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On Non-Linear Realizations of the Group $SU(2)$

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Abstract. The non-linear realizations of compact connected Lie groups are considered mainly from the point of view of algebraic topology. In particular, all homogeneous spaces of the group $SU(2)$ are listed, the construction of a few non-linear realizations of $SU(2)$ is given and the orbit structure of linear and non-linear realizations are discussed.

I. Introduction

Recently the method of effective Lagrangians was used to fit the experimental data [1]. The effective Lagrangians have been considered partially invariant under non-linear realizations of some chiral group. Consequently the problem of non-linear realizations of Lie groups has arisen and has begun to be studied by physicists [2]. In contradistinction to them, we deal with the problem globally by means of the theory of homogeneous spaces. In particular, non-linear realizations of the group $SU(2)$ are treated in detail from the point of view of algebraic topology.

First, in Section II, we formulate the problem and define basic notions. Realizations and transitive realizations as well as two concepts of equivalence of realizations are introduced. Any realization of a given group can be written as a union of transitive realizations and therefore, in order to find all realizations of the group, we have to answer the following questions. How many transitive realizations exist for a given group? What is their structure and dimension? How to construct other realizations of the group from the transitive realizations.

Since these questions are, in general, difficult, we restrict ourselves to the group $SU(2)$. However, the method used for constructing its non-linear realizations can be applied to the other compact Lie groups as well. In Section III we find all transitive realizations of $SU(2)$, that is, all homogeneous spaces of $SU(2)$, by listing all subgroups of the group

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$SU(2)$. Section IV is devoted to the study of orbit structure of linear realizations of $SU(2)$. Section V deals with non-linear realizations of $SU(2)$ and with an example of their construction on spheres and on Euclidean spaces. Since the homotopy theory – a subject less familiar to physicists – is used in the paper, a review of some definitions from this theory is given in the Appendix.

II. Realizations of a Topological Group

Let G be a locally compact group with a countable basis and X a locally compact Hausdorff space.

Definition 1. *A realization of a group G in X is a homomorphism of G into the topological transformation group $\hat{G}$ of the space X, i.e., a homomorphism*

$$g \to f_g, \quad g \in G, f_g \in \hat{G},$$

where f_g is a homeomorphism [1] *of X into itself.*

Definition 2. *Let us recall that $\hat{G}$ is a topological transformation group of X if each element f_g of $\hat{G}$ is a homeomorphism of X into itself, i.e.,*

$$f_g : x \to x' = f_g(x), \quad x, x' \in X, f_g \in \hat{G},$$

such that

(i) *$f_{g_1} f_{g_2}(x) = f_{g_1}(f_{g_2}(x))$ for $x \in X$ and $f_{g_1}, f_{g_2} \in \hat{G}$ and*

(ii) *the mapping $(f_g, x) \to f_g(x)$ is a continuous (even simultaneously in $x \in X$ and $f_g \in \hat{G}$) mapping of $\hat{G} \times X$ into X.*

In connection with these definitions let us remark that:

1. From (i) and the fact that f_g is a homeomorphism of X it follows that $f_e(x) = x$ and that $f_{g^{-1}}(x) = f_g^{-1}(x)$ for all $x \in X$.

2. If $g = e \in G$ is the only element in G for which f_g leaves all x of X fixed G is said to act *effectively* on X or, simply, G is effective.

3. In many cases $\hat{G}$ is a Lie group and its elements are special homeomorphisms, namely, diffeomorphisms [2] or even analytic homeomorphisms. In these cases the group $\hat{G}$ is said to be a *Lie transformation group* of X.

After specifying what we mean by realization of G in X we may define when two realizations are equivalent. We introduce two concepts of equivalence – continuous and differentiable equivalence.

Definition 3. *Two realizations of the group G, $g \to f_g^{(1)}$ and $g \to f_g^{(2)}$ in $X^{(1)}$ and $X^{(2)}$, respectively, are said to be continuously (differentiably) equivalent if a homeomorphism (diffeomorphism) $\varphi : X^{(1)} \to X^{(2)}$ exists such that*

$$f_g^{(2)}(\varphi(x)) = \varphi(f_g^{(1)}(x)) \quad \textit{for every } x \in X^{(1)} \textit{ and } g \in G.$$

[1] f is a homeomorphism iff f and f^{-1} are one-to-one continuous transformations.
[2] f is a diffeomorphism iff f and f^{-1} are one-to-one differentiable transformations.

In other words, the diagram

$$\begin{array}{ccc} X^{(1)} & \xrightarrow{f_g^{(1)}} & X^{(1)} \\ \varphi\downarrow & & \downarrow\varphi \\ X^{(2)} & \xrightarrow[f_g^{(2)}]{} & X^{(2)} \end{array}$$

is commutative.

Since any diffeomorphism is a homeomorphism, differentiable equivalence is finer that the continuous one. For example, let us considere the carrier spaces $X^{(1)}$ and $X^{(2)}$ of two trivial representations $g \to f_e^{(i)}$, $i = 1, 2$, of a group G to be two 11-dimensional spheres S^{11}. Since every two spheres S^{11} are topologically equivalent, i.e., homeomorphic, the realizations $g \to f_e^{(1)}$ and $g \to f_e^{(2)}$ are continuously equivalent. On the other hand, it is known [3] that there are 992 spheres $S^{11}_{(i)}$, $i = 1, 2, \ldots, 992$, which are not diffeomorphic! Therefore, if we take $S^{11}_{(i)}$ and $S^{11}_{(j)}$, $i \neq j$, as the spaces $X^{(1)}$ and $X^{(2)}$, respectively, the representations $g \to f_e^{(1)}$ and $g \to f_e^{(2)}$ are *not* differentiably equivalent.

Among realizations of G there are so-called transitive ones which have particular properties.

Definition 4. *The realization of the group G in X is transitive if for every two points $x_1, x_2 \in X$ there exists $g \in G$ such that $f_g \in \hat{G}$ maps x_1 to x_2, i.e.,*

$$f_g(x_1) = x_2$$

(in other words, if, for any $x_0 \in X$, the orbit $\hat{G}x_0$, i.e., the set consisting of all $f_g(x_0)$, is exactly the space X).

Since the transitive realization of a locally compact group G with a countable basis in a locally compact Hausdorff space X is a homomorphism of G into the transformation group $\hat{G}$ of X and since G

Theorem 1. *Acts on X transitively, the space X is homeomorphic to a coset space G/H, where H is a closed subgroup of G* [3]. *For the proof of this theorem see Ref.* [4], *p. 111, Theorem 3.2.*

Moreover, it can be easily shown that two homogeneous spaces G/H and G/H', where $H' = gHg^{-1}$, g fixed element of G, are homeomorphic. If G acts on X as a Lie transformation group, then in fact $G/H \overset{d}{\approx} X$ if X is a transitive manifold.

Hence, in order to classify all transitive realizations of the group G we must find all inequivalent homogeneous spaces, that is, all possible conjugacy classes of closed subgroups of G.

[3] The subgroup H is closed if, considered as a set, it is closed. For this it is sufficient that H be an isotropy group of some point of X.

However, to classify more general realizations of G on some

Theorem 2. *space X turns out to be very difficult. It is true that any carrier space X of the realization of G can be written as a union of orbits, each being a carrier space of a transitive realization of G in X* [4], *but due to possible different action of G on the same space this union is not unique.*

Thus, even if we take as G the group $SU(2)$ and as a space X some conventional manifold, as for example the Euclidean space R^n, the sphere S^n or the unit disc D^n, the result is far from complete. Only in some special cases, e.g. when the dimension of the principal orbit or the fixed point set is almost the same as the dimension of the considered manifold on which the group acts, are there fairly good results for compact groups G (see, for example, Ref. [5]).

In general, it has been shown [6] that there are at most a countable number of differentiable inequivalent realizations of a compact Lie group on a compact differentiable manifold. Both the compactness of the manifold and the differentiability of the action are necessary assumptions. If one of them is broken we obtain an uncountable number of realizations [6]. The non-compact case is much more difficult and therefore only a few theorems concerning the actions of non-compact groups are available.

III. Subgroups and Homogeneous Spaces of $SU(2)$

As we have already mentioned, all homogeneous spaces for the group $SU(2)$ can be obtained by finding the conjugacy classes of closed subgroups of $SU(2)$. This can be done, for instance, by using a method of Murnaghan [7]. We have, eventually, the following list of conjugacy classes of the proper closed subgroups of $SU(2)$:

(i) The unitary subgroup $U(1)$.

(ii) The subgroup $N[U(1)]$ – the normalizer of the group $U(1)$.

(iii) The cyclic subgroups C_n, $n=1,2,\ldots$, of order n.

(iv) The subgroups $\tilde{D}_{2n}$, $n=1,2,\ldots$, whose factor groups $\tilde{D}_{2n}/Z_2$ are isomorphic to the dihedral group D_n of order $2n$, $n=1,2,\ldots$, respectively.

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[4] The proof is trivial. We can form the set $\{X_x \mid X_x = \hat{G}x,\ x \in X\}$ such that $X = \bigcup_{x\in X} X_x$. Now if x_1 and x_2 are elements of X, the orbits $\hat{G}x_1$ and $\hat{G}x_2$ either coincide or have no element in common. Therefore we may consider a subset A of X for which the sets X_x, $x \in A$, are disjoint that is, if $x_1, x_2 \in A \subset X$ and $x_1 \neq x_2$, $X_{x_1} \cap X_{x_2} = \varphi$. Then, obviously, $X = \bigcup_{x\in A} X_x$. Q.e.d.

The decomposition of any realization into transitive realizations does not mean that any realization of G in X is "completely reducible" since one has a union of transitive realizations rather than a direct sum of "irreducible representations".

(v) The subgroup $\tilde{T}$ whose factor group $\tilde{T}/Z_2$ is isomorphic to the tetrahedral group T of order 12.

(vi) The subgroup $\tilde{O}$ whose factor group $\tilde{O}/Z_2$ is isomorphic to the octahedral group O of order 24.

(vii) The subgroup $\tilde{Y}$ whose factor group $\tilde{Y}/Z_2$ is isomorphic to the icosahedral group of order 60.

Here, $Z_2 \equiv \{e, -e\}$ denotes the centre of the group $SU(2)$. We see that we have at our disposal two proper continuous subgroups and five types of discrete (crystalographic or molecular) subgroups of $SU(2)$.

Let us discuss now the corresponding homogeneous spaces.

1. First let us give those which are *three-dimensional* manifolds. They are:

$$SU(2) \approx S^3; \quad SU(2)/C_n \approx L(n, 1), \quad n=1,2,\dots;$$
$$SU(2)/\tilde{D}_{2n} \equiv \tilde{L}_{2n}, \quad n=1,2,\dots; \quad SU(2)/\tilde{T} \equiv M_1;$$
$$SU(2)/\tilde{O} \equiv M_2 \quad \text{and} \quad SU(2)/\tilde{Y} \equiv M_3 .$$

$SU(2)$, as is well known, is homeomorphic to the three-dimensional sphere S^3, and $SU(2)/C_n$ is homeomorphic to the Lens space $L(n, 1)$ (for the definition of $L(n, 1)$ see, e.g., Ref. [8]). The homogeneous space M_3 is sometimes called the Poincaré space. To the best of our knowledge, the other three-dimensional homogeneous spaces are not homeomorphic to some known manifolds and thus $\tilde{L}_{2n}$, $n=1,2,\dots, M_1, M_2$ and M_3 are their abbreviated denotation.

It follows easily from Ref. [9] that the fundamental group of any of our homogeneous spaces with discrete stability subgroup is isomorphic to this stability subgroup. Therefore the above-mentioned spaces are homotopically, hence also topologically, non-equivalent.

2. Then we have two *two-dimensional* homogeneous spaces:

$$SU(2)/U(1) \approx S^2 \quad \text{and} \quad SU(2)/N[U(1)] \approx RP^2 .$$

Here S^2 is the two-dimensional sphere and RP^2 is the two-dimensional real projective plane.

3. Finally, there is a zero-dimensional homogeneous space homeomorphic to the point p,

$$SU(2)/SU(2) \approx p .$$

IV. Orbit Structure of Linear Unitary Irreducible Representations of $SU(2)$

In order to construct and better understand the non-linear realizations it is useful to be a little familiar with the orbit structure of linear representations. Although this is well known to specialists, no comprehensive study is available to the best of our knowledge. So let us give a brief study of orbit structure of linear unitary irreducible representations (UIR) here.

First we shall prove a simple theorem.

Theorem 3. *The orbit G/K, where K is a closed subgroup of the group G, is contained in the linear representation $g \to T_g$ of the group G in the n-dimensional vector space V_n over the field F iff:*

(i) *$g \to T_g$ when reduced with respect to K contains at least one one-dimensional trivial representation of K;*

(ii) *among the one-dimensional trivial representations of K there is at least one, say $K \to T_1$, operating on $V_1 \subset V_n$ such that no other subgroup K', $K \subset K' \subset G$, $K \neq K'$ operates trivially on V_1.*

Proof. To prove the necessity, assume $G/K \subset V_n$. Then $z_0 \in V_n$, $z_0 \neq 0$ exists [5] such that $K z_0 = z_0$ and the subspace $V_1 = \{z \mid z = \lambda z_0, \ \lambda \in F\}$ forms a basis for a trivial representation of K. No other subgroup $K' \neq K$, $K \subset K' \subset G$ can operate trivially on V_1 because otherwise the stability subgroup at z_0 would be larger than K.

To prove the sufficiency let us suppose that there is a trivial representation T_1 of K in $V_1 \subset V_n$ with the above-mentioned properties. Then there is a point $z_0 \in V_1$ such that $K z_0 = z_0$. But there is no larger subgroup K' of G such that $K' z_0 = z_0$ because otherwise K' would operate trivially on the space $V_1 = \{z \mid z = \lambda z_0, \lambda \in F\}$. Hence K is the stability subgroup at z_0.

Let us return to the case $G = SU(2)$.

Theorem 4. *In the spin half-integer case ($l = \frac{1}{2}, \frac{3}{2}, \ldots$) only the orbit types $S^3, L(p, 1)$ where $p = 3,5, \ldots, 2l$, and the fixed point (in the origin) occur.*

Proof. The group C_2 is represented by the matrices $\{\mathbf{1}, -\mathbf{1}\}$ and the only fixed point is the origin. Therefore, we need only to study subgroups of $SU(2)$ which do not contain C_2; these are the C_p's with p odd. The group C_p is generated by the element $c_p = e^{i\frac{2\pi}{p}}$. In an UIR of $SU(2)$ we have $c_p |lm\rangle = e^{im\frac{4\pi}{p}} |lm\rangle$. From this one can conclude that $|lm\rangle$ is invariant under C_{2m} but not invariant under C_p, $p > 2m$. It follows that only the stability subgroups $C_p, p = 1, 3, 5, \ldots, 2l$, are present.

Remark. In connection with the linear representations of $SU(2)$ there exists also fibrations of S^n which contain only spheres S^3; these are associated with reducible representations. Compare to the Hopf-fibration $\{S^{4n-1}, QP^{n-1}, SU(2), S^3\}$, [10].

The orbit structure of integer-spin representation is much more complicated. Besides a fixed point, the following orbits are contained:

[5] The case $z_0 = 0$ is trivial because the stability group K is then G itself.

Theorem 5. *If integer spin l is even, the two-dimensional orbits are real projective planes RP^2 and if l is odd the two-dimensional orbits are spheres S^2.*

Proof. The subspace in which $U(1)$ acts trivially is spanned by $|l0\rangle$. The subgroup $N[U(1)]$ contains an element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in D_n$ which is represented by matrix A acting on a basis vector $|l0\rangle$ in the following way:

$$A|l0\rangle = (-1)^l |l0\rangle .$$

Thus in the $l=$ even case the vector $|l0\rangle$ carries a trivial representation of $N[U(1)]$ and in the $l=$ odd case $U(1)$ is the "maximal trivial group" on $|l0\rangle$. Then Theorem 5 follows from Theorem 3.

Theorem 6. (i) *If $l \geqq 30$ the orbits M_1, M_2, and M_3 are always contained in the carrier space of unitary irreducible representation $\mathscr{D}^{(l)}$ of $SU(2)$.*
(ii) *If $l<30$ the orbits M_1, M_2, and M_3 are contained in $\mathscr{D}^{(l)}$ in the cases listed in Table 1.*

Proof. Let n_i^l, $i=1,2,3$, denote the number of trivial representations of $\tilde{T}$, $\tilde{O}$, $\tilde{Y}$ respectively in $\mathscr{D}^{(l)}$. The numbers n_i^l, $i=1,2,3$, can easily be calculated by using the orthogonality property of characters [11]. We obtain

$$n_1^l = \frac{1}{12}\left\{(2l+1) + 3\cdot(-1)^l + \frac{16}{\sqrt{3}}\cdot \sin\left[\left(l+\frac{1}{2}\right)\frac{2\pi}{3}\right]\right\},$$

$$n_2^l = \frac{1}{24}\left\{(2l+1) + 9\cdot(-1)^l + \frac{16}{\sqrt{3}}\cdot \sin\left[\left(l+\frac{1}{2}\right)\frac{2\pi}{3}\right] + 6\sqrt{2}\sin\left[\left(l+\frac{1}{2}\right)\frac{\pi}{2}\right]\right\},$$

$$n_3^l = \frac{1}{60}\left\{(2l+1) + 15\cdot(-1)^l + \frac{40}{\sqrt{3}}\cdot \sin\left[\left(l+\frac{1}{2}\right)\frac{2\pi}{3}\right]\right.$$
$$\left. + 12\cdot\left(\sin\frac{\pi}{5}\right)^{-1}\cdot \sin\left[\left(l+\frac{1}{2}\right)\frac{2\pi}{5}\right] + 12\cdot\left(\sin\frac{2\pi}{5}\right)^{-1}\cdot \sin\left[\left(l+\frac{1}{2}\right)\frac{4\pi}{5}\right]\right\}.$$

The results are listed in Table 2. The rest of the proof is elementary and is left to the reader. Note that in the cases $l=4, 8$ the only trivial representation of $\tilde{T}$ is trivial also with respect to $\tilde{O}$ because of $\tilde{T} \subset \tilde{O}$; this is the reason why the orbits M_1 are not included in these cases.

Theorem 7. *The orbit $L(k, 1)$ is contained in $\mathscr{D}^{(l)}$ iff $k=2,4,6,\ldots,2l$. The orbit $\tilde{L}_k$ is contained in $\mathscr{D}^{(l)}$ iff $k=2,6,10,\ldots,2l$ for l odd and iff $k=4,8,12,\ldots,2l$ for l even.*

Proof. The proof is straightforward. First we note that the vector $|lm\rangle$ is invariant under C_{2m} [6] but it is not invariant under C_k, $k>2m$.

[6] Note that $\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$ corresponds to a rotation about an angle 2ϕ round the 3-axis.

Table 1. *The orbit $M_i(=SU(2)/\tilde{T},\ SU(2)/\tilde{O},\ SU(2)/\tilde{Y})$ is contained in the representation $\mathcal{D}^{(l)}$ of $SU(2)$ in the crossed cases. M_1 is not contained in $\mathcal{D}^{(4)}$ or $\mathcal{D}^{(8)}$ because $n_1=n_2$ when $l=4, 8$ (see Table 2). If $l\geqq 30$ then M_i, $i=1, 2, 3$, is always contained in $\mathcal{D}^{(l)}$*

l	M_1	M_2	M_3	l	M_1	M_2	M_3
1				16	×	×	×
2				17	×	×	
3	×			18	×	×	×
4		×		19	×	×	
5				20	×	×	×
6	×	×	×	21	×	×	×
7	×			22	×	×	×
8		×		23	×	×	
9	×	×		24	×	×	×
10	×	×	×	25	×	×	×
11	×			26	×	×	×
12	×	×	×	27	×	×	×
13	×	×		28	×	×	×
14	×	×		29	×	×	
15	×	×	×	30	×	×	×

Under the element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ of $\tilde{D}_{2m}$ the vector $|lm\rangle$ is transformed to $\pm\,|l-m\rangle$. If m is sufficiently small, C_{2m} can be a subgroup of $\tilde{T}$, $\tilde{O}$ or $\tilde{Y}$. But, using the explicit form

$$\mathcal{D}^l_{mn}(\alpha,\beta,\gamma)=e^{im\alpha}\,P^l_{mn}(\cos\beta)\cdot e^{in\gamma}$$

Table 2. *n_i gives the number of trivial representations of K_i $(=\tilde{T}, \tilde{O}, \tilde{Y})$ contained in the representation $\mathcal{D}^{(l)}$ of $SU(2)$. If $l\geqq 30$ then $n_i\neq 0$ and $n_1>n_2$, $n_1>n_3$*

l	n_1	n_2	n_3	l	n_1	n_2	n_3
1	0	0	0	16	3	2	1
2	0	0	0	17	2	1	0
3	1	0	0	18	4	2	1
4	1	1	0	19	3	1	0
5	0	0	0	20	3	2	1
6	2	1	1	21	4	2	1
7	1	0	0	22	4	2	1
8	1	1	0	23	3	1	0
9	2	1	0	24	5	3	1
10	2	1	1	25	4	2	1
11	1	0	0	26	4	2	1
12	3	2	1	27	5	2	1
13	2	1	0	28	5	3	1
14	2	1	0	29	4	2	0
15	3	1	1	30	6	3	2

Table 3. *The orbits contained in $\mathscr{D}^{(l)}$, $l=1,2,\ldots,10$. p is the fixed point (in origin), S^2 and S^3 the 2- and 3-spheres, P^2 and P^3 the real projective spaces in dimensions 2 and 3, respectively. The others are:* $L(k,1)\approx SU(2)/C_k$, $\tilde{L}_{2k}\approx SU(2)/\tilde{D}_{2k}$, $M_1\approx SU(2)/\tilde{T}$, $M_2\approx SU(2)/\tilde{O}$, $M_3\approx SU(2)/\tilde{Y}$

l	Orbits
1	$p,\ S^2,\ L(2,1)\approx P^3,\ \tilde{L}_2$
2	$p,\ P^2,\ P^3,\ L(4,1),\ \tilde{L}_4$
3	$p,\ S^2,\ P^3,\ L(4,1),\ L(6,1),\ \tilde{L}_2,\ \tilde{L}_6,\ M_1$
4	$p,\ P^2,\ P^3,\ L(4,1),\ L(6,1),\ L(8,1),\ \tilde{L}_4,\ \tilde{L}_8,\ M_2$
5	$p,\ S^2\ P^3,\ L(4,1)\ldots L(10,1),\ \tilde{L}_2,\ \tilde{L}_6,\ \tilde{L}_{10}$
6	$p,\ P^2,\ P^3,\ L(4,1)\ldots L(12,1),\ \tilde{L}_4,\ \tilde{L}_8,\ \tilde{L}_{12},\ M_1,\ M_2,\ M_3$
7	$p,\ S^2,\ P^3,\ L(4,1)\ldots L(14,1),\ \tilde{L}_2,\ \tilde{L}_6,\ \tilde{L}_{10},\ \tilde{L}_{14},\ M_1$
8	$p,\ P^2,\ P^3,\ L(4,1)\ldots L(16,1),\ \tilde{L}_4,\ \tilde{L}_8,\ \tilde{L}_{12},\ \tilde{L}_{16},\ M_2$
9	$p,\ S^2,\ P^3,\ L(4,1)\ldots L(18,1),\ \tilde{L}_2,\ \tilde{L}_6,\ \tilde{L}_{10},\ \tilde{L}_{14},\ \tilde{L}_{18},\ M_1,\ M_2$
10	$p,\ P^2,\ P^3,\ L(4,1)\ldots L(20,1),\ \tilde{L}_4,\ \tilde{L}_8,\ \tilde{L}_{12},\ \tilde{L}_{16},\ \tilde{L}_{20},\ M_1,\ M_2,\ M_3$

for the matrix elements of $SU(2)$, one can show by direct calculation that a $k_i\in K_i$ ($K_i=\tilde{T},\ \tilde{O},\ \tilde{Y}$) exists such that $T_{k_i}|lm\rangle \neq |lm\rangle$. The rest of the proof goes along similar lines and is left to the reader. Theorems 4–7 give a complete characterization of orbit types of linear unitary irreducible representations of $SU(2)$. The list of orbit types of integer-spin representations is given in Table 3 up to $l=10$.

V. Examples of Non-Linear Realizations of $SU(2)$ on Spheres and Euclidean Spaces

Even if we choose only the modest goal of constructing some examples of non-linear realizations of $SU(2)$, the problem is not trivial. In order to get non-linear realizations of $SU(2)$ on some manifold M, e.g., $M=R^n$, we must "fill up" M using the homogeneous spaces listed in Section III in a tricky way. For example, R^3 can be filled up by spheres S^2 and a fixed point in the origin $R^4\approx C^2$ can be filled by S^3's and a fixed point, and so on, but then, without a remarkable imagination the resulting action of $SU(2)$ will be linear (and very simple). Thus, more sophisticated topological methods are needed.

By topological methods it is possible to construct non-linear realizations starting from the linear ones (see Refs. [12–16]). That the constructed realizations are really non-equivalent to linear representations and to each other can be seen by comparing the orbit structure of the constructed actions with the known orbit structure of linear representations. Usually it is enough to show that the homology groups[7] of the fixed point sets are different.

[7] See Appendix.

New non-linear actions can be obtained again by standard topological methods from those already constructed.

In the following we shall build a series of actions of $SU(2)$ on spheres starting from an example of W. C. Hsiang and W. Y. Hsiang [14], which in turn is based on an example of Bredon [13]). All actions in this section will be differentiable if not stated otherwise (in many cases they are even analytic).

Note that if we have a differentiable action of any group G on a sphere S^n, we can also construct a differentiable action of G in R^{n+1} by just filling R_0^{n+1} in the usual way with spheres S^n and putting a fixed point in the origin.

First we briefly sketch Bredon's construction. For more details see Ref. [13].

Take the diagonal, orthogonal linear action of $SO(2n+1)$ in $R^{2n+1} \times R^{2n+1}$; that is, $g(x, y) \equiv (gx, gy)$, $g \in SO(2n+1)$, $(x, y) \in R^{2n+1} \times R^{2n+1}$. First, Bredon forms a set of equivariant, norm-preserving analytic diffeomorphisms ψ_k, $k = 0, 1, 2, \ldots$

$$\begin{aligned} &\psi_k(g(x, y)) = g(\psi_k(x, y)), \\ &\psi_k(x, y) = (x', y'), \quad \|x\| = \|x'\| \quad \text{and} \quad \|y\| = \|y'\|, \end{aligned} \tag{5.1}$$

that is, ψ_k carries $S^{2n} \times S^{2n}$ into itself.

Then take the unit sphere S^{4n+1} in $R^{2n+1} \times R^{2n+1}$. For each integer k let X_k^{4n+1} be a copy of $\{(x, y) \in S^{4n+1} \mid y \neq 0\}$ and Y_k^{4n+1} a copy of $\{(x, y) \in S^{4n+1} \mid x \neq 0\}$. Let

$$U_k = \{(x, y) \in X_k^{4n+1} \mid x \neq 0\} \quad \text{and} \quad V_k = \{(x, y) \in Y_k^{4n+1} \mid y \neq 0\}.$$

ψ_k induces an analytic diffeomorphism $U_k \to V_k$, say f_k.

Denote by M_k^{4n+1} the analytic $(4n+1)$-manifold obtained from X_k^{4n+1} and Y_k^{4n+1} by identifying U_k with V_k via $f_k: U_k \to V_k$. The orthogonal action of $SO(2n+1)$ in $R^{2n+1} \times R^{2n+1}$ induces an analytic action on M_k^{4n+1} without fixed points.

It can be shown that M_k^{4n+1} is a closed, connected, simply connected manifold with integral homology groups the same as those of the $(4n+1)$-sphere:

$$H_q(M_k^{4n+1}; Z) = \begin{cases} Z, & q = 0, 4n+1 \\ 0, & q \neq 0, 4n+1. \end{cases} \tag{5.2}$$

It then follows from a result by S. Smale [17] that M_k^{4n+1} is homeomorphic to the $(4n+1)$-sphere.

In the same way, starting from $R^{2n} \times R^{2n}$ instead of $R^{2n+1} \times R^{2n+1}$, it is possible to obtain analytic actions of $SO(2n)$ on an analytic manifold

M_k^{4n+1} with the following integral homology groups:

$$H_q(M_k^{4n-1};Z) = \begin{cases} Z, & q=0,4n-1 \\ Z_{2k+1}, & q=2n-1 \\ 0, \text{ otherwise.} & \end{cases} \tag{5.3}$$

Let us explain now our methode of constructing non-linear realizations. Let $\mathscr{D}$ be a real, linear and orthogonal representation of $SU(2)$ in R^{2p+1}, $p=1,2,3,\ldots$, such that $\mathscr{D}$ does not contain the trivial representation. Take the direct sum of $\mathscr{D}$ and $(2n+1)-(2p+1)=2(n-p)$ copies of trivial representations in $R^{2p+1}\times R^{2(n-p)}=R^{2n+1}$. By the previous construction this induces an analytic action of $SU(2)$ on M_k^{4n+1}; one only needs to make the embedding

$$\mathscr{D}\oplus\sum(2(n-p) \text{ trivial representation})\subset SO(2n+1) \tag{5.4}$$

in the natural way. In $R^{2n+1}\times R^{2n+1}$ the fixed point set of $SU(2)$ is $R^{2(n-p)}\times R^{2(n-p)}$. Thus the fixed point set $F[SU(2);M_k^{4n+1}]\approx M_k^{4(n-p)-1}$.

Let $n>p$. Then the fixed point set on M_k^{4n+1} is not empty. Let x_0 be a fixed point. Then the action of $SU(2)$ in some neighbourhood around x_0 is equivalent to a linear orthogonal action[8]. Thus we can form the connected sum $M_k^{4n+1}\# M_k^{4n+1}$ (see Appendix) without destroying the action of the group. From Ref. [3] it follows that $M_k^{4n+1}\# M_k^{4n+1}$ is diffeomorphic to the standard sphere S^{4n+1}. Now we have a differentiable action of $SU(2)$ on S^{4n+1} with fixed point set $M_k^{4(n-p)}\# M_k^{4(n-p)}$. In general, if we take the connected sum

$$M_k^{4n+1}\# M_k^{4n+1}\#\cdots\# M_k^{4n+1}\equiv M_k^{kn+1}(2j)=M\approx S^{4n+1} \tag{5.5}$$

with an even number, say $2j$, of terms we get a differentiable action of $SU(2)$ on S^{4n+1} with fixed set

$$M_k^{4(n-p)-1}\#\cdots\# M_k^{4(n-p)-1}\equiv F_k^{4(n-p)-1}(2j)=F\,. \tag{5.6}$$

We denote this action by $(\alpha)=(n,k,p,j)$. Next we show that these actions are all non-equivalent if $n>p+1$[9], i.e., $(\alpha)\approx(\alpha)'\Rightarrow n=n'$, $k=k'$, $p=p'$, $j=j'$. From the dimension of M and M' (F and F' respectively) it is clear that $(\alpha)\approx(\alpha)'\Rightarrow n=n'$ ($p=p'$ respectively). $H_q(F;Z)$, $4(n-p)-1>q>1$, can be calculated using the Mayer-Vietoris sequence [19]. For example

$$H_{2(n-p)-1}(F_k^{4(n-p)-1}(2j);Z)=\sum_{2j\text{ terms}}\oplus Z_{2k+1}\,. \tag{5.7}$$

From (5.7) it follows that in order to have same homology groups of the fixed set we must also put $j=j'$ and $k=k'$.

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[8] See: Bochner (Ref. [18], 1945) the simple proof can also be found in Ref. [2] by Coleman *et al.*

[9] This requirement is only for technical convenience.

Another method of obtaining new differentiable actions of $SU(2)$ on spheres is to take some differentiable, connected and simply connected contractible m-manifold Y[10] and the unit disc D^n and form the action on $Y \times D^n$ induced by the trivial action on Y and some action on S^{n-1}. (This is the simplest case; in general we can also have non-trivial actions on Y.) If $n+m$ is even and $\neq 4$, it follows from Ref. [17] that $Y \times D^n$ is diffeomorphic to D^{n+m}. If we take the boundary, we get a differentiable action on S^{n+m-1} with fixed point set $Y \times F(SU(2); D^n)$.

We are not going to dwell on any more details; we only note that the constructions represented in this section are also applicable to compact groups other than $SU(2)$.

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Appendix

1. Homotopy Groups

Let X and Y be two topological spaces, $A \subset X$, and f, g two maps from X into Y such that $f(x) = g(x)$ for all $x \in A$. We say that the map f is *homotopic* to g relative to A, denoted $f \simeq g \operatorname{rel} A$, if a map

$$F: X \times I \to Y \quad (I \text{ is the unit interval})$$

exists such that

(i) $F(x, 0) = f(x)$ for $\forall x \in X$,
(ii) $F(x, 1) = g(x)$ for $\forall x \in X$,
(iii) $F(x, t) = f(x) = g(x)$ for $\forall x \in A$, $\forall t \in I$.

Roughly speaking, f can be deformed to g. It is easy to show that homotopy is an equivalence relation.

Let us now define the homotopy group π_n. Let I^n be the unit cube in n dimensions with co-ordinates $x = (x_1, x_2, \ldots, x_n)$, $0 \leqq x_i \leqq 1$. An $(n-1)$-face of I^n is a submanifold with some x_i equal to 0 or 1. The union of $(n-1)$-faces is the *boundary* ∂I^n of I^n.

Denote by $F^n = F^n(X, p_0)$ the set of maps

$$f: I^n \to X, \quad f(\partial I^n) = p_0,$$

where X is some topological space and p_0 a point of X. We shall denote by $\pi_n(X, x_0)$ the set of homotopy equivalence classes of these maps. We can define the addition in π_n in the following way. Let f and g be the maps representing the classes $[f]$ and $[g]$. First define the sum of f

[10] For example, the manifolds Y_i in Ref. [12] are useful.

and g:

$$(f+g)(x)=\begin{cases} f(2x_1, x_2, \ldots, x_n) & \text{if } 0 \leqq x_1 \leqq \tfrac{1}{2}, \\ g(2x_1-1, x_2, \ldots, x_n) & \text{if } \tfrac{1}{2} \leqq x_1 \leqq 1. \end{cases}$$

Then define the class $[f]+[g]$ by

$$[f]+[g]=[f+g].$$

It can be shown that the addition depends only upon the classes $[f]$ and $[g]$. The constant map is defined by $f(x)=p_0$ for all $x \in I^n$. Denote by $[0]$ the corresponding homotopy class. Consider the map $\theta: I^n \to I^n$, $\theta(x)=(1-x_1, x_2, \ldots, x_n)$. It can be shown that the maps $f+f\theta$ and $f\theta+f$ are homotopic to the identity map 0, so that we can define the inverse of $[f]$ by $[f]^{-1}=[f\theta]$. Then $\pi_n(X, x_0)$ is a group with respect to the operation of addition introduced above called the *n-th homotopy group* of the manifold X with respect to the base point x_0. If $n=1$, this is just the fundamental group of the manifold X with respect to the base point x_0.

2. The Homology Modules

Consider the submanifolds Δ_q of R^∞, defined by

$$\Delta_q=\left\{x \in R^\infty \mid x=(x_1, x_2, \ldots, x_q, 0, 0, \ldots), \quad \text{all } x_i \geqq 0, \quad \sum_{i=1}^{q} x_i \leqq 1\right\},$$
$$\Delta_0=(0,0,0,\ldots).$$

Thus Δ_1 is the unit interval, Δ_2 is the triangle including its interior, Δ_3 is a tetrahedron, etc. In general, Δ_q is called *the standard q simplex.*

Given a space X, a *singular q-simplex* in X is a map $\Delta_q \to X$. For $q=0$ it can be identified with a point in X, for $q=1$ with a path in X, etc.

Let R be a commutative ring (usually the ring of real numbers or integers). Define $S_q(X)$ to be the free R-module generated by the singular q-simplexes. That is, every element of $S_q(X)$ is a formal sum $\sum_\sigma v_\sigma \sigma$, where σ runs through singular q-simplexes and $v_\sigma \in R$. The elements of $S_q(x)$ are called *singular q-chains.*

For $q>0$ define $F_q^i: \Delta_{q-1} \to \Delta_q$, $0 \leqq i \leqq q$, as follows:

$$F_q^i(x_1, x_2, \ldots, x_{q-1})=(x_1, \ldots, x_{i-1}, 0, x_i, x_{i+1}, \ldots, x_q).$$

If σ is a singular q-simplex in X, then the *i-th face* $\sigma^{(i)}$ of σ is by definition the singular $(q-1)$-simplex $\sigma \circ F_q^i$.

The *boundary* of a singular q-simplex, σ, is by definition the singular $(q-1)$-chain

$$\partial(\sigma)=\sum_{i=0}^{q}(-1)^i \sigma^{(i)}.$$

Then ∂ can be extended to a module homomorphism $S_q(X) \to S_{q-1}(X)$ by writing

$$\partial\left(\sum_\sigma v_\sigma \sigma\right) = \sum_\sigma v_\sigma \partial(\sigma).$$

It can be shown that the boundary of the boundary of any singular q-chain vanishes, i.e., $\partial\partial = 0$.

A singular q-chain c such that $\partial(c) = 0$ is called a *cycle*. The cycles form a submodule Z_q of $S_q(X)$. Denote by B_q the module of boundaries

$$B_q = \{c \mid c = \partial(c'), \quad c' \text{ is a singular } (q+1)\text{-chain}\}.$$

From $\partial\partial = 0$ follows that $B_q \subset Z_q$. Now the q-the *singular homology module* is by definition

$$H_q(X; R) = Z_q / B_q.$$

If no confusion can arise we simply write $H_q(X)$. Let us remark that if R is the ring of integers, $H_q(X; R)$ are in fact *homology groups*.

3. *Some Constructions of Spaces*

In this paragraph we list some standard topological constructions used to derive non-linear actions of groups.

a) *Suspension.* Let X be a Hausdorff space. Take the product space $X \times I$, where I is the unit interval, and identify the subspace $X \times 0$ to one point and the subspace $X \times 1$ to another. The resulting space, denoted by SX, is called the *suspension of* X. For example $S(S^n) \approx S^{n+1}$. The following is true:

$$H_q(SX; R) \approx \begin{cases} H_{q-1}(X; R), & q > 1, \\ R, & q = 0, \\ 0, & q = 1. \end{cases}$$

b) *Join.* Let X and Y be Hausdorff spaces, $x \in X$, $y \in Y$ such that x, y have contractible[11] open neighbourhoods. The space obtained by identifying x and y is called *the join* $X \vee Y$ of X and Y at (x, y). For join it is true that

$$H_q(X \vee Y) \approx \begin{cases} H_q(X) \oplus H_q(Y) & q > 0 \\ R^{c+d-1} & q = 0 \end{cases}$$

where $c(d)$ is the number of disconnected parts of $X(Y)$, respectively.

c) *Connected Sum.* Let X and Y be two connected n-manifolds and D^n the unit disc $\{x \in R^n \mid \|x\| \leqq 1\}$. Choose the embeddings

$$i: D^n \to X, \quad j: D^n \to Y$$

[11] If M is a space such that the identity map on M is homotopic to a constant map on some point in M, we say M is contractible.

so that i preserves orientation and j reverses orientation. By definition, $X \# Y$ is obtained from the disjoint sum

$$(X - i(0)) + (Y - j(0))$$

by identifying $i(tu)$ with $j((1-t)u)$ for every $0 < t < 1$ and every unit vector $u \in S^{n-1}$. Because the correspondence $i(tu) \rightarrow j((1-t)u)$ preserves orientation, it is possible to choose the orientation for $X \# Y$ so that it is compatible with that of X and Y. The following lemma has been used in the text:

Lemma. *The connected sum operation is well defined, associative and commutative up to orientation-preserving diffeomorphism. The sphere S^n serves as identity element (see Ref.* [3] *for proof).*

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