B}E \oplus (e_{\mathfrak{B}} - E)\,\mathfrak{B}(e_{\mathfrak{B}} - E)$ and $\mathfrak{B}/\mathfrak{J}$ may be identified with $(e_{\mathfrak{B}} - E)\,\mathfrak{B}(e_{\mathfrak{B}} - E)$. Let J be the self-adjoint part of $\mathfrak{J}$. Then, $J + iJ = \mathfrak{J}$ and $V^*(\mathfrak{B})/J$ may be identified with $(e_{\mathfrak{B}} - E)\,V^*(\mathfrak{B})\,(e_{\mathfrak{B}} - E)$. The canonical mapping

$$\phi : V^*(\mathfrak{B}) \to V^*(\mathfrak{B})/J = (e_{\mathfrak{B}} - E)\,V^*(\mathfrak{B})\,(e_{\mathfrak{B}} - E) \tag{5.8}$$

is clearly an order homomorphism and hence J is an Archimedean ideal in $V^*(\mathfrak{B})$. Moreover, the order unit norm in $((e_{\mathfrak{B}} - E)\,V^*(\mathfrak{B})\,(e_{\mathfrak{B}} - E)$, $e_{\mathfrak{B}} - E)$ clearly coincides with its quotient norm and $\phi([0, e_{\mathfrak{B}}]) = [0, e_{\mathfrak{B}} - E]$. Hence J satisfies the conditions of Theorem 4.1. Conversely, let J be an ultraweakly closed Archimedean ideal in $V^*(\mathfrak{B})$ and let $\phi : V^*(\mathfrak{B}) \to V^*(\mathfrak{B})/J$ be the canonical order homomorphism. Let $\mathfrak{J} = J + iJ$, $\Phi = \phi + i\phi$. Then Φ satisfies the conditions of Lemma 5.1 of [27] and hence $\mathfrak{J}$ is an ultraweakly closed two-sided ideal in $\mathfrak{B}$. Hence, the following results has been proved.

Theorem 5.4. *Let J be an ultraweakly closed subspace of the complete order unit space $(V^*(\mathfrak{B}), e_{\mathfrak{B}})$ of self-adjoint elements of the Von Neumann algebra $\mathfrak{B}$ and let $\phi : V^*(\mathfrak{B}) \to V^*(\mathfrak{B})/J$ be the canonical mapping. Then J is a strongly Archimedean ideal of characteristic unity satisfying $\phi([0, e_{\mathfrak{B}}]) = [0, \phi(e_{\mathfrak{B}})]$ if and only if J is the self-adjoint part of an ultraweakly closed two-sided ideal in $\mathfrak{B}$.*

Let H be a norm closed invariant extremal subset of $K(\mathfrak{B})$. Then, Theorem 4.10 of [11] shows that $L = H - H$ is a norm closed invariant subspace of $V(\mathfrak{B})$, and it follows from Theorems 4.6, 5.2 of [11] that $H^0 \cap K^*(\mathfrak{B}) = L^0 \cap K^*(\mathfrak{B})$ is an ultraweakly closed invariant extremal set in $K^*(\mathfrak{B})$ and that $(H^0)_0 = L$. Proposition 1.1.6 of [6] shows that H^0 is the self-adjoint part of an ultraweakly closed two-sided ideal in $\mathfrak{B}$. Hence, from Theorem 5.4, H^0 is strongly Archimedean and of characteristic unity satisfying $\phi([0, e_{\mathfrak{B}}]) = [0, \phi(e_{\mathfrak{B}})]$ where $\phi: V^*(\mathfrak{B}) \to V^*(\mathfrak{B})/H^0$ is the canonical order homomorphism. Since $(H^0)_0 = L$, it follows from Theorem 4.1 that H is a norm closed extremal subset of $K(\mathfrak{B})$ satisfying the conditions of Postulate 6. Conversely, let H be a norm closed extremal subset of $K(\mathfrak{B})$, let $L = H - H$ and suppose H satisfies the conditions of Postulate 6. Then, from Theorems 4.1 and 5.4 H^0 is the self-adjoint part of an ultraweakly closed two-sided ideal in $\mathfrak{B}$ and $(H^0)_0 = L$. Hence, Theorems 4.6, 5.2 of [11] show that H is an invariant extremal subset of $K(\mathfrak{B})$. Therefore, the following result has been proved.

Theorem 5.5. *Let H be a norm closed extremal subset of the cone $K(\mathfrak{B})$ of positive elements of the pre-dual $\mathfrak{B}_*$ of the Von Neumann algebra $\mathfrak{B}$. Then H satisfies the conditions of Postulate* 6 *if and only if H is $\mathfrak{B}$-invariant.*

Notice that Theorems 5.4, 5.5 hold for any Von Neumann algebra $\mathfrak{B}$ and not just the envelope of $\mathfrak{A}$. It is for this reason that, in the proof, H has been merely said to be invariant rather than $\mathfrak{A}$ or $\mathfrak{B}$-invariant. When $\mathfrak{B}$ is an arbitrary Von Neumann algebra no confusion arises and when $\mathfrak{B}$ is the Von Neumann envelope of $\mathfrak{A}$, the two notions are identical for H.

§ 6. Algebraic Models

The results of § 5 suggest two possible models for the set K of states of a physical system. Maintaining the nomenclature of § 1,

(a) The set $K(\mathfrak{B})$ of positive normal linear functionals on the Von Neumann algebra $\mathfrak{B}$ with identity $e_{\mathfrak{B}}$.

(c) The set $C^*(\mathfrak{A})$ of positive linear functionals on a C^*-algebra $\mathfrak{A}$ having identity $e_{\mathfrak{A}}$.

(a) is said to be the Von Neumann algebra model and (c) is said to be the C^*-algebra model. Using Postulate 3, the sets $\mathcal{Q}$ of simple observables for the models (a), (c) are

(a) The set of positive operators T in $\mathfrak{B}$ such that $0 \leqq T \leqq e_{\mathfrak{B}}$.

(c) The set of positive operators T in the Von Neumann envelope $\mathfrak{B}$ of $\mathfrak{A}$ such that $0 \leqq T \leqq e_{\mathfrak{B}}$.

It follows from Proposition 5.3 that model (c) is merely a particular case of model (a) since $C^*(\mathfrak{A})$ can be identified with $K(\mathfrak{B})$ where $\mathfrak{B}$ is the Von Neumann envelope of $\mathfrak{A}$. This is the key to the whole situation regarding algebraic theories since it explains why the observables of an unrestricted system described by a C^*-algebra $\mathfrak{A}$ are chosen to be self-adjoint operators in its Von Neumann envelope. In the following, models (a) and (c) will be studied together except on those occasions when special results hold in the more restrictive situation of model (c).

It is not immediately obvious in either case what the operations are in general, though it is clear from Postulate 5 that the observables are merely measures $\mathscr{A}$ on some Borel space $(\mathscr{S}, \mathscr{B})$ taking values in $\mathscr{Q}$ and satisfying $\mathscr{A}(\mathscr{S}) = e_{\mathfrak{B}}$.

It is instructive to consider at this stage how conventional algebraic quantum theory fits into this description. The set of states chosen here is precisely the same as that in the conventional description. However, the sets of simple observables and observables described by Postulates 3 and 5 are much larger than the usual sets. Conventionally, the simple observables are merely projection operators in $\mathfrak{B}$ whilst the observables are self-adjoint operators in $\mathfrak{B}$ and hence may be identified with projection-valued measures on $\mathbb{R}$. In fact, the set of projection operators in $\mathfrak{B}$ may be identified with the set of extreme points of the compact convex set $\mathscr{Q} = [0, e]$ (see Theorem 4, [18]). It follows that the conventional simple observables are the extreme points of $\mathscr{Q}$ whilst the conventional observables are Borel measures concentrated on compact subsets of $\mathbb{R}$ and taking values in the set of extreme points of $\mathscr{Q}$. It follows that the conventional simple observables, at least, have an abstract characterization.

The restrictions of the physical system described by models (a) and (c) are now examined. It follows from Theorems 5.4, 5.5 that it suffices to study the ultraweakly closed two-sided ideals in $\mathfrak{B}$. For each such ideal $\mathfrak{J}$, let E be the projection in the centre of $\mathfrak{B}$ such that $\mathfrak{J} = E\mathfrak{B}E$. Then, it is clear from Postulate 6 that the set of states of the corresponding restricted system is $H = J_0 \cap K(B)$ and that the set of simple observables of the restricted system is $[0, e_{\mathfrak{B}} - E]$ in $(e_{\mathfrak{B}} - E)\,\mathfrak{B}(e_{\mathfrak{B}} - E)$. Hence,

$$H = \{\omega : \omega \in K(\mathfrak{B}), \omega(ETE) = 0, \forall T \in \mathfrak{B}\}\,. \tag{6.1}$$

Using the expansion (5.2) for $\omega \in H$, it follows that H may be identified with $K((e_{\mathfrak{B}} - E)\,\mathfrak{B}(e_{\mathfrak{B}} - E))$ the set of positive normal linear functionals on the Von Neumann algebra $(e_{\mathfrak{B}} - E)\,\mathfrak{B}(e_{\mathfrak{B}} - E)$.

Theorem 6.1. *In the Von Neumann algebra and the C^*-algebra models, there exists a one-one correspondence between restrictions of the system and projections in the centre of $\mathfrak{B}$. For each such projection E, the set of* *states of the corresponding restricted system is* $K((e_{\mathfrak{B}}-E)B(e_{\mathfrak{B}}-E))$ *and the set of simple observables of the restricted system is the set* $[0, e_{\mathfrak{B}}-E]$ *in* $(e_{\mathfrak{B}}-E)\mathfrak{B}(e_{\mathfrak{B}}-E)$.

In the C^*-algebra model it is possible to say a little more. It was remarked in § 5 that there exists a one-one correspondence between projections in the centre of $\mathfrak{B}$ and quasi-equivalence classes of representations of $\mathfrak{A}$. It was pointed out that for each essential representation ψ in a particular quasi-equivalence class $K(\psi(\mathfrak{A})^{-})=K(\psi^n(\mathfrak{B}))$ was the same and could be identified with a norm closed extremal set in $K(\mathfrak{B})$, namely the set

$$\{\omega:\omega\in K(\mathfrak{B}):\omega(e_{\mathfrak{B}})=\omega(E')\}$$

where E' is the central projection corresponding to the equivalence class. Hence $K(\psi(\mathfrak{A})^{-})$ may be identified with $K(E'\mathfrak{B}E')$. Moreover, the kernel of ψ^n is uniquely determined by the quasi-equivalence class to which ψ belongs which implies that $\psi(\mathfrak{A})^{-}$ is defined uniquely by the quasi-equivalence class containing ψ up to isomorphism. Hence, using Theorem 6.1, the following result holds.

Theorem 6.2. *In the C^*-algebra model, there exists a one-one correspondence between restrictions of the physical system and quasi-equivalence classes of representations of $\mathfrak{A}$. The set of states of a restricted system may be identified with $K(\psi(\mathfrak{A})^{-})$, where ψ is any representation in the corresponding equivalence class, and the set of simple observables of the restricted system may be identified with the set $[0, \psi(e_{\mathfrak{A}})]$ in $\psi(\mathfrak{A})^{-}$.*

This is the principle result which associates the abstract formulation with the usual approach to algebraic quantum theory.

§ 7. Examples

The results so far obtained have been for C^*-algebras with identity. However, the existence of the identity plays no fundamental rôle in the theory since it is always possible to adjoin an identity with little significant change in the results obtained. In the two examples considered the C^*-algebra has no identity.

(i) Let $\mathfrak{A}=\mathfrak{LC}(Y)$, the C^*-algebra of compact operators on some Hilbert space Y. Then $\mathfrak{A}^*$ may be identified with the space $\mathfrak{TC}(Y)$ of trace class operators on Y (see 4.1, 12.1 of [7]), where for $f\in\mathfrak{A}^*$, the corresponding element ϱ_f of $\mathfrak{TC}(Y)$ satisfies

$$f(a)=Tr(\varrho_f a)\qquad \forall a\in\mathfrak{A}.$$

The mapping $f\mapsto\varrho_f$ is isometric for the trace norm in $\mathfrak{TC}(Y)$ and maps $C^*(\mathfrak{A})$ onto the set $\mathfrak{TC}(Y)^+$ of positive elements of $\mathfrak{TC}(Y)$ and $S(\mathfrak{A})$ onto those elements of $\mathfrak{T}\mathfrak{C}(Y)^+$ of trace unity. In addition, the dual $\mathfrak{T}\mathfrak{C}(Y)^*$ of $\mathfrak{T}\mathfrak{C}(Y)$ may be identified with $\mathfrak{L}(Y)$ since every bounded linear functional on $\mathfrak{T}\mathfrak{C}(Y)$ is of the form $\varrho \mapsto Tr(T\varrho)$ for some unique $T \in \mathfrak{L}(Y)$. Hence the Von Neumann envelope $\mathfrak{B}$ of $\mathfrak{A}$ may be identified with $\mathfrak{L}(Y)$. Hence, in the C^*-algebra model corresponding to $\mathfrak{A}$, the set of states is represented by the set of positive trace class operators on Y whilst the set of observables is represented by the set of measures on Borel spaces $(\mathscr{S}, \mathscr{B})$ taking values in the set of self-adjoint operators T in $\mathfrak{L}(Y)$ such that $0 \leqq T \leqq I$, the identity in $\mathfrak{L}(Y)$. This model is obviously closely connected to the conventional model for quantum mechanics. The only difference occurs in the set of observables which in the C^*-algebra model is larger than in the conventional model.

Further, since $\mathfrak{L}(Y)$ is a factor, there are no non-trivial projections in its centre and hence no restrictions of the associated quantum mechanical system.

(ii) Let Ω be a separable locally compact Hausdorff space and let $\mathfrak{A}$ be the C^*-algebra $\mathscr{C}_0(\Omega)$ of continuous functions on Ω taking arbitrarily small values outside compact subsets of Ω. Then $\mathfrak{A}^*$ may be identified with $\mathscr{M}(\Omega)$, the Banach space of complex Borel measures of finite total variation on Ω (see [17]). Then $U^*(\mathfrak{A})$ is the subset of $\mathscr{M}(\Omega)$ consisting of real measures, $C^*(\mathfrak{A})$ is the subset of $U^*(\mathfrak{A})$ consisting of positive measures and $S(\mathfrak{A})$ is the set of probability measures. The dual $\mathscr{M}(\Omega)^*$ of $\mathscr{M}(\Omega)$, which may be identified with the Von Neumann envelope $\mathfrak{B}$ of $\mathfrak{A}$ is a large, unwieldy space. Hence, the set of all possible observables is difficult to describe. However, $\mathscr{M}(\Omega)^*$ contains as a closed subspace the set $\mathscr{B}(\Omega)$ of bounded Borel functions on Ω. Hence, whilst $K(\mathfrak{B})$ describes the set of states of the conventional model for classical probability theory, the observables of the C^*-algebra model form a much larger set than the set of random variables in the conventional model.

Since $\mathfrak{B}$ is commutative, it possesses many central projections. It follows that plenty of restrictions exist in general, though it is by no means clear what they all are. However, it is clear that the characteristic function χ_M of each Borel subset M of Ω is such a projection. The corresponding set of states of the restricted system consists of those measures on Ω, with support contained in M.

§ 8. Concluding Remarks

The principle feature of the theory described above is that, as was shown in § 8, the sets of states of the conventional models for quantum and classical probability theories occur as the sets of states in two C^*-algebra models. Therefore, although the C^*-algebra approach is merely an example of the general formulation, it is sufficiently general as to contain the most important physical examples.

In examining the examples in § 8, certain interesting questions arise. The observables as described by the C^*-algebra approach are far more numerous than those described by the conventional models. One might reasonably ask why this feature of the theory occurs. Firstly, conventional observables are described by projection-valued measures and not by positive operator-valued measures which occur in the C^*-algebra approach. It is an essential part of the operational approach that instruments should be regarded as $\mathscr{P}$-valued measures, since it is only in this case that compositions of instruments can be formed. However, it has been remarked that the conventional observables are described as extreme $\mathscr{Q}$-valued measures. Secondly, the conventional observables in quantum mechanics have as their measurement space some subset of $\mathbb{R}$, whilst the operational approach allows arbitrary Borel spaces. In fact this is an unimportant difference since the indications are that the Borel spaces must be standard and every standard Borel space is Borel isomorphic either to a countable set or the unit interval. The third and most important difference between the conventional models and the associated C^*-algebra models lies in the choice of C^*-algebra in which the operator-valued measures take values. In the quantum case the algebra in which the measures take values is $\mathfrak{L}(Y)$ in both the conventional and the C^*-algebra approach. However, in the classical case the algebra in which the conventional observables are defined is $\mathscr{B}(\Omega)$ whilst in the C^*-algebra approach the observables are defined in a much larger space. However, it is interesting to note that the σ-envelope of $\mathfrak{L}\mathfrak{C}(Y)$ coincides with its Von Neumann envelope $\mathfrak{L}(Y)$ whilst the σ-envelope of $\mathscr{C}_0(\Omega)$ is $\mathscr{B}(\Omega)$. It follows that the conventional observables are measures taking values in the σ-envelopes of the related C^*-algebras in both cases.

In the abstract, it is possible to place further conditions on (V, B) making more restrictive classes of operation, instrument and observable available. Some of these conditions have some physical motivation. In fact, one can imitate Haag and Kastler's notion of physical equivalence of states to define a topology on (V, B) and if, in addition, the existence of basic, or pure, states is assumed, it is a short step to make (V, B) into the dual of an order unit space (U, e). This is, of course, the case in the C^*-algebra model. By doing this, a large number of different classes of operations and simple observables are automatically defined depending upon their $\sigma(V, U)$ continuity properties. In particular, since the simple observables may be regarded as positive affine functionals on B of norm not greater than unity, the two relevant classes of simple observables are those $\sigma(V, U)$ continuous on B and those which are the pointwise limits of $\sigma(V, U)$ continuous functions on B. In the C^*-algebra model corresponding to $\mathfrak{A}$, these lead to positive operators of norm not greater than unity in $\mathfrak{A}$ and its σ-envelope $\mathfrak{A}^\sigma$, respectively.

Associated with each class of simple observables, there is a different class of restrictions. Since the whole problem summarised above is intimately connected with subsets of the set of operations, the details will not be given here but will be the subject of a further paper.

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