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A $\lambda\phi^{2n}$ Field Theory without Cutoffs*

LON ROSEN

Courant Institute of Mathematical Sciences, New York University

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Abstract. We consider a self-interacting scalar boson field in two-dimensional space-time with self-interaction given by an arbitrary Wick polynomial of even degree in the field. It is shown that the field theory can be constructed in a Hilbert space of physical states. The hamiltonian is a positive self-adjoint operator possessing a physical vacuum. The method of proof consists of imposing and then removing three cutoffs: a box cutoff, an ultraviolet cutoff, and a space cutoff. As the first two are removed the resolvents of the cutoff hamiltonians converge uniformly and this leads to the self-adjointness of the spatially cutoff hamiltonian.

§ 1. Introduction and Discussion of Results

In this paper we consider the self-interacting boson field in two-dimensional space-time with hamiltonian given formally as

$$H_{\text{formal}} = H_0 + \lambda \int :P(\phi(x)):dx \tag{1.1}$$

where H_0 is the free hamiltonian for the mass $m > 0$,

P is an arbitrary polynomial of even degree, $P(y) = y^{2n} + b_{2n-1}y^{2n-1} + \cdots + b_0$,

and

λ, the coupling constant, is taken equal to 1 in this paper unless otherwise indicated.

From perturbation theory considerations we expect this model, when rigorously treated, to provide a Lorentz-covariant local quantum field theory with nontrivial scattering. This is the motivation for this study in which we take the first steps towards a field theory.

As has been emphasized by Wightman (see, for example, [21]), a formal expression for the hamiltonian like (1.1) is highly singular and must be "cutoff" or "butchered" if we wish to use the interaction picture or to work in Fock space. For instance, a version of Haag's theorem [21, § VI] states that either we must destroy the translation invariance of the density $:P(\phi(x)):$ in (1.1) or else we must work with a strange representation of the commutation relations (and cannot use Fock space).

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We butcher. Two standard methods are to introduce into the interaction hamiltonian a spatial cutoff function $g(x)$ whose physical meaning is that particles interact only in a bounded region of space, or to place the world in a large box. For convenience we do both, and, in addition, impose an ultraviolet cutoff on the momentum the particles are allowed to have. This last cutoff also makes the hamiltonian less singular; in particular, it produces an interaction term that is bounded below. Finally, of course, we must remove all the cutoffs and investigate the sense in which the cutoff or approximate theories converge, thereby deducing properties of the theory without cutoffs.

We now describe the spatial cutoff. The boson Fock space $\mathscr{F}$ is the direct sum $\mathscr{F}=\sum_n \oplus \mathscr{F}^n$ where $\mathscr{F}^n$ is the space of symmetric square integrable functions of n (momentum) variables. The field at $t=0$ is

$$\phi(x)=\frac{1}{(4\pi)^{1/2}}\int e^{ikx}[a^*(-k)+a(k)]\,\mu(k)^{-1/2}dk \tag{1.2}$$

where $\mu(k)=(k^2+m^2)^{1/2}$, and the annihilation and creation operators are normalized by

$$[a(k), a^*(k')]=\delta(k-k').$$

The power of the field, $:\phi^p(x):$, Wick ordered by placing creators on the left and annihilators on the right, lacks meaning as an operator on $\mathscr{F}$, but is a densely defined bilinear form. However when smeared with $g(x)$,

$$:\phi^p(g):\ \equiv \int :\phi^p(x):\, g(x)\, dx$$

is a densely defined operator on $\mathscr{F}$ (by (1.3) and Lemma 4.1) where we take $g(x)$ to be an infinitely differentiable function of compact support, equal to 1 on a large set $(-X, X)$, and satisfying $0\leqq g(x)\leqq 1$. By taking Fourier transforms we find that

$$\begin{aligned}:\phi^p(g):\ =(4\pi)^{-p/2}\sum_{j=0}^{p}\binom{p}{j}\int & a^*(-k_1)\dots a^*(-k_j)\\ &\cdot a(k_{j+1})\dots a(k_p)\,\hat{g}(k_1+\cdots+k_p)\\ &\cdot\prod_i \mu(k_i)^{-1/2}dk_i\,.\end{aligned} \tag{1.3}$$

We replace the formal hamiltonian of (1.1) with the spatially cutoff

$$H(g)=H_0+\lambda H_I(g)=H_0+\lambda\int g(x)\,:P(\phi(x)):\,dx\,, \tag{1.4}$$

which is a densely defined operator on $\mathscr{F}$.

The completely cutoff interaction hamiltonian $H_{I,K,V}(g)$ and full hamiltonian $H_{K,V}(g)$ are defined in the next section. The subscript V

stands for the volume of the enclosing box, and K for the ultraviolet cutoff on the modes that contribute to the hamiltonian.

By unravelling the Wick ordering it is seen [2, Lemma 5.2] that $H_{I,K,V}(g)$ is bounded below, $H_{I,K,V}(g) \geqq -A(\ln K)^n$, where A is a constant independent of K and V. (The assumption that $P(y)$ has even degree has entered here.) We can by a unitary transformation bring $H_{K,V}(g)$ into the form

$$-\Delta + U(p_1, \ldots, p_M)$$

on $L^2(\boldsymbol{R}^M)$, where $\Delta = \frac{\partial^2}{\partial p_1^2} + \cdots + \frac{\partial^2}{\partial p_M^2}$, M is the number of modes that are not cutoff, and U is a polynomial that is bounded below. By a theorem of Jaffe [12, p. 35], we then deduce that $H_{K,V}(g)$ is essentially self-adjoint on the domain generated from the Fock vacuum $\Omega_0 = (1, 0, 0, \ldots)$ by the cutoff field algebra.

The next step is to remove the momentum cutoffs by letting K and V approach infinity. In [2] Glimm proved that $H(g)$ is semibounded by using a method of Nelson [16]. This method consists of studying the approximate hamiltonians $H_{K,V}(g)$ in a representation which permits the use of the Feynman-Kac formula. We employ the same technique to demonstrate that as $K, V \to \infty$, the resolvents of $H_{K,V}(g)$, $R_{K,V}(z) = (H_{K,V}(g) - z)^{-1}$, converge uniformly (Theorem 5.3). This enables us to prove that $H(g)$, with momentum cutoffs removed, is essentially self-adjoint on its natural domain of definition, $\mathscr{D} = D(H_0) \cap D(H_I(g))$ (Theorem 6.3).

Glimm and Jaffe, in a series of papers [5–7], have carried out a program for the ϕ^4 model in two dimensions, with all cutoffs removed, in which they verify most of the axioms of the Haag-Kastler scheme [9, 10], and several of the Wightman axioms [19]. By their methods, many of which are not restricted to ϕ^4, we can, as a consequence of our theorems, remove the spatial cut-off for the $P(\phi)$ model and verify a number of the axioms.

First, we can obtain a dynamics, independent of the cutoff, for bounded functions of the free fields localized in a bounded region of space. More precisely, let B be a bounded open region of space, and define $\mathscr{R}(B)$ as the von Neumann algebra of operators generated by the spectral projections of all the fields $\phi(f)$ and $\pi(f)$ where $\operatorname{supp} f \subset B$ and π is the field canonically conjugate to ϕ,

$$\pi(x) = \frac{i}{(4\pi)^{1/2}} \int e^{ikx} [a^*(-k) - a(k)] \, \mu(k)^{1/2} \, dk \,. \tag{1.5}$$

We time translate $A \in \mathscr{R}(B)$ in the Heisenberg picture

$$\sigma_t(A) = e^{itH(g)} A e^{-itH(g)} \,. \tag{1.6}$$

Then by a theorem of Segal [18, Theorem 3] the essential self-adjointness of $H(g)$ on $\mathscr{D}$ implies (i) that $\sigma_t(A)$ is independent of $g(x)$ provided that $g(x)=1$ on $\overline{B_t}$ and (ii) that $\sigma_t(A) \in \mathscr{R}(B_t)$, where B_t is the region B expanded by t (the speed of light has been taken to be one). A proof of this theorem is given in [5] except that it is shown only that $\sigma_t(A) \in \mathscr{R}(B_t')'$ where $B_t' = \boldsymbol{R} - B_t$ and $\mathscr{R}'$ is the commutant of $\mathscr{R}$; the proof is completed by the duality theorem of Araki, $\mathscr{R}(B') = \mathscr{R}(B)'$. Thus the spatial cutoff has been removed as far as the dynamics of local algebras is concerned, and the resulting theory is local.

The fact that the resolvents of $H_{K,V}(g)$ converge uniformly to the resolvent of $H(g)$ as $K, V \to \infty$ leads to the existence of a unique vacuum vector Ω_g for $H(g)$ (Theorem 6.4). The proof of this statement, which is given in [6, § 2], again illustrates how we take advantage of the more manageable cutoff hamiltonians. The result is first proved for $H_{K,V}(g)$ and then transferred to $H(g)$ by means of the uniform convergence of the resolvents.

After Glimm and Jaffe (see [6, § 5] and [7] for full details), we now indicate how the space cutoff $g(x)$ can be completely removed from the theory with the aid of the vacuum vectors Ω_g. By Haag's theorem, the theory without any cutoffs cannot exist on Fock space and the phenomenon of changing Hilbert spaces arises.

Let E_g be the ground state energy of $H(g)$, $H(g)\Omega_g = E_g \Omega_g$. Then as $g(x) \to 1$ on $(-\infty, \infty)$, $E_g \to -\infty$ [8], and (in first order perturbation theory) Ω_g converges weakly to zero, difficulties anticipated from Haag's theorem. On the other hand we do get better behavior from the expectation values

$$\omega_g(A) = (\Omega_g, A\Omega_g)\,, \tag{1.7}$$

where A is in the C^*-algebra $\mathscr{A}$ of bounded functions of the local field ϕ. (The self-adjoint elements of $\mathscr{A}$ are the observables of the theory.) Now ω_g is a positive linear functional of norm one in $\mathscr{A}^*$, the dual of $\mathscr{A}$; that is, ω_g is a state in the sense of C^*-algebras. From compactness arguments we deduce that as $g(x) \to 1$, a subsequence of the ω_g converges:

$$\omega_{g_n} \to \omega \in \mathscr{A}^*\,. \tag{1.8}$$

According to the Gelfand-Segal-Naimark construction [15], ω defines an inner product on a new Hilbert space $\mathscr{F}_{\text{ren}}$ where the operators A of $\mathscr{A}$ are represented by operators A_{ren}. $\mathscr{F}_{\text{ren}}$ is the physical Hilbert space for the renormalized or cutoff-free theory.

The Heisenberg dynamics (1.6) obtained above gives rise to a dynamics in $\mathscr{F}_{\text{ren}}$:

$$U(t)^* A_{\text{ren}} U(t) = (\sigma_t(A))_{\text{ren}}$$

where $U(t)$ is a one-parameter strongly continuous group of unitary operators in $\mathscr{F}_{\text{ren}}$. The physical hamiltonian H is defined as the generator of this group, $U(t) = e^{-itH}$; H is positive and has Ω as a vacuum vector, $H\Omega = 0$. Ω is the physical vacuum. In addition, it is possible to obtain space translations and a physical momentum operator with the expected properties. It should be mentioned that in order to accomplish the above construction, Glimm and Jaffe found it necessary to modify slightly the definition (1.7) of ω_g by averaging over space.

In a sense this summary of the physical theory is deceiving because it ignores how remote the theory is on $\mathscr{F}_{\text{ren}}$. We can ask: Is the vacuum Ω unique? Does Ω belong to an invariant domain for the fields in $\mathscr{F}_{\text{ren}}$? Does the energy-momentum spectrum permit the application of the Haag-Ruelle scattering theory? Such questions will ultimately be answered by means of the tenuous link between the physical theory and the cutoff theory expressed by the limit (1.8). As a first step towards controlling this limit, Glimm and Jaffe [6, Theorem 5.2.1] have bounded the rate at which the vacuum energy E_g diverges:

$$|E_g| \leqq \text{const.}\,|\text{supp}_1 g|\,,$$

where $\text{supp}_1 g$ is the set of points whose distance from $\text{supp}\, g$ is less than 1. As a consequence of this estimate, the physical representation $A \to A_{\text{ren}}$ is locally Fock [7]; that is, for A localized in a bounded region B of space-time, $A_{\text{ren}} = U_B A U_B^*$ where U_B is some unitary transformation from Fock space to $\mathscr{F}_{\text{ren}}$. In fact, the locally Fock property has already been invoked in the above construction to produce the convergent subsequence of (1.8) from a convergent subnet and to extend the physical representation to all of $\mathscr{A}$; its practical significance is that it enables us to define physical fields and to establish local properties of the physical theory by working in Fock space.

§ 2.. The Approximate Hamiltonians $H_{K,V}(g)$ and Q-space

In the remainder of the paper the hamiltonian always includes the spatial cutoff; however we shall frequently suppress the cutoff function g in the notation.

If we place the system in a large box of volume V (in one space dimension the "box" is the line segment $[-V/2, V/2]$), and impose periodic boundary conditions on the wave functions, then the momentum space variables $k \in \boldsymbol{R}$ are replaced by discrete variables $k \in \Gamma_V$ where the lattice

$$\Gamma_V = \left\{k \,|\, k = n\frac{2\pi}{V},\ n = 0, \pm 1, \pm 2, \ldots\right\}.$$

The appropriate Fock space is $\mathscr{F}_V = \sum_n \oplus \mathscr{F}_V^n$ where $\mathscr{F}_V^n$ is the space of symmetric square summable functions on $(\Gamma_V)^n = \Gamma_V \times \Gamma_V \times \cdots \times \Gamma_V$. For our purposes it is more convenient to regard $\mathscr{F}_V$ as a subspace of $\mathscr{F}$ consisting of functions which are constant around lattice points; that is,

$$\Psi = (\Psi_0, \Psi_1 \ldots) \in \mathscr{F}_V \quad \text{if} \quad \Psi_n(k_1, \ldots k_n) = \Psi_n([k_1]_V, \ldots [k_n]_V),$$

where $[k]_V$ is the lattice point closest to k:

$$[k]_V = l \in \Gamma_V, \qquad -\frac{\pi}{V} < l - k \leqq \frac{\pi}{V}.$$

Let $\mathscr{F}_{K,V}$ be the subspace of $\mathscr{F}_V$ consisting of functions with support cutoff at $K > 0$; that is, $\Psi_n(k_1, \ldots k_n) = 0$ if $[k_i]_V \notin \Gamma_{K,V}$ for some $1 \leqq i \leqq n$, where

$$\Gamma_{K,V} = \{k \mid k \in \Gamma_V, |k| \leqq K\}.$$

The annihilation and creation operators in the box we define as

$$a_V^{\#}(k) = \left(\frac{V}{2\pi}\right)^{1/2} \int_{-\pi/V}^{\pi/V} a^{\#}(k + l)\, dl \tag{2.1}$$

where $k \in \Gamma_V$ and $a^{\#}$ stands for a or a^*.

We can now define the fully butchered hamiltonian. The cutoff free hamiltonian is

$$H_{0,K,V} = \sum_{k \in \Gamma_{K,V}} \mu(k)\, a_V^*(k)\, a_V(k).$$

We approximate the field (1.2) and its powers by

$$\phi_{K,V}(g) = (2V)^{-1/2} \sum_{k \in \Gamma_{K,V}} [a_V^*(-k) + a_V(k)]\, \hat{g}_V(k)\, \mu(k)^{-1/2}$$

and

$$\begin{aligned} :\phi_{K,V}^r(g): = (2V)^{-r/2} \sum_j \binom{r}{j} \sum_{k_i \in \Gamma_{K,V}} a_V^*(-k_1) \ldots a_V^*(-k_j)\, a_V(k_{j+1}) \\ \ldots a_V(k_r)\, \hat{g}_V(k_1 + \cdots + k_r)\, [\mu(k_1) \ldots \mu(k_r)]^{-1/2}, \end{aligned} \tag{2.2}$$

where $\hat{g}_V(k) = \int_{-V/2}^{V/2} e^{ikx} g(x)\, dx$.

(For all boxes $[-V/2, V/2]$ containing the support of g, $\hat{g}_V(k)$ no longer depends on V; in the following we assume that we are dealing with large enough V and we drop the subscript on $\hat{g}_V$.)

The cutoff interaction hamiltonian is

$$H_{I,K,V}(g) = :\phi_{K,V}^{2n}(g): + b_{2n-1} :\phi_{K,V}^{2n-1}(g): + \cdots + b_0. \tag{2.3}$$

Finally the fully cutoff hamiltonian is

$$H_{K,V}(g) = H_{0,K,V} + H_{I,K,V}(g). \tag{2.4}$$

We note that $a_V^\sharp$, $H_{0,K,V}$, $H_{I,K,V}(g)$, and $H_{K,V}(g)$ are operators on the whole space $\mathscr{F}$ but that they leave $\mathscr{F}_{K,V}$ invariant.

We shall determine the sense in which $H_{K,V}(g)$ approximates $H(g)$: as a first step it is easy to verify that $H_{K,V}(g)$ converges in the sense of bilinear forms to $H(g)$ as $K, V \to \infty$, on the set of states with a finite number of particles and bounded momentum.

We now introduce "Q-space" [1, 2] which is a new representation of the Hilbert space $\mathscr{F}$ in which the interaction hamiltonian $H_I(g)$ becomes a multiplication operator while the free hamiltonian becomes a differential operator. In this sense the hamiltonian in Q-space is mathematically more accessible and, in particular, we obtain an explicit expression for the semigroup $e^{-tH_{K,V}(g)}$ (see the next section).

Physically, Q-space is also a familiar object, for it is obtained by regarding the fields $\phi(x)$ and $\pi(x)$, defined in (1.2) and (1.5), as a collection of coupled harmonic oscillators each expressed in the Schroedinger representation. More precisely, we expand the cutoff fields in terms of sines and cosines.

$$\phi_{K,V}(x) = (2V)^{-1/2}\left[\sum_{\substack{k\in\Gamma_{K,V}\\ k\neq 0}} (q_{|k|}\cos kx + q_{-|k|}\sin|k|x) + \sqrt{2}\,q_0\right],$$

$$\pi_{K,V}(x) = (2V)^{-1/2}\left[\sum_{\substack{k\in\Gamma_{K,V}\\ k\neq 0}} (p_{|k|}\cos kx + p_{-|k|}\sin|k|x) + \sqrt{2}\,p_0\right]$$

where for $k \neq 0$,

$$q_{|k|} = \frac{1}{2}\mu(k)^{-1/2}[a_V^*(-k) + a_V(k) + a_V^*(k) + a_V(-k)],$$

$$q_{-|k|} = \frac{i}{2}\mu(k)^{-1/2}[a_V^*(|k|) + a_V(-|k|) - a_V^*(-|k|) - a_V(|k|)],$$

$$p_{|k|} = \frac{i}{2}\mu(k)^{1/2}[a_V^*(-k) - a_V(k) + a_V^*(k) - a_V(-k)],$$

$$p_{-|k|} = \frac{1}{2}\mu(k)^{1/2}[a_V^*(-|k|) - a_V(|k|) - a_V^*(|k|) + a_V(-|k|)],$$

and

$$q_0 = (2m)^{-1/2}[a_V^*(0) + a_V(0)], \qquad p_0 = i\left(\frac{m}{2}\right)^{1/2}[a_V^*(0) - a_V(0)].$$

It is easily checked that $H_{I,K,V}(g)$ is a polynomial in the q_k's, while

$$H_{0,K,V} = \frac{1}{2}\sum_{k\in\Gamma_{K,V}} [p_k^2 + \mu(k)^2 q_k^2 - \mu(k)].$$

The set of p_k and q_k defined in this way are canonical variables for the kth harmonic oscillator:

$$[q_k, p_{k'}] = i\delta_{kk'}, \quad \text{for} \quad k, k' \in \Gamma_{K,V}.$$

The Schroedinger representation for these oscillators is realized on $L^2(\boldsymbol{R}^M)$ where q_k is multiplication by q_k and p_k is the operator $\frac{1}{i}\frac{\partial}{\partial q_k}$. Here M is the number of oscillators, or number of points in $\Gamma_{K,V}$. Since we are interested in the limit $M \to \infty$ we normalize the total measure on $\boldsymbol{R}$ to be 1 so that infinite products are well-defined. We introduce $Q(k) = (\boldsymbol{R}, \varrho_k(q_k)\, dq_k)$, the real line with gaussian measure

$$\varrho_k(q_k)\, dq_k = \left(\frac{\mu(k)}{\pi}\right)^{1/2} e^{-\mu(k) q_k^2} dq_k,$$

$Q_{K,V} = \bigotimes_{k \in \Gamma_{K,V}} Q(k)$, $\boldsymbol{R}^M$ with product measure

$$\varrho_{K,V}(q)\, dq = \prod_{k \in \Gamma_{K,V}} \varrho_k(q_k)\, dq_k,$$

$Q_V = \bigotimes_{k \in \Gamma_V} Q(k)$ with product measure

$$\varrho_V(q)\, dq = \prod_{k \in \Gamma_V} \varrho_k(q_k)\, dq_k.$$

By von Neumann's uniqueness theorem for irreducible representations of a finite number of canonical variables we know that there is a unitary equivalence $W_{K,V}$ between $\mathscr{F}_{K,V}$ and $L_2(Q_{K,V})$ which sends q_k into multiplication by q_k in the factor $Q(k)$ and p_k into the operator $\varrho_k^{-1/2} \frac{1}{i}\frac{\partial}{\partial q_k} \varrho_k^{1/2}$ acting in $Q(k)$. We shall be concerned with the limit $K, V \to \infty$. For convenience we shall remove the box cutoff with a sequence of boxes whose volumes are $\{V_0, 2V_0, \ldots 2^j V_0, \ldots\}$ for some fixed V_0. This is not essential but it allows us, for $V < V'$, to regard $\mathscr{F}_V$ as a subspace of $\mathscr{F}_{V'}$ and Q_V as a subspace of $Q_{V'}$. (The final results are independent of the particular sequence used.) Thus we are dealing with a sequence of spaces and unitary maps $W_{K,V}: L_2(Q_{K,V}) \to \mathscr{F}_{K,V}$, where for $V \leqq V'$, $K \leqq K'$ we have the inclusions $\mathscr{F}_{K,V} \subset \mathscr{F}_{K',V'}$, $L_2(Q_{K,V}) \subset L_2(Q_{K',V'})$, $W_{K,V} \subset W_{K',V'}$; moreover the sequence $\mathscr{F}_{K,V}$ is dense in $\mathscr{F}$. This suggests that the $Q_{K,V}$ converge to some measure space Q and $L_2(Q)$ is unitarily equivalent to $\mathscr{F}$. Such a unitary equivalence is explicitly constructed in [1]. In fact, its existence is guaranteed by a theorem of Gelfand and Naimark [15]: the operators $\{e^{i\alpha q_k} | \alpha \in \boldsymbol{R}, k \in \Gamma_{K,V}$ for some $K, V\}$ form a maximal abelian C^*-algebra $\mathscr{A}$ on $\mathscr{F}$ and thus there exist a positive measure $d\nu$ on the set $\mathscr{M}$ of maximal ideals of the algebra $\mathscr{A}$ and a unitary map W from $L_2(\mathscr{M}, d\nu)$ onto $\mathscr{F}$ which diagonalizes $\mathscr{A}$. Actually we never need the whole space $Q = (\mathscr{M}, d\nu)$.

We transform the above discussion into the sketch of a proof of a theorem by stating:

Theorem 2.1. *There is a unitary equivalence W between $\mathscr{F}$ and $L_2(Q)$ for some (positive) measure space Q,*

$W: L_2(Q) \to \mathscr{F}$, with the following properties:

(i) *the space Q contains Q_V and $Q_{K,V}$ as factors;*

(ii) *$W_V = W|_{L_2(Q_V)}$ is a unitary equivalence between $\mathscr{F}_V$ and $L_2(Q_V)$;*

(iii) *$W_{K,V} = W_V|_{L_2(Q_{K,V})}$ is a unitary equivalence between $\mathscr{F}_{K,V}$ and $L_2(Q_{K,V})$;*

(iv) *$W_V 1 = \Omega_0$ where $1 \in L_2(Q_V)$ is the constant function and Ω_0 the Fock vacuum;*

(v) *$W_V^{-1} q_k W_V$ is multiplication by q_k in the factor $L_2(Q(k))$, $k \in \Gamma_V$;*

(iv) *$W_V^{-1} p_k W_V$ is the operator $\varrho_k^{-1/2} \frac{1}{i} \frac{d}{dq_k} \varrho_k^{1/2}$ in the factor $L_2(Q(k))$, $k \in \Gamma_V$;*

(vii) *$W_V^{-1} H_{0,K,V} W_V = \sum_{k \in \Gamma_{K,V}} H_{\mu(k)}$ where*

$$H_{\mu(k)} = -\frac{1}{2}\left(\frac{d}{dq_k}\right)^2 + \mu(k) q_k \frac{d}{dq_k}$$

acting on $L_2(Q(k))$;

(viii) *$W_V^{-1} :\phi_{K,V}^p(g): W_V$ is a polynomial in $Q_{K,V}$.*

By virtue of this theorem we identify $\mathscr{F}_V$ with $L_2(Q_V)$, $\mathscr{F}_{K,V}$ with $L_2(Q_{K,V})$, $H_{I,K,V}$ with a polynomial $H_{I,K,V}(q)$ in $Q_{K,V}$, and $H_{0,K,V}$ with the differential operator in $L_2(Q_{K,V})$ defined in (vii).

§ 3. The Feynman-Kac Formula [2, 16]

In the Q-space representation there is an explicit expression for the semigroup $e^{-tH_{K,V}(g)}$ as an integral over path space.

Let C be the space of continuous paths $q(s)$ in $Q_{K,V}$, $0 \leqq s < \infty$. We assign a measure $dQ_{K,V}$ to C that is intrinsically associated with the semigroup $\exp(-tH_{0,K,V})$. As shown in [16], for $\psi \in L_2(Q(k))$

$$(e^{-tH_{\mu(k)}}\psi)(q) = \int p_k^t(q, q') \psi(q') \varrho_k(q') dq' , \tag{3.1}$$

where

$$p_k^t(q, q') = (1 - e^{-2\mu t})^{-1/2} \exp\left[-\frac{\mu(q' - e^{-\mu t} q)^2}{1 - e^{-2\mu t}} + \mu q'^2\right]. \tag{3.2}$$

Thus for a path $q_k(s)$ in $Q(k)$ we define the conditional probability

$$Pr\{q_k' \leqq q_k(t) < q_k' + dq_k' \mid q_k(0) = q_k\} = p_k^t(q_k, q_k') \varrho_k(q_k') dq_k' .$$

On $Q_{K,V} = \bigotimes_{k \in \Gamma_{K,V}} Q(k)$ we use the product measure and assign a gaussian distribution to the initial point $q(0)$ to arrive at the following definition: Let B be the set of paths $q(s)$ in C satisfying $q(s_i) \in B_i$, $1 \leqq i \leqq N$, where B_i is a Borel set in $Q_{K,V}$, and $0 = s_1 < s_2 < \cdots < s_N$. Then the measure of B is

$$\int dQ_{K,V} \equiv \int_{B_1 \times \cdots \times B_N} \varrho_K(q(0))\, dq(0)\, p_K^{s_2 - s_1}(q(s_1), q(s_2))$$
$$\cdot \varrho_K(q(s_2))\, dq(s_2) \ldots p_K^{s_N - s_{N-1}}(q(s_{N-1}), q(s_N))\, \varrho_K(q(s_N))\, dq(s_N)$$

where

$$p_K^t(q, q') = \prod_{k \in I_{K,V}} p_k^t(q_k, q_k'). \tag{3.3}$$

These sets B generate the σ-field of measurable subsets of C.

It is easy to show that $H_{I,K,V} \in L_p(Q_{K,V})$ for all $p < \infty$ (see Lemma 4.3), that [2, Lemma 6.1]

$$I_{K,V}(t) = \int_0^t H_{I,K,V}(q(s))\, ds \tag{3.4}$$

belongs to $L_p(C, dQ_{K,V})$ for all $p < \infty$, and that

$$\|I_{K,V}\|_j \leqq t\, \|H_{I,K,V}\|_j$$

where j is a positive integer and $\|\cdot\|_j$ is the L_j norm in $L_j(C, dQ_{K,V})$ or $L_j(Q_{K,V})$ depending on the context.

The Feynman-Kac formula expresses the operator $\exp(-tH_{K,V})$ in terms of an integral over path space (see, for example, [12]):

$$(\Phi, \exp(-tH_{K,V})\,\Psi) = \int \overline{\Phi(q(0))}\, \exp(-I_{K,V}(t))\, \Psi(q(t))\, dQ_{K,V}, \tag{3.5}$$

where Φ, Ψ belong to $L_2(Q_{K,V})$, or equivalently to $\mathscr{F}_{K,V}$.

We shall frequently omit the subscripts on $dQ_{K,V}$ in a path integral like (3.5), and by dQ we mean that the integration takes place over the paths C in some $Q_{K',V'}$ that contains all the Q spaces involved in the integrand. Thus in (3.5), Φ, $I_{K,V}$, or Ψ may be functions only of the variables in smaller Q spaces. When variables appear nowhere in the integrand they are simply integrated out and make no contribution; this last statement is a simple consequence of the relation

$$\int p_k^t(q, q')\, \varrho_k(q')\, dq' = 1.$$

Formula (3.5) may also be written in vector form

$$(\exp(-tH_{K,V})\,\Psi)(q) = \int dQ_q\, e^{-I_{K,V}(t)}\, \Psi(q(t)) \tag{3.6}$$

where dQ_q is the measure induced by dQ on paths starting at q, $q(0) = q$; in terms of dQ, $dQ = \varrho(q)\, dq\, dQ_q$.

§ 4. Properties of the Approximate Hamiltonians

In this section we describe the convergence of $H_{I,K,V}$ as K and $V\to\infty$, and we summarize the properties of $H_{K,V}$ and $H_{I,K,V}$ which are used in the sequel and which were proved in [2].

Lemmas 4.1 and 4.2 provide estimates for the kernels of $H_{I,K,V}$. Lemma 4.4 is essentially Lemma 5.3 of [2] except that an error in the rate of convergence is corrected: the rate is given by the factor $(\ln K)^{2n-1}/K$ of Lemma 4.2, and not by $1/K$. This error produces no essential change in the final conclusions of [2] and these are reproduced in Lemmas 4.6 to 4.8, and Theorem 4.9.

Lemma 4.1. *Let*

$$G(k)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dk_1\ldots dk_{n-1}}{\mu(k_1)\,\mu(k_2-k_1)\ldots\mu(k-k_{n-1})}.$$

Then

$$G(k)=O\left(\frac{(\ln|k|)^n}{|k|}\right)\quad as\quad |k|\to\infty\,.$$

Proof. By a long and direct proof it can be shown that $G(k)$ is actually $O((\ln|k|)^{n-1}/|k|)$, but it is simpler to sacrifice a power of $\ln|k|$ and to use the modified Bessel function

$$K_0(mx)=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ikx}dk}{(m^2+k^2)^{1/2}}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{ikx}dk}{\mu(k)}. \tag{4.1}$$

$K_0(x)$ and $K_0'(x)$ are well-behaved functions at infinity,

$$K_0(x)=e^{-|x|}\left(\frac{\pi}{2|x|}\right)^{1/2}\left[1+O\left(\frac{1}{|x|}\right)\right],$$

$$K_0'(x)=-e^{-|x|}\left(\frac{\pi}{2|x|}\right)^{1/2}\left[1+O\left(\frac{1}{|x|}\right)\right],\quad\text{as}\quad |x|\to\infty\,,$$

while

$$K_0(x)=-\ln\frac{1}{2}|x|+\gamma+o(x)\quad\text{as}\quad |x|\to 0\,. \tag{4.2}$$

Here γ is Euler's constant and the term $o(x)$ represents an absolutely convergent power series.

Since $1/\mu(k)$ is in L_p for all $p>1$, it follows from Young's inequality [20, § 4.2, Lemma γ] that $G(k)$ is in L_2. Thus by taking Fourier transforms we find from (4.1) that

$$G(k)=a\int_{-\infty}^{\infty}e^{-ikx}[K_0(mx)]^n dx\,, \tag{4.3}$$

where a is some constant. We split up the integral into two parts:

$$G(k) = G_1(k) + G_2(k)$$
$$\equiv a \int_{-1}^{1} e^{-ikx} K_0^n(mx)\,dx + a \int_{|x|>1} e^{-ikx} K_0^n(mx)\,dx\,.$$

Since outside of any neighborhood of the origin $K_0^{n-1}(x)\,K_0'(x)$ is absolutely integrable, we can integrate by parts to obtain $G_2(k) = O(1/|k|)$. For G_1 we substitute from (4.2), integrate by parts, and find that

$$G_1(k) = a \int_{-1}^{1} e^{-ikx}\left(\ln\frac{m}{2}|x| + \gamma\right)^n dx + O\left(\frac{1}{|k|}\right).$$

Now

$$\int_0^1 e^{-ikx}(\ln x)^n dx = \frac{1}{k}\int_0^k e^{-iy}(\ln y - \ln k)^n dy \qquad (k>0)$$
$$= \sum_{r=0}^{n} \binom{n}{r} (-\ln k)^{n-r} \frac{1}{k} \int_0^k e^{-iy} (\ln y)^r dy\,.$$

Since $(\ln y)^r$ is integrable at zero,

$$\int_0^k e^{-iy}(\ln y)^r dy = \text{const.} + \int_1^k e^{-iy}(\ln y)^r dy$$
$$= O((\ln k)^r)\,,$$

by integrating by parts. Therefore

$$\int_0^1 e^{-ikx}(\ln x)^n dx = O\left(\frac{(\ln k)^n}{k}\right),$$

$$G_1(k) = O\left(\frac{(\ln |k|)^n}{|k|}\right),$$

and the lemma follows. q.e.d.

Lemma 4.1 implies, in particular, that the kernels appearing in (2.2) are in L_2. It should be mentioned that a weaker form of the next lemma, namely, $F(k,\lambda) = O(\lambda^{-1+\varepsilon})$, is sufficient for the rest of the paper and can be proved with less trouble [17].

Lemma 4.2.

$$F(k,\lambda) \equiv \int_{|k_1|>\lambda} dk_1 \int_{-\infty}^{\infty} dk_2 \ldots \int_{-\infty}^{\infty} dk_{n-1} \frac{1}{\mu(k_1)\,\mu(k_2-k_1)\ldots\mu(k-k_{n-1})}$$
$$= O\left(\frac{\ln^{n-1}\lambda}{\lambda}\right), \quad \textit{uniformly in } k.$$

Proof. Again a direct proof is possible, but by taking Fourier transforms we reduce the lemma to an exercise in advanced calculus.

The proof proceeds as in the previous lemma, except that instead of (4.3) we obtain

$$F(k, \lambda) = \text{const.} \int_{-\infty}^{\infty} e^{-ikx}[K_0(mx)]^{n-1} f(x, \lambda)dx\,, \tag{4.4}$$

where

$$f(x, \lambda) = \int_{|k_1|>\lambda} \frac{dk_1}{\mu(k_1)} e^{ixk_1}.$$

Integrating by parts,

$$f(x, \lambda) = \frac{e^{ixk_1}}{ix} \frac{1}{\mu(k_1)}\bigg]_{\lambda}^{-\lambda} + \int_{|k_1|>\lambda} \frac{k_1}{[\mu(k_1)]^3} \frac{e^{ixk_1}}{ix}.$$

For $|x| \geqq 1$ we can estimate

$$|f(x, \lambda)| \leqq \frac{2}{\mu(\lambda)} + \int_{|k_1|>\lambda} \frac{dk_1}{k_1^2} = O\left(\frac{1}{\lambda}\right), \tag{4.5}$$

uniformly in x. To estimate $f(x, \lambda)$ for $0<x<1$, we expand:

$$\begin{aligned} f(x, \lambda) &= \frac{1}{x} \int_{|y|>x\lambda} \frac{dy}{\mu(y/x)} e^{iy} = \int_{|y|>x\lambda} \frac{e^{iy} dy}{\left(1+\dfrac{m^2x^2}{y^2}\right)^{1/2}} \\ &= \int_{|y|>x\lambda} dy\, e^{iy}\left[\frac{1}{y} - \frac{1}{2}\frac{m^2x^2}{y^3} + \frac{3}{8}\left(\frac{m^2x^2}{y^2}\right)^2 \frac{1}{y} + \cdots\right], \end{aligned}$$

where the series converges uniformly in y when $\lambda > m$. Since for $r \geqq 1$,

$$\int_{x\lambda}^{\infty} \left(\frac{x^2}{y^2}\right)^r \frac{dy}{y} = \frac{1}{2r}\left(\frac{x^2}{y^2}\right)^r\bigg]_{\infty}^{xy} = \frac{1}{2r}\frac{1}{\lambda^{2r}},$$

we find that

$$f(x, \lambda) = \int_{|y|>x\lambda} \frac{e^{iy}}{y} dy + O\left(\frac{1}{\lambda^2}\right), \tag{4.6}$$

uniformly for x in (0, 1).

The same estimate holds for $-1<x<0$ since $f(x, \lambda) = f(|x|, \lambda)$. Finally, the function

$$g(\xi) \equiv \int_{|y|>\xi} \frac{e^{iy}}{y} dy = 2\int_{\xi}^{\infty} \frac{\cos y}{y} dy$$

behaves as $\left(2\ln\frac{1}{\xi}+O(1)\right)$ for small ξ, and $\left[-\frac{2\sin\xi}{\xi}+O\left(\frac{1}{\xi^2}\right)\right]$ for large ξ. From (4.4), (4.5), and (4.6),

$$\begin{aligned}|F(k,\lambda)| &\leqq \text{const.}\left[\int_{|x|>1}|K_0(mx)|^{n-1}O\left(\frac{1}{\lambda}\right)dx\right.\\ &\quad\left.+\left|\int_{|x|<1}e^{-ikx}K_0(mx)^{n-1}g(x\lambda)\,dx\right|\right]+O\left(\frac{1}{\lambda^2}\right)\\ &\leqq \text{const.}\left|\int_{|x|<1}e^{-ikx}K_0(mx)^{n-1}g(x\lambda)\,dx\right|+O\left(\frac{1}{\lambda}\right).\end{aligned}$$

We treat the integral in two parts:

$$\begin{aligned}&\int_{|x|<1}e^{-ikx}K_0(mx)^{n-1}g(x\lambda)\,dx=\frac{1}{\lambda}\int_{|y|<\lambda}e^{-iky/\lambda}K_0\left(\frac{my}{\lambda}\right)^{n-1}g(y)\,dy\\ &=\frac{1}{\lambda}\int_{|y|\leqq 1}e^{-iky/\lambda}K_0(my/\lambda)^{n-1}g(y)\,dy+\frac{1}{\lambda}\int_{1<|y|<\lambda}K_0\left(\frac{my}{\lambda}\right)^{n-1}g(y)e^{-iky/\lambda}\,dy.\end{aligned}$$

From (4.2) and the behavior of $g(y)$ at zero, the first term is dominated by

$$\frac{1}{\lambda}\int_{|y|\leqq 1}\left|-2\ln\left(\frac{|y|}{\lambda}\right)+\text{const.}\right|^{n-1}(-2\ln|y|+\text{const.})\,dy=O\left(\frac{(\ln\lambda)^{n-1}}{\lambda}\right).$$

The second term equals

$$\begin{aligned}&\frac{1}{\lambda}\int_{1<|y|<\lambda}K_0\left(\frac{my}{\lambda}\right)^{n-1}\left[-\frac{2\sin y}{y}+O\left(\frac{1}{y^2}\right)\right]e^{-iky/\lambda}\,dy\\ &\qquad=\frac{1}{\lambda}\int_{1<|y|<\lambda}K_0\left(\frac{my}{\lambda}\right)^{n-1}\left(-\frac{2\sin y}{y}\right)e^{-iky/\lambda}\,dy+O\left(\frac{(\ln\lambda)^{n-1}}{\lambda}\right).\end{aligned}$$

Therefore

$$|F(k,\lambda)|\leqq O\left(\frac{(\ln\lambda)^{n-1}}{\lambda}\right)+\frac{\text{const.}}{\lambda}\left|\int_1^{\lambda}K_0\left(\frac{my}{\lambda}\right)^{n-1}\frac{\sin y}{y}\cos\frac{ky}{\lambda}\,dy\right|.$$

We substitute for K_0 its series and, for simplicity, treat only the worst term, i.e., $\frac{1}{\lambda}\int_1^{\lambda}\left[-\ln\frac{y}{\lambda}\right]^{n-1}\frac{\sin y}{y}\cos\frac{ky}{\lambda}\,dy$. The other terms are obviously dominated by similar arguments. Since

$$\left(-\ln\frac{y}{\lambda}\right)^{n-1}=\sum\binom{n-1}{r}(-\ln y)^r(\ln\lambda)^{n-1-r},$$

it remains to prove that

$$h_r(\lambda, k) \equiv \int_1^\lambda (\ln y)^r \frac{\sin y}{y} \cos\frac{ky}{\lambda}\, dy$$
$$= O((\ln\lambda)^r), \quad \text{uniformly in } k\,.$$

Since $h_r(\lambda, k) = h_r(\lambda, -k)$, we restrict ourselves to $k \geqq 0$. We consider separately the two cases $|k-\lambda| \geqq 1$ and $|k-\lambda| < 1$. For $|k-\lambda| \geqq 1$, we write

$$h_r(\lambda, k) = \frac{1}{2}\,[h_r^+(\lambda, k) + h_r^-(\lambda, k)]\,,$$

where

$$h_r^\pm(\lambda, k) = \int_1^\lambda (\ln y)^r \frac{\sin(y a_\pm)}{y}\, dy \quad \text{and} \quad a_\pm = 1 \pm \frac{k}{\lambda}\,.$$

We integrate by parts in the expression for $h_r^+(\lambda, k)$, using

$$-\int_y^\infty \frac{\sin a\eta}{\eta}\, d\eta = -\int_{ay}^\infty \frac{\sin\eta}{\eta}\, d\eta = \frac{\cos(ay)}{ay} + O\left(\frac{1}{(ay)^2}\right)$$

for $ay > 1$. Thus

$$h_r^+(\lambda, k) = (\ln y)^r \left(\frac{\cos(a_+ y)}{a_+ y} + O\left(\frac{1}{(a_+ y)^2}\right)\right)\Bigg]_1^\lambda - \int_1^\lambda \frac{r(\ln y)^{r-1}}{y}\, O\left(\frac{1}{a_+ y}\right) dy\,.$$

Since $a_+ > 1$, we obtain

$$h_r^+(\lambda, k) = O\left(\frac{(\ln\lambda)^r}{\lambda}\right) + O((\ln\lambda)^{r-1})$$

where we have estimated the integral by const. $\int_1^\lambda (\ln\lambda)^{r-1} \frac{dy}{y^2}$. For h^-, this method works for $k > 2\lambda$, that is, when $|a_-| > 1$; for k in $[0, \lambda - 1] \cup [\lambda + 1, 2\lambda]$, where $\frac{1}{\lambda} \leqq |a_-| \leqq 1$, we argue as follows;

$$h_r^-(\lambda, k) = (\operatorname{sgn} a_-) \int_{|a_-|}^{|a_-|\lambda} \left(\ln\frac{y}{|a_-|}\right)^r \frac{\sin y}{y}\, dy$$
$$= (\operatorname{sgn} a_-) \int_{|a_-|}^{|a_-|\lambda} \sum_{s=0}^r (\ln y)^{r-s} (-\ln|a_-|)^s \binom{r}{s} \frac{\sin y}{y}\, dy\,.$$

Since $\left|\int_0^\infty (\ln y)^{r-s} \frac{\sin y}{y}\, dy\right| < \infty$,

$$|h^-| \leqq \text{const.} \sum_{s=0}^r \left(\ln\frac{1}{|a_-|}\right)^s = O((\ln\lambda)^r)\,.$$

Finally, for $|k-\lambda|<1$, we write

$$\begin{aligned}
|h_r(\lambda,k)| &= \left|\int_{1/\lambda}^{1} (\ln \lambda y)^r \frac{\sin \lambda y}{y} \cos k y \, dy\right| \\
&= \left|\int_{1/\lambda}^{1} (\ln \lambda y)^r \frac{\sin \lambda y}{y} [\cos \lambda y + (\lambda - \mathrm{k}) y \sin \xi] \, dy\right| \\
&\quad \text{(where } \xi \text{ lies between } ky \text{ and } \lambda y) \\
&\leqq \left|\frac{1}{2} \int_{1/\lambda}^{1} (\ln \lambda y)^r \frac{\sin 2\lambda y}{y} \, dy\right| + \int_{1/\lambda}^{1} |\ln \lambda y|^r \, dy \\
&= \frac{1}{2} \left|\int_{1}^{\lambda} (\ln y)^r \frac{\sin 2y}{y} \, dy\right| + O((\ln \lambda)^r) \quad \text{(by the usual argument)} \\
&\leqq \text{const.} + O((\ln \lambda)^r) .
\end{aligned}$$

Thus $h_r(\lambda, k) = O((\ln \lambda)^r)$, uniformly in k. q.e.d.

Lemma 4.3. *If* $W = 2^\nu V$, ν *a positive integer,*

$$\|H_{I,K,V} - H_{I,K,W}\|_{2j}^{2j} \leqq (2nj)! \left(\frac{K_2}{V}\right)^{2j},$$

where j *is a positive integer and* K_2 *a constant independent of* V, W, K.

Proof. The method of proof is exactly that of Lemma 5.3 in [2], to which the reader is referred for full details. We observe that

$$\|H_{I,K,V} - H_{I,K,W}\|_{2j}^{2j} = \|(H_{I,K,V} - H_{I,K,W})^j \Omega_0\|^2$$

and thus calculate in Fock space. From (2.3) ($b_{2n} = 1$),

$$H_{I,K,V} - H_{I,K,W} = \sum_r b_r (:\phi_{K,V}^r(g): - :\phi_{K,W}^r(g):),$$

and from (2.1) and (2.2),

$$\begin{aligned}
:\phi_{K,V}^r(g): - :\phi_{K,W}^r(g): = \sum_{j=0}^{r} \binom{r}{j} \sum_{k_i \in \Gamma_{K,W}} & a_W^*(-k_1) \dots a_W^*(-k_j) \\
& \cdot a_W(k_{j+1}) \dots a_W(k_r) f_r(k_1, \dots k_r)
\end{aligned}$$

where

$$f_r(k_1, \dots k_r) = \left(\frac{2\pi}{W}\right)^{r/2} \left[\frac{\hat{g}(\sum [k_i]_V)}{\prod_i \mu^{1/2}([k_i]_V)} - \frac{\hat{g}(\sum k_i)}{\prod_i \mu^{1/2}(k_i)}\right].$$

Taking into account the numerical factors associated with $a_W^\#$ we estimate

$$\|(H_{I,K,V} - H_{I,K,W})^j \Omega_0\|^2 \leqq K_0 (2nj)! \max_r \|f_r\|_2^{2j} \tag{4.7}$$

where K_0 is a constant independent of K, V, W, j, r, and $\|f_r\|_2$ is the l_2-norm of f_r;

$$\|f_r\|_2 = \left\{\left(\frac{2\pi}{W}\right)^r \sum_{k_i \in \Gamma_{K,W}} \left|\frac{\hat{g}(\sum[k_i]_V)}{\prod \mu^{1/2}([k_i]_V)} - \frac{\hat{g}(\sum k_i)}{\prod \mu^{1/2}(k_i)}\right|^2\right\}^{1/2}$$

$$= \left\{\left(\frac{2\pi}{W}\right)^r \sum_{k_i \in \Gamma_{K,V}} \sum_{\substack{k_i' \in \Gamma_{K,W} \\ [k_i']_V = k_i}} \left|\frac{\hat{g}(\sum k_i)}{\prod \mu^{1/2}(k_i)} - \frac{\hat{g}(\sum k_i')}{\prod \mu^{1/2}(k_i')}\right|^2\right\}^{1/2}$$

$$\leqq \left\{2^{-\nu r}\left(\frac{2\pi}{V}\right)^r \sum_{k_i \in \Gamma_{K,V}} 2^{\nu r}\left(\frac{2\pi r}{V}\right)^2 \sup_{[l_i]_V = k_i} \left|\frac{\partial}{\partial l_1} \frac{\hat{g}(\sum l_i)}{\prod \mu^{1/2}(l_i)}\right|^2\right\}^{1/2}$$

Since $\hat{g} \in C_0^\infty$, $\left|\frac{\partial}{\partial l_1} \mu(l_1)^{-1/2}\right| \leqq \mu(l_1)^{-3/2}$, and by Lemma 4.1, we conclude that $\left[\frac{\partial}{\partial l_1} \frac{\hat{g}(\sum l_i)}{\prod \mu^{1/2}(l_i)}\right] \in L_2$. Therefore the Riemann sums

$$\sum_{k_i \in \Gamma_{K,V}} \left(\frac{2\pi}{V}\right)^r \sup_{[l_i]_V = k_i} \left|\frac{\partial}{\partial l_1} \frac{\hat{g}(\sum l_i)}{\prod \mu^{1/2}(l_i)}\right|^2$$

are bounded functions of V, and it follows that $\|f_r\|_2^2 \leqq K_1 V^{-2}$ where K_1 is a constant independent of K, V, W. Eq. (4.7) then yields the lemma.

Lemma 4.4. *If* $L \leqq K$, $\|H_{I,K,V} - H_{I,L,V}\|_{2j}^{2j} \leqq (2nj)! \left[\frac{K_3 (\ln L)^{2n-1}}{L}\right]^j$ *where j is a positive integer, and K_3 a constant independent of K, L and V.*

Proof. This lemma is proved in the same manner as the previous lemma. From (2.2),

$$:\phi_{K,V}^r(g): - :\phi_{L,V}^r(g): = \sum_{j=0}^{r} \binom{r}{j} \int_R dk_1 \dots dk_r a^*(-k_1) \dots a(k_r) \frac{\hat{g}(\sum[k_i]_V)}{\prod \mu^{1/2}([k_i]_V)},$$

where the integration extends over the region

$$R = \{(k_1 \dots k_r) \mid |k_i| \leqq K \,\forall i, \text{ at least one } |k_i| > L\}.$$

The proof continues as before and relies on the estimate of Lemma 4.2.

Corollary. *For* $p < \infty$, $\|H_{I,K,V} - H_{I,L,W}\|_p \to 0$ *as* $K, L, V, W \to \infty$.

The previous two lemmas lead at once to a better convergence result for $H_{K,V}$.

Lemma 4.5. *For* $\Psi \in D(H_0)$, *we have* $\Psi \in D(H_{0,K,V})$ *for all* K, V *and* $H_{0,K,V}\Psi \underset{s}{\to} H_0\Psi$ *as* $K, V \to \infty$. *For* Ψ *in* $D(N^n)$, *where* N *is the number operator, we have* Ψ *in* $D(H_I)$ *and* $D(H_{I,K,V})$ *for all* K, V, *and*

$$H_{I,K,V}\Psi \underset{s}{\to} H_I\Psi.$$

Proof. The verification of the first statement is trivial and uses the familiar relation $|\mu([k]_V) - \mu(k)| \leqq \text{const.}/V$.

To check the second statement, consider a typical term in $:\phi^{2n}(g):$,

$$W = \int dk_1 \dots dk_{2n} a^*(-k_1) \dots a^*(-k_j)\, a(k_{j+1}) \dots a(k_{2n})\, w(k_1, \dots k_{2n}),$$

$$w(k_1, \dots, k_{2n}) = \hat{g}(\textstyle\sum k_i)\, [\mu(k_1) \dots \mu(k_{2n})]^{-1/2},$$

and its correspondent in $:\phi^{2n}_{K,V}(g):$,

$$W_{K,V} = \int dk_1 \dots dk_{2n} a^*(-k_1) \dots a(k_{2n})\, w_{K,V}(k_1, \dots, k_{2n}),$$

$$\begin{aligned} w_{K,V}(k_1, \dots, k_{2n}) = {} & \chi_K(k_1) \dots \chi_K(k_{2n}) \\ & \cdot \hat{g}(\textstyle\sum [k_i]_V)\, [\mu([k_1]_V) \dots \mu([k_{2n}]_V)]^{-1/2}, \end{aligned}$$

where χ_K is the characteristic function for the interval $[-K, K]$. We use the inequality $\|W(N+1)^{-n}\| \leqq \text{const.}\, \|w\|_2$, to estimate

$$\begin{aligned} \|(W - W_{K,V})\Psi\| &= \|(W - W_{K,V})(N+1)^{-n}(N+1)^n \Psi\| \\ &\leqq \text{const.}\, \|w - w_{K,V}\|_2\, \|(N+1)^n \Psi\|, \end{aligned}$$

and this goes to zero by the proofs of Lemmas 4.3 and 4.4. The other terms in H_I are similarly handled. q.e.d.

Corollary. *$H_{K,V} \xrightarrow[s]{} H$ on $D_n = D(N^n) \cap D(H_0)$.*

The following results, which we list here for convenience, are due to Nelson [16] and Glimm [2].

Lemma 4.6. [2, Lemma 5.1]. *$\exp(-tH_{0,K,V})$ is a contraction operator on $L_r(Q_{K,V})$, $1 \leqq r \leqq \infty$. For $1 < p$ and $r < \infty$, it is a contraction from L_p to L_r for large t, $t \geqq T$ where T depends on p and r, but not on K and V. If p is bounded away from one and r is bounded, then T does not depend on p and r.*

Corollary [2, Lemma 6.2]. *Let $r \in [1, 2)$, Φ and $\Psi \in L_2(Q_{K,V})$. Then there is a T independent of K and V such that if $t \geqq T$, then $\Phi(q(0))\, \Psi(q(t))$ belongs to $L_r(C, dQ_{K,V})$ and $\|\Phi(q(0))\, \Psi(q(t))\|_r \leqq \|\Phi\|_2\, \|\Psi\|_2$.*

Let Pr denote the probability measure on C defined by dQ: $\Pr\{C\} = 1$. Then the lower bound on $H_{I,K,V}$ leads to the following estimates for $I_{K,V}$ (Eq. (3.4)), which are proved in [2, pp. 23–24] except for the correction implied by Lemma 4.4.

Lemma 4.7. $\Pr\{I_{K,V} \leqq -X - 1\} \leqq \exp\left(-K_1 \dfrac{e^{K_2 X^{1/n}}}{X^{\left(\frac{1}{n} - \frac{1}{2n^2}\right)}}\right)$ *where K_1 and K_2 are positive constants that depend on t but are independent of V and K.*

Lemma 4.8. *For each $p < \infty$, $\|\exp(-I_{K,V})\|_p$ is bounded uniformly in V, K.*

From Lemma 4.8 and the Feynman-Kac formula follows the main result of [2]:

Theorem 4.9. *The hamiltonian $H(g)$ and its approximations $H_{K,V}(g)$ are bounded below by a constant depending on g but independent of V and K.*

For convenience we redefine H_I and $H_{I,K,V}$ by adding in this constant ($H_I \to H_I + c$, $H_{I,K,V} \to H_{I,K,V} + c$) so that H and $H_{K,V}$ are positive.

§ 5. Resolvent Convergence of $H_{K,V}(g)$

As positive self-adjoint operators, the $H_{K,V}$ generate strongly continuous contraction semigroups,

$$U_{K,V}(t) = \exp(-tH_{K,V})$$
$$\|U_{K,V}(t)\| \leqq 1 .$$

We show that the $U_{K,V}(t)$ converge uniformly for each t as $K, V \to \infty$. For convenience we are taking the sequence of $\{V\}$ to be

$$\{V_0, V_1 = 2V_0, \ldots V_j = 2^j V_0 \ldots\}$$

for a fixed V_0.

Theorem 5.1. *For each $t \geqq 0$, $U_{K,V}(t)$ converges uniformly as $K, V \to \infty$ to a strongly continuous contraction semigroup $U(t)$.*

Proof. We first prove the theorem for large t. For vectors Φ, Ψ in $\bigcup_{K,V} L_2(Q_{K,V})$, we have by the Feynman-Kac formula (3.5),

$$(\Phi, (U_{K,V} - U_{L,W})\Psi) = \int \overline{\Phi(q(0))}\, \Psi(q(t))\, (e^{-I_{K,V}} - e^{-I_{L,W}})\, dQ\,, \quad (5.1)$$

where $W \geqq V$ and $L \geqq K$. We apply Hölder's inequality with indices 4/3 and 4,

$$\begin{aligned}|(\Phi, [U_{K,V}(t) - U_{L,W}(t)]\Psi)| &\leqq \|\Phi(q(0))\, \Psi(q(t))\|_{4/3}\, \|e^{-I_{K,V}(t)} - e^{-I_{L,W}(t)}\|_4 \\ &\leqq \|\Phi\|_2\, \|\Psi\|_2\, \|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4\end{aligned}$$

for $t \geqq T$ by the corollary to Lemma 4.6.

Since Φ and Ψ of the form $L_2(Q_{K,V})$ are dense in $\mathscr{F}$ and $\|U_{K,V}(t)\| \leqq 1$, we conclude that

$$\|U_{K,V}(t) - U_{L,W}(t)\| \leqq \|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4 .$$

To show the semigroups converge it is sufficient to prove the convergence to zero of the differences

$$\|e^{-I_{K,V}} - e^{-I_{L,V}}\|_4 \quad \text{and} \quad \|e^{-I_{K,V}} - e^{-I_{K,W}}\|_4$$

where one of the two parameters (the volume V or the momentum cutoff K) is held fixed.

We start with $\|e^{-I_{K,V}}-e^{-I_{L,V}}\|_4$. Let $\varepsilon>0$ be a given arbitrarily small number. We wish to truncate $e^{-I_{K,V}}$ and $e^{-I_{L,V}}$ when they become large, so we write, for $I=I_{K,V}$ and $I_{L,V}$,

$$e^{-I}=e^{-I}\theta(I+M)+e^{-I}[1-\theta(I+M)]$$

where $\theta(x)$, the heaviside function, equals 1 for $x>0$, and vanishes for $x\leqq 0$, and M is a large integer to be determined. By the triangle inequality,

$$\begin{aligned}\|e^{-I_{K,V}}-e^{-I_{L,V}}\|_4 &\leqq \|e^{-I_{K,V}}\theta(I_{K,V}+M)-e^{-I_{L,V}}\theta(I_{L,V}+M)\|_4\\ &\quad+\|e^{-I_{K,V}}(1-\theta(I_{K,V}+M))\|_4+\|e^{-I_{L,V}}(1-\theta(I_{L,V}+M))\|_4 .\end{aligned}$$

By Lemma 4.7, each of the last two terms can be dominated by

$$A=\left\{\sum_{j=M}^{\infty} e^{(j+2)4}\exp\left(-K_1 e^{K_2 j^{1/n}}\times j^{\left(\frac{1}{2n^2}-\frac{1}{n}\right)}\right)\right\}^{1/4} .$$

Since this sum is convergent we can fix M so large that A, $\Pr\{I_{K,V}<-M\}$, and $\Pr\{I_{L,V}<-M\}$ are all less than ε. Then

$$\|e^{-I_{K,V}}-e^{-I_{L,V}}\|_4\leqq\|e^{-I_{K,V}}\theta(I_{K,V}+M)-e^{-I_{L,V}}\theta(I_{L,V}+M)\|_4+2\varepsilon .$$

The integration in this last expression is split into the two regions $\{q(s)\mid |I_{K,V}-I_{L,V}|>\varepsilon\}$ and $\{q(s)\mid |I_{K,V}-I_{L,V}|\leqq\varepsilon\}$.

The first region has measure dominated by

$$\begin{aligned}\int\frac{|I_{K,V}-I_{L,V}|^2}{\varepsilon^2}\,dQ &\leqq\frac{1}{\varepsilon^2}\,t\,\|H_{I,K,V}-H_{I,L,V}\|_2^2 \quad [2, \text{Lemma } 6.1]\\ &\leqq\frac{t(2n)!}{\varepsilon^2}\,K_3\,\frac{(\ln K)^{2n-1}}{K} \quad (\text{Lemma } 4.4) .\end{aligned}$$

Thus

$$\begin{aligned}&\left\{\int_{|I_{K,V}-I_{L,V}|>\varepsilon}|e^{-I_{K,V}}\theta(I_{K,V}+M)-e^{-I_{L,V}}\theta(I_{L,V}+M)|^4\,dQ\right\}^{1/4}\\ &\qquad\leqq\left\{(2e^M)^4\,\frac{t(2n)!}{\varepsilon^2}\,K_3\,\frac{(\ln K)^{2n-1}}{K}\right\}^{1/4}\end{aligned}$$

We choose K so large that this is less than ε. Then by Schwartz

$$\begin{aligned}&\|e^{-I_{K,V}}-e^{-I_{L,V}}\|_4\\ &\leqq 3\varepsilon+\left\{\int_{|I_{K,V}-I_{L,V}|\leqq\varepsilon}|e^{-I_{K,V}}\theta(I_{K,V}+M)-e^{-I_{L,V}}\theta(I_{L,V}+M)|^4\,dQ\right\}^{1/4}\\ &\leqq 3\varepsilon+\|e^{-I_{K,V}}\|_8\left\{\int_{|I_{K,V}-I_{L,V}|\leqq\varepsilon}|\theta(I_{K,V}+M)-e^{(I_{K,V}-I_{L,V})}\theta(I_{L,V}+M)|^8\,dQ\right\}^{1/8}\end{aligned}$$

According to Lemma 4.8, $\|e^{-I_{K,V}}\|_8 \leqq K_4$, a constant independent of K, V. We again split the remaining integration into the complementary regions $\{q(s) \mid I_{K,V} > -M$ and $I_{L,V} > -M\}$ and $\{q(s) \mid$ at least one of $I_{K,V}$ and $I_{L,V} < -M\}$. The second region has measure less than 2ε by the choice of M, and in this region the maximum value of the integrand is $e^{8\varepsilon}$. In the first region the integrand is dominated by $(1-e^{\varepsilon})^8$. Therefore

$$\|e^{-I_{K,V}} - e^{-I_{L,V}}\|_4 \leqq 3\varepsilon + K_4[(1-e^{\varepsilon})^8 + 2\varepsilon e^{8\varepsilon}]^{1/8},$$

and

$$\|e^{-I_{K,V}} - e^{-I_{L,V}}\|_4 \to 0 \quad \text{as} \quad K \to \infty .$$

The proof that $\|e^{-I_{K,V}} - e^{-I_{K,W}}\|_4 \to 0$ as $V \to \infty$ is exactly analogous. The only difference, in fact, comes when we estimate $\|H_{I,K,V} - H_{I,K,W}\|_2^2 \leqq (2n)!\,(K_2/V)^2$ by Lemma 4.3. Then for V large enough

$$\left\{ \int_{|I_{K,V} - I_{K,W}| > \varepsilon} |e^{-I_{K,V}} \theta(I_{K,V} + M) - e^{-I_{K,W}} \theta(I_{K,W} + M)|^4 \, dQ \right\}^{1/4} < \varepsilon .$$

This completes the proof that $U_{K,V}(t)$ converges uniformly as $K, V \to \infty$ for $t \geqq T$. For $0 \leqq t < T$ we obtain convergence by observing that $U_{K,V}(t/2) = [U_{K,V}(t)]^{1/2}$, and that if a sequence of uniformly bounded self-adjoint operators converges uniformly then so do their square roots. This last statement follows from the fact that polynomials in the operators converge uniformly, and that on the spectra the square root function can be uniformly approximated by polynomials.

The remaining assertions of the theorem are trivial to verify. Let $U(t)$ be the limiting operator, $U(t) = \lim_{K,V \to \infty} U_{K,V}(t)$. Then the uniform convergence implies that

(i) $\|U(t)\| \leqq 1$ since $\|U_{K,V}(t)\| \leqq 1$;

(ii) $U(t)\,U(s) = U(t+s)$ since $U_{K,V}(t)\,U_{K,V}(s) = U_{K,V}(t+s)$; and

(iii) $U(t)\Psi$ is strongly continuous in t since $U_{K,V}(t)\Psi$ is, and, as a check of the above estimates shows, $U_{K,V}(t) \xrightarrow{s} U(t)$ uniformly in t in any finite time interval. For intervals about $t = 0$ we can directly show strong convergence uniformly in t, by choosing Ψ in the dense set $\bigcup_{K,V} L_4(Q_{K,V})$ and noting that from (5.1)

$$\begin{aligned} |(\Phi, (U_{K,V} - U_{L,W})\Psi)| &\leqq \|\Phi(q(0))\|_2 \, \|\Psi(q(t))\|_4 \, \|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4 \\ &= \|\Phi\|_2 \, \|e^{-tH_{0,K,V}} \Psi\|_4 \, \|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4 \\ &\leqq \|\Phi\|_2 \, \|\Psi\|_4 \, \|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4, \quad \text{by Lemma 4.6.} \end{aligned}$$

q.e.d.

We make use of the notions of graph and resolvent convergence [4].

Definition. Graph or G-convergence.

The operators C_n converge strongly to C_∞ in the sense of graphs ($C_n \underset{G}{\to} C$) if

$$G_\infty = \{(\theta, \psi) \mid \theta = s\text{-}\lim_n \theta_n, \theta_n \in D(C_n) \text{ and } \psi = s\text{-}\lim_n C_n \theta_n\}$$

is the graph of a densely defined operator C_∞.

Definition. Resolvent or R-convergence.

$$C_n \underset{R}{\to} C \quad \text{if the resolvents} \quad R_n(z) = (C_n - z)^{-1}$$

converge strongly to an operator $R(z)$ which has a densely defined inverse.

The properties of R- and G-convergence that we require are summarized in the following lemma.

Lemma 5.2. *Let C_n be a sequence of self-adjoint operators.*

(i) *If $C_n \underset{G}{\to} C_\infty$, then C_∞ is maximal symmetric.*

(ii) *If $C_n \underset{R}{\to} C_\infty$, then $C_n \underset{G}{\to} C_\infty$.*

(iii) *If $C_n \underset{s}{\to} C$ on a dense set D, then the sequence has a graph limit, $C_n \underset{G}{\to} C_\infty$, and C_∞ is a symmetric extension of C.*

(iv) *If $C_n \underset{s}{\to} C$ on a dense set D, and if $C_n \underset{R}{\to} C_\infty$, then C_∞ is a maximal symmetric extension of C.*

(v) *Let C be self-adjoint with core D, that is, $\overline{(C|_D)} = C$. If $C_n \underset{s}{\to} C$ on D, then $C_n \underset{R}{\to} C$.*

Proof. Statements (i), (ii), and (iii) are proved in [4, § 3]. (iv) follows from (ii) and (iii). (v) is stated as Corollary 1.6 in [14, p. 429]; (v) implies via (ii) that if $\theta_n \to \theta \in D(C)$, and $C_n \theta_n$ converges, then $C_n \theta_n \to C\theta$.

Let T be the generator of the semigroup $U(t)$ of Theorem 5.1 (see [11] for the proofs of the properties of T). Then T is obtained from $U(t)$ by the Laplace formula

$$(T-z)^{-1} = \int_0^\infty e^{zt} U(t) dt, \qquad \operatorname{Re} z < 0 .$$

T is a densely defined closed operator with the negative real axis in its resolvent set.

Theorem 5.3. *$T =$ uniform $\underset{K,V\to\infty}{R\text{-}\lim} H_{K,V}$ and T is a positive self-adjoint extension of $H|_{D_n}$ where $D_n = D(N^n) \cap D(H_0)$.*

Proof. The first statement follows at once from Theorem 5.1 and the Lebesgue dominated convergence theorem applied to the equation,

$$(T-z)^{-1} - (H_{K,V} - z)^{-1} = \int_0^\infty e^{zt} [U(t) - U_{K,V}(t)] dt, \qquad \operatorname{Re} z < 0 .$$

That T is symmetric is readily seen from the defining equation

$$(T-x)^{-1}=\int_0^\infty e^{xt}U(t)dt$$

and the self-adjointness of $U(t)$. As a closed symmetric operator with the negative real axis in its resolvent set, T must be self-adjoint [14, p. 271].

Since $H_{K,V}\underset{s}{\rightarrow}H$ on D_n (Lemma 4.5, Corollary), we conclude that $H|D_n\subset T$ (Lemma 5.2, (iv)). q.e.d.

§ 6. Essential Self-adjointness of $H(g)$

In this section we prove that, in fact,

$$T=\overline{(H|_{\mathscr{D}})}$$

where $\mathscr{D}=D(H_0)\cap D(H_I)$, so that H is essentially self-adjoint on its natural domain of definition $\mathscr{D}$. The idea of the proof is to establish the existence of a core for T that is contained in $\mathscr{D}$.

Lemma 6.1. *Let* $\mathscr{S}=\bigcup_{L,W}L_\infty(Q_{L,W})$. *For all* $t\geqq 0$ *and for* $\operatorname{Re}z$ *sufficiently negative,* $H_{I,K,V}U_{K,V}(t)$, $H_{I,K,V}R_{K,V}(z)$, *and* $H_{0,K,V}R_{K,V}(z)$ *all converge strongly on* $\mathscr{S}$ *as* $K, V\to\infty$.

Remark. It is not much more difficult to prove the lemma on the set $\mathscr{S}_p=\bigcup_{L,W}L_p(Q_{L,W})$ where $p>2$.

Proof. We show that $H_{I,K,V}U_{K,V}$ converges much as in Theorem 5.1 by using path space integrals, and the convergence of $H_{I,K,V}R_{K,V}$ and $H_{0,K,V}R_{K,V}$ follows readily. By the Feynman-Kac formula (3.6),

$$(U_{K,V}(t)\Psi)(q)=\int dQ_q e^{-I_{K,V}(t)}\Psi(q(t))$$

where we take $\Psi\in L_\infty(Q_{L',W'})$ for some L', W'. Let

$$\Phi=[H_{I,K,V}U_{K,V}(t)-H_{I,L,W}U_{L,W}(t)]\Psi,$$

where $K\leqq L$, $V\leqq W$. We show that $\|\Phi\|\to 0$ as $K, V\to\infty$. Now,

$$\begin{aligned}\Phi(q)=[H_{I,K,V}(q)-H_{I,L,W}(q)]\int dQ_q e^{-I_{K,V}}\Psi(q(t))\\+H_{I,L,W}(q)\int dQ_q(e^{-I_{K,V}}-e^{-I_{L,W}})\Psi(q(t)).\end{aligned}$$

By the triangle and Schwartz inequalities,

$$\begin{aligned}\|\Phi(q)\|\leqq \|H_{I,K,V}-H_{I,L,W}\|_4\,\|\Psi\|_\infty\,[\int\varrho(q)\,dq(\int dQ_q e^{-I_{K,V}})^4]^{1/4}\\+\|H_{I,L,W}\|_4\,\|\Psi\|_\infty\,[\int\varrho(q)dq\,|\int dQ_q(e^{-I_{K,V}}-e^{-I_{L,W}})|^4]^{1/4}.\end{aligned}\tag{6.1}$$

Since $\int dQ_q = 1$ we have by Hölder's inequality that

$$\int dQ_q |f(q(s))| \leqq [\int dQ_q |f|^p]^{1/p} \tag{6.2}$$

for $p > 1$. Therefore,

$$[\int \varrho(q)\, dq (\int dQ_q e^{-I_{K,V}})^4]^{1/4} \leqq [\int \varrho(q)\, dq \int dQ_q e^{-4I_{K,V}}]^{1/4} = \|e^{-I_{K,V}}\|_4 .$$

A similar estimate for the second term in (6.1) yields

$$\begin{aligned} \|\Phi\| \leqq \|\Psi\|_\infty [&\|H_{I,K,V} - H_{I,L,W}\|_4 \, \|e^{-I_{K,V}}\|_4 \\ &+ \|H_{I,L,W}\|_4 \, \|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4] . \end{aligned} \tag{6.3}$$

By Lemmas 4.3 and 4.7, $\|H_{I,L,W}\|_4 = \|H^2_{I,L,W} \Omega_0\|^{1/2}$ and $\|e^{-I_{K,V}}\|_4$ are bounded independently of K and V; by Lemma 4.4 (Corollary), and the proof of Theorem 5.1, $\|H_{I,K,V} - H_{I,L,W}\|_4$ and $\|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4$ go to zero as $K, V \to \infty$. We conclude from (6.3) that $\|\Phi\| \to 0$; that is, $H_{I,K,V} U_{K,V}(t)$ converges strongly on $\mathscr{S}$ as $K, V \to \infty$.

By the Laplace formula,

$$\|[H_{I,K,V} R_{K,V}(z) - H_{I,L,W} R_{L,W}(z)] \Psi\| = \|\int e^{zt} \Phi \, dt\| \leqq \int e^{xt} \|\Phi\| \, dt , \tag{6.4}$$

where $x = \operatorname{Re} z$.

For each t, the integrand in (6.4) approaches zero as $K, V \to \infty$ by the previous argument. It remains to bound the integrand by an integrable function of t. Now,

$$\int dQ_q e^{-4I_{K,V}(t)} = (e^{-tH^{(4)}_{K,V}} \Omega_0)(q)$$

where $H^{(4)}_{K,V} = H_{0,K,V} + 4H_{I,K,V}$.

Theorem 4.9 applies to $H^{(4)}_{K,V}$ as well as to $H_{K,V}$; for some constant d independent of K and V, $H^{(4)}_{K,V} \geqq -d$. Thus,

$$\begin{aligned} \|e^{-I_{K,V}}\|_4 &= [\int \varrho(q)\, dq \int dQ_q e^{-4I_{K,V}}]^{1/4} \\ &= (\Omega_0, e^{-tH^{(4)}_{K,V}} \Omega_0)^{1/4} \\ &\leqq e^{dt/4} . \end{aligned}$$

Likewise,

$$\begin{aligned} \|e^{-I_{K,V}} - e^{-I_{L,W}}\|_4 &\leqq \|e^{-I_{K,V}}\|_4 + \|e^{-I_{L,W}}\|_4 \\ &\leqq 2 e^{dt/4} . \end{aligned}$$

Thus by (6.3) the integrand in (6.4) is bounded by const. $\times e^{(x+d/4)t}$; we choose x sufficiently negative and conclude by the Lebesgue dominated convergence theorem that $H_{I,K,V} R_{K,V}(z)$ converges strongly on $\mathscr{S}$.

Finally, since $R_{K,V}(z)$ converges (Theorem 5.3),

$$H_{0,K,V} R_{K,V}(z) = 1 + z R_{K,V}(z) - H_{I,K,V} R_{K,V}(z) .$$

also converges strongly on $\mathscr{S}$. q.e.d.

The resolvent convergence of $H_{K,V}$ required a substantial proof (§ 5) and leads to the self-adjointness of H (Theorem 6.3); on the other hand, the resolvent convergence of $H_{0,K,V}$ and $H_{I,K,V}$ and the value of the limiting operator follow readily from the *known* self-adjointness of H_0 and H_I.

Lemma 6.2. *As $K, V\to\infty$,*

$$H_0 = R\text{-lim}\, H_{0,K,V} = G\text{-lim}\, H_{0,K,V}$$

and

$$H_I = R\text{-lim}\, H_{I,K,V} = G\text{-lim}\, H_{I,K,V}\,.$$

Proof. In [5, § III] it was proved that for a ϕ^4 theory, H_I is essentially self-adjoint on $D_0 = \bigcap_n D(H_0^n)$. There is nothing in the proof peculiar to ϕ^4; all that is used is that H_I is a Wick polynomial in ϕ. Thus for $P(\phi)$, H_I is essentially self-adjoint on D_0. The same conclusion holds for the approximate interactions $H_{I,K,V}$.

According to Lemma 4.5, $H_{I,K,V} \underset{s}{\to} H_I$ on D_n and hence on $D_0 \subset D_n$ as $K, V\to\infty$. Therefore by (v) of Lemma 5.2, $H_I = R\text{-lim}\, H_{I,K,V}$, and by (ii) of the same lemma, $H_I = G\text{-lim}\, H_{I,K,V}$.

Similarly, H_0 is essentially self-adjoint on D_0 (its set of analytic vectors is contained in its set of C^∞ vectors) and by Lemma 4.5, $H_{0,K,V} \underset{s}{\to} H_0$ on D_0. Thus,

$$H_0 = R\text{-lim}\, H_{0,K,V} = G\text{-lim}\, H_{0,K,V} \qquad \text{q.e.d.}$$

We can now prove that H is essentially self-adjoint on $\mathscr{D}$. For ϕ^4 this result can be strengthened to essential self-adjointness on D_0 [5, § IV], but this has not been proved for ϕ^{2n}.

Theorem 6.3. $T = \overline{H|_{\mathscr{D}}}$; *that is, H is essentially self-adjoint on $\mathscr{D} = D(H_0)\cap D(H_I)$.*

Proof. $\mathscr{S}$ (technically, its identification in $\mathscr{F}$) is dense in $\mathscr{F}$ since $L_\infty(Q_{L,W})$ is dense in $L_2(Q_{L,W})$ and the subspaces $\mathscr{F}_{L,W}$ are dense in $\mathscr{F}$. Therefore $\mathscr{C} = R(z)\mathscr{S}$ is a core for T [14, p. 166].

We choose z to give convergence in Lemma 6.1. Let Ψ be an arbitrary vector in $\mathscr{C}$; $\Psi = R(z)\Phi$ where $\Phi\in\mathscr{S}$. By Theorem 5.3

$$\Psi_{K,V} = R_{K,V}(z)\Phi \to \Psi\,,$$

and by Lemma 6.1, $H_{I,K,V}\Psi_{K,V}$ and $H_{0,K,V}\Psi_{K,V}$ both converge. Therefore by Lemma 6.2, $\Psi\in D(H_0)\cap D(H_I) = \mathscr{D}$, and we obtain by adding that

$$(H_{0,K,V} + H_{I,K,V})\Psi_{K,V} \to (H_0 + H_I)\Psi\,.$$

On the other hand, $H_{K,V}$ R-converges to T and so by Lemma 5.2, (ii),

$$(H_{0,K,V} + H_{I,K,V})\Psi_{K,V} \to T\Psi\,.$$

Therefore $T = H_0 + H_I$ on $\mathscr{C} \subset \mathscr{D}$. Taking closures, we find that

$$T = \overline{T|_{\mathscr{C}}} = \overline{H|_{\mathscr{C}}} \subset \overline{H|_{\mathscr{D}}}$$

and obtain a symmetric extension of a self-adjoint operator. Thus $T = \overline{H|_{\mathscr{D}}}$. q.e.d.

As a consequence of Theorems 5.3 and 6.3 we can draw a number of conclusions regarding the spectrum of $H(g)$. Let E_g be the infimum of the spectrum of $H(g) = H_0 + H_I(g)$, where now by $H_I(g)$ we mean the original interaction hamiltonian which has not been redefined by the addition of a constant (see Theorem 4.9). It is known that $E_g \to -\infty$ as $g \to 1$ [8]. Since the arguments of Glimm and Jaffe [6, § 2] extend to the $P(\phi)$ model, we conclude that $H(g)$ has a unique (up to phase) vacuum with a gap above the ground state energy. We state this more completely:

Theorem 6.4. *$H(g)$ has compact spectrum on $[E_g, m)$ and E_g is a simple eigenvalue.*

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L. Rosen
Courant Institute of
Mathematical Sciences
New York University
New York, N.Y. 10012, USA