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The Monodromy Rings of a Class of Self-Energy Graphs

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Abstract. The monodromy rings of self-energy graphs, with two vertices and an arbitrary number of connecting lines, are determined.

§ 1. Introduction

This paper is the first of a series of publications in which we hope to elucidate in a systematic way the properties of Feynman integrals. The motivation for this work is clear: we hope to develop sufficiently the methods of investigating functions of several complex variables defined by integrals to give a basis for the determination of the analytic structure of the S-matrix itself. This is admittedly not an easy task and one whose outcome we cannot guarantee. An ideal research program should be carried out in three steps:

I) The individual contributions of each perturbation order should be separately investigated. These are functions of the Nilsson class[1] and therefore their analytic structure admits a simple qualitative description – to each function corresponds a certain group, the fundamental group of its domain, and a finite dimensional representation of this group by linear transformations of the vector space spanned by the determinations of the function in the neighbourhood of a nonsingular point. This representation may be extended to a representation of the group ring of the fundamental group which we term the monodromy ring of the function[2]. These rings are to be explicitly determined.

This point of the program is well under way and has been completed for the single loop Feynman relativistic amplitudes (FRA) and for the class of self-energy FRA considered in the present paper.

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[1] See § 3.3.

[2] These concepts are developed in detail in § 2.

It is important to know to what extent a function is defined by its monodromy ring. This question was posed already by Riemann who initiated the qualitative approach to the theory of functions. For functions of a single complex variable Riemann's problem has been given a complete and beautiful solution [1]:

Given a representation of the group ring of the fundamental group of the complex plane minus a finite set of points, which is finite dimensional and has a cyclic vector, there exist n functions having these points as branch points such that

a) These functions have the given representation as their monodromy ring.

b) They are linearly independent over the field of rational functions.

c) Any function having the given representation as its monodromy ring is a linear combination of the n basic functions with rational functions as coefficients.

We expect that this theorem remains valid also for functions of several complex variables.

II) The second step is heuristic.

The aim is to determine the simplest possible analytic structure for the S-matrix which would be consistent with known physical principles. For an account of the work which has been done in this direction we refer to [2]. We remark only that the unitarity condition dictates that the S-matrix must have a very complicated structure. The close correspondence between unitarity integrals and Feynman integrals indicates that the S-matrix has an analytic structure which is closely related to that of the terms in the perturbation series. We expect that the techniques developed in I) will be essential to obtain the full implication of the unitarity condition and so to give a precise statement of the analytic structure of the S-matrix.

III) The problem of constructing an S-matrix which has the structure suggested by II) must be solved. This problem can be regarded as a far-reaching generalization of the problem of Riemann mentioned above.

Even in the case in which a complete solution of I) is obtained, a complete solution of II), III) would almost certainly remain out of the question. However, a partial solution – which would be in effect an extension of the original calculational scheme of Mandelstam to deal with many particle processes – would be of great interest.

We conclude this introduction with some remarks on the technical ideas introduced in this paper.

a) In contrast with previous work aimed at the determination of the monodromy rings of Feynman integrals [3, 4], we are able to avoid the use of homology theory. It was remarked already by Pham [4] that a knowledge of the fundamental group gives considerable information about the Kronecker indices which must be determined in the homological method. We have found that taken together with the information which is obtained by a purely *local* analysis in the integration space, the fundamental group determines completely the monodromy ring. We have no proof that this is the case for an arbitrary graph but it is not necessary to know in advance that the method will be successful in order to apply it.

b) We consider all the parameters which enter into the FRA as independent complex variables. In this way we exploit the fact that the FRA may be analytically continued in all these variables.

c) We use a device introduced by one of us (E. S.) in dealing with renormalization theory whereby the ordinary Feynman propagators $(p^2 - m^2 + i\varepsilon)^{-1}$ are replaced by $(p^2 - m^2 + i\varepsilon)^{-\lambda}$, λ complex. In this way we achieve a double goal.

i) We are free of worries about divergent FRA. Renormalized FRA can be obtained by the Speer method, and their analytic structure determined.

ii) The λ variables – we introduce a different λ for each line – serve to label the lines. In this way we can see clearly the influence of each line in the whole structure. In particular we are able to state and prove a prove a theorem which describes the effect of cutting one line of the graph. If this theorem can be proved for an arbitrary graph it could play a key role in the construction of the monodromy rings of complicated graphs. For it would then be sufficient to construct the monodromy rings of the complete graphs on an arbitrary number of vertices. (The monodromy ring of a graph with multiple lines can be computed from the monodromy ring of the corresponding graph without multiple lines and the monodromy rings determined in the present paper.)

1.2. The Fundamental Group – General Theorems

For the definition of the fundamental group of a topological space we refer to any standard text on topology, for example [5]. In this section we cite a number of theorems on the fundamental group $\pi_1(\mathbb{P}^m - L; B)$ of the complement in a projective space $\mathbb{P}^m$ over the field $\mathbb{C}$ of complex numbers of an algebraic variety L of complex dimension $m-1$. B denotes the base point for the loops defining π_1.

Definition 1.2.1. A line $\ell \subset \mathbb{P}^m$ *is generic with respect to* L if ℓ intersects L in a finite set of points equal in number to the degree d of L.

Theorem 1.2.2 (Picard-Severi). $\pi_1(\mathbb{P}^m - L; B)$ *is generated by the elements* $\alpha_1, \ldots, \alpha_d$ *defined by elementary loops in* ℓ *around the points of intersection of a generic line* ℓ *through* B *with* L *(Fig. 1).*

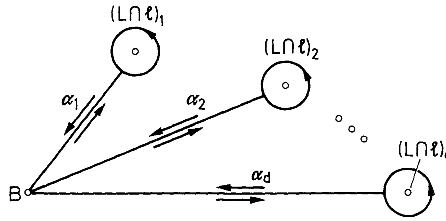

Fig. 1. The complex line ℓ

Proof. See e.g. Pham [6].

Note that $\pi_1(\ell - L; B)$ is the quotient of the free group F_d on $\alpha_1, \ldots, \alpha_d$ by the normal subgroup generated by $\alpha_1 \ldots \alpha_d$.

Definition 1.2.3 (Zariski). If $O \in \mathbb{P}^m - L$, and $N \cong \mathbb{P}^{m-1}$ is a linear subspace of $\mathbb{P}^m$ not containing O, the *branch variety V of L relative to O* is the intersection with N of the cone of singular (i.e. non-generic) lines on O with respect to L. A plane $\pi \cong \mathbb{P}^2 \subset \mathbb{P}^m$ *is generic with respect to L* if there is some point $O \in \pi$ such that if N is a linear subspace of $\mathbb{P}^m$ of dimension $m-1$ not containing O, the line $\pi \cap N$ is generic with respect to the branch variety V of L relative to O.

Theorem 1.2.4 (Zariski [7]). *If π is generic with respect to L the natural injection*

$$i: \pi_1(\pi - L; B) \to \pi_1(\mathbb{P}^m - L; B)$$

(which by 1.2.2 *is onto) is an isomorphism.*

Combining Zariski's theorem with the Picard-Severi theorem we see that $\pi_1(\mathbb{P}^m - L; B)$ is a finitely generated group with generators $\alpha_1, \ldots, \alpha_d$ and that all the relations on these generators may be found by considering homotopies within a generic plane π on B.

Let π be generic for L. We may choose the base point B to be the point O of Zariski's definition and the generic line ℓ used to construct generators for $\pi_1(\mathbb{P}^m - L; B)$ to be the line BQ_0, where $Q_0 \in \pi \cap N$ is any point of $\pi \cap N$ other than the s points $Q_1 \ldots Q_s$ in which $\pi \cap N$ intersects the branch variety V of L relative to B. Denote by u a complex variable parametrising $\pi \cap N$ so that Q_i corresponds to $u = u_i$ $0 \leqq i \leqq s$. For $u \neq u_1, \ldots u_s$, the line $BQ(u)$ intersects L in d points $P_1(u), \ldots, P_d(u)$. Let γ be a line segment in the u-plane connecting points R, S. We write $\gamma = \{\gamma(t)\}$ so $R = \gamma(0)$, $S = \gamma(1)$. Then the motion of the points $P_1(\gamma(0)) \ldots P_d(\gamma(0))$ of intersection of BR with L into the points $P_1(\gamma(1)) \ldots P_d(\gamma(1))$ of $BS \cap L$ defined by continuously varying t from 0 to 1 may be extended to a motion which carries an arbitrary point of the line BR into a corresponding point of BS[3]. The motion may be constructed so that B remains fixed. It therefore defines an isomorphism

$$\pi_1(BR-L;B)\to\pi_1(BS-L;B).$$

With the help of this construction we may assign to each piecewise linear loop β on Q_0 in $\pi\cap N$ an automorphism

$$h(\beta):\pi_1(\ell-L;B)\to\pi_1(\ell-L;B).$$

This automorphism depends only on the homotopy class of the loop β in $\pi_1(\pi\cap N-V;Q_0)$. For any $\alpha\in\pi_1(\ell-L;B)$ and $\beta\in\pi_1(\pi\cap N-V;Q_0)$ we clearly have

$$i(\alpha)=i(h(\beta)\alpha)$$

where i denotes the injection $i:\pi_1(\ell-L;B)\to\pi_1(\mathbb{P}^m-L;B)$.

Theorem 1.2.5 (van Kampen [9]). *The kernel of the injection* i

$$i:\pi_1(\ell-L;B)\to\pi_1(\mathbb{P}^m-L;B)$$

is generated by the elements $\alpha[h(\beta)\alpha]^{-1}$ *where*

$$\alpha\in\pi_1(\ell-L;B);\qquad \beta\in\pi_1(\pi\cap N-V;Q_0)$$

are arbitrary.

$\pi_1(\pi\cap N-V;Q_0)$ is generated by elements $\beta_1\ldots\beta_s$ defined by elementary loops around $Q_1\ldots Q_s$. We may therefore combine Theorem 1.2.5 with the remark following 1.2.2 to give:

Proposition 1.2.6. $\pi_1(\mathbb{P}^m-L;B)$ *is a finitely presented group – the quotient of the free group* F_d *on generators* $\alpha_1,\ldots\alpha_d$ *by the normal subgroup generated by*

$$\alpha_1\ldots\alpha_d,$$
$$\alpha_i[h(\beta_j)\alpha_i]^{-1};\qquad 1\leqq i\leqq d,\qquad 1\leqq j\leqq s.$$

[In Proposition 1.2.6 $h(\beta_j)\alpha_i$ as an element of F_d is not uniquely defined but is understood to be any element of F_d which maps onto $h(\beta_j)\alpha_i$ considered as an element of $\pi_1(\ell-L;B)$.]

We refer to the relations $\alpha_i[h(\beta_j)\alpha_i]^{-1}=1$, $1\leqq i\leqq d$ as the van Kampen relations for the branch point. If Q_j is the intersection with $\pi\cap N$ of the line BP joining B to a singular point P of L we may also refer to these relations as the van Kampen relations for the singular point P.

[3] The practised reader will recognize this as an ambiant isotopy [8].

Remark. The ambiant isotopy construction may be used to assign uniquely to each $\beta \in \pi_1(\pi \cap N - V; Q_0)$ an automorphism $h'(\beta)$ of the free group F_d by identifying this group with $\pi_1(\ell - L \cup B)$. Then in 1.2.6 we could write $\alpha_i[h'(\beta_j)\alpha_i]^{-1}$. It is interesting to note that the automorphisms $h'(\beta)$ map each generator α_i into a conjugate of some other generator α_j and map $\alpha_1 \ldots \alpha_d$ into itself. They therefore define elements of the braid group B_d on d braids [10].

Fig. 2. The anticlockwise convention

In the application of the above results to the study of Feynman integrals we are concerned with a variety L defined by a polynomial whose coefficients are real. We denote by $L_r \subset \mathbb{P}^m(R)$ the real part of L, $L_r = \mathbb{P}^m(R) \cap L$. As a set of real points L_r may have components of varying topological dimension. However, the set $L_r - S = M$ (S the set of singular points of L), which in our case will always be nonempty is a manifold of real dimension $m-1$. We denote by M_i $1 \leqq i \leqq c$ its connected components. Choose a real base point B. With each point $P \in M$ we associate the element $\alpha(P) \in \pi_1(\mathbb{P}^m - L; B)$ defined by an elementary loop in the complex line BP which follows the real line interval $\overline{BP}$ except for small anticlockwise detours to avoid the intersections of BP with L interior to $\overline{BP}$ and circles L anticlockwise at P (Fig. 2). Let X be a connected subset of M. We say that X is *good* with respect to B if any two points P, P' in X may be connected by a path $\gamma = \{\gamma(t)\}$ in M such that as t runs from 0 to 1 no complex intersection of $BP = B\gamma(0)$ crosses the interval $B\gamma(t)$. We then have $\alpha(P) = \alpha(P')$, i.e. we may define $\alpha(X) = \alpha(P)$ for any $P \in X$. An optimal choice of base point B is clearly one such that each component M_i $1 \leqq i \leqq c$ is good with respect to B. For the Landau variety of a single loop diagram such a base point exists [11]. For the multi-loop graphs studied in the present paper the Landau varieties do not admit such a choice of base point. However, we have been able to choose B so that a large number of the components M_i are good with respect to B. This choice is successful in the sense that the corresponding generators $\alpha(M_i)$ generate $\pi_1(\mathbb{P}^m - L; B)$ and such that the relations on these generators obtained by writing down van Kampen relations for certain real branch points completely define the group. Note that the completeness of the set of elements $\{\alpha(M_i)\}$ is essential for our purpose, but that the completeness of the set of relations which we write down is not essential. It is sufficient to have sufficient relations to reconstruct uniquely the representation $\mathscr{L}$ of the fundamental group.

A Landau variety L is a reducible algebraic variety. The singular points of a generic plane section π of L are therefore expected to be of the following types

(i) transverse intersection,
(ii) tacnode,
(iii) cusp.

We are interested particularly in *real* singular points (since we try to avoid having to write down van Kampen relations for complex singular points). As a singular point of the real section a transverse intersection or node can appear either as a transverse intersection of two branches of L_r (crunode) or as an isolated realpoint (acnode). Since we wish to consider only van Kampen relations which can be written down as relations on elements of the fundamental group defined by elementary loops around points of L_r we do not consider the acnode case. This gives us the three cases illustrated in Figs. 3, 4, 5. In the neighbourhood of the singular point P local coordinates u_1, u_2 may be chosen so that L has local equation

(i) $(u_1 - u_2)(u_1 + u_2) = 0$,
(ii) $(u_1^2 - u_2)u_2 = 0$,
(iii) $u_2^2 - u_1^3 = 0$.

We choose a base point B such that P is the only singular point of L on BP and such that BP does not touch L, i.e. we choose B to be a point O satisfying the conditions of 1.2.3. We also choose B so that in cases (ii), (iii) B stands in the relative position to the real section of L in the neighbourhood of P indicated in Figs. 4 and 5. Then if U is a sufficiently small neighbourhood of P the connected components K_i of $U \cap (L_r - P)$ are good with respect to B. We denote by $\alpha_i = \alpha(K_i)$ the corresponding elements of the fundamental group constructed by the anticlockwise convention, the labelling being carried out as shown in the figures. We will work out in detail the relations between these generators for the tacnode. Choose as the generic line ℓ the line BP' for some $P' \in K_1$, and let $\pi \cap N$ be a line defined as in the general discussion preceding the statement of the van Kampen theorem (Fig. 6). Let $\pi \cap N$ intersect BP, $BP' = \ell$ in Q, Q_0. The construction of the van Kampen theorem now gives us two kinds of relations

(a) identifications obtained by taking for the path γ in $\pi \cap N$ a path from $Q_0 = R$ to a point S on the opposite side of Q circling Q anticlockwise. By following the motion of the intersections of $B\gamma(t)$ with L as t traces this path we can express the generators defined by BS in terms of those defined by ℓ. For the remaining $d-2$ generators this results in trivial identifications but we do obtain two non-trivial relations expressing α_3, α_4 in terms of α_1, α_2;

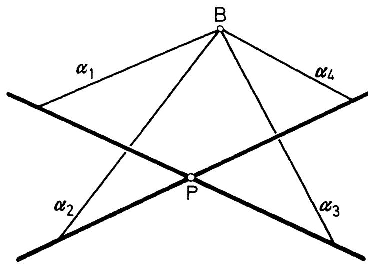

Fig. 3. Crunode

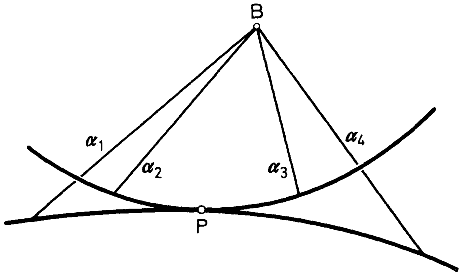

Fig. 4. Tacnode*

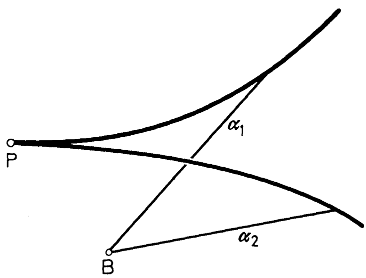

Fig. 5. Cusp

(b) the van Kampen relations for P obtained by taking a loop β around Q in $\pi \cap N$.

The task of following the motion of the loops in $B\gamma(t)$ obtained by deformation of α_1, α_2 may be reduced to successive applications of

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* The labels α_1 and α_2 in Fig. 4 should be interchanged.

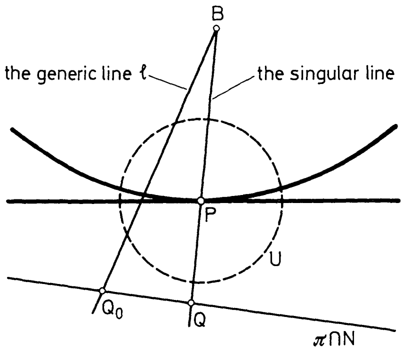


Fig. 6. The neighbourhood of a tacnode

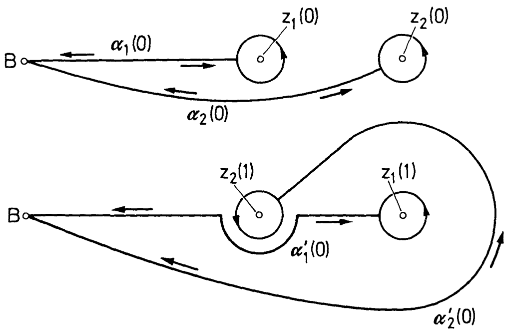


Fig. 7. z-plane

Lemma 1.2.7 (Fig. 7). *Let* $z_1(u)$, $z_2(u)$ *be two points of the complex* z*-plane depending continuously on a parameter* u*. Suppose that for* $u=1,0$ *the points are real and satisfy*

$$z_1(0)<z_2(0), \qquad z_2(1)<z_1(1).$$

Suppose also that $\mathrm{Im}(z_2(u)-z_1(u))$ *is positive for any* u, $0<u<1$ *(so that the points circle just once). Choose a base point* B *far away on the negative real axis and denote by* $\alpha_1(0)$, $\alpha_2(0)$; $\alpha_1(1)$, $\alpha_2(1)$ *elements of* $\pi_1(\mathbb{C}^2-\{z_1\}\cup\{z_2\})$ *defined by loops around* z_1, z_2 *constructed according to the anti-clockwise convention. Then as* u *varies from* 0 *to* 1 $\alpha_1(0), \alpha_2(0)$ *are carried into*

$$\alpha_1'(0)=\alpha_1(1), \tag{1.2.8}$$

$$\alpha_2'(0)=\alpha_1^{-1}(1)\alpha_2(1)\alpha_1(1). \tag{1.2.9}$$

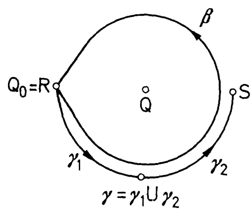


Fig. 8. The complex line $\pi \cap N$

Proof. (1.2.8) is evident. For (1.2.9) we note that the loop $\alpha_1(0)\alpha_2(0)$ is mapped into the loop $\alpha_2(1)\alpha_1(1)$, i.e.

$$\alpha_1'(0)\alpha_2'(0) = \alpha_2(1)\alpha_1(1)$$

or

$$\alpha_2'(0) = \alpha_1^{-1}(1)\alpha_2(1)\alpha_1(1)$$

in view of (1.2.9).

To obtain the identification relations for the tacnode we write the path γ as the union of paths γ_1, γ_2, arcs circling one quarter the way round Q (Fig. 8). Lemma 1.2.7 gives the relations between the generators in the lines $B\gamma_1(0)$, $B\gamma_1(1)$ and between those on $B\gamma_2(0) = B\gamma_1(1)$ and $B\gamma_2(1)$. Hence we obtain

$$\alpha_1 = \alpha_4^{-1}\alpha_3\alpha_4\,, \tag{1.2.10}$$

$$\alpha_2 = (\alpha_3\alpha_4)^{-1}\alpha_4(\alpha_3\alpha_4)\,. \tag{1.2.11}$$

To obtain the van Kampen relations we write down the identification relations obtained by linking S to R by the semicircular arc $\beta - \gamma$

$$\alpha_3 = \alpha_2^{-1}\alpha_1\alpha_2\,, \tag{1.2.12}$$

$$\alpha_4 = (\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2)\,, \tag{1.2.13}$$

and eliminate α_3, α_4 to give

$$\alpha_1 = (\alpha_1\alpha_2)^{-1}\alpha_2^{-1}(\alpha_1\alpha_2)(\alpha_2^{-1}\alpha_1\alpha_2)(\alpha_1\alpha_2)^{-1}\alpha_2(\alpha_1\alpha_2)\,,$$
$$\alpha_2 = (\alpha_1\alpha_2)^{-2}\alpha_2(\alpha_1\alpha_2)^2\,.$$

The first of these relations is a consequence of the second which may be written

$$(\alpha_1\alpha_2)^2 = (\alpha_2\alpha_1)^2\,. \tag{1.2.14}$$

In view of (1.2.14), (1.2.13) simplifies to

$$\alpha_4 = \alpha_1\alpha_2\alpha_1^{-1}\,. \tag{1.2.15}$$

If two elements α_1, α_2 of a group satisfy (1.2.14) we say that they *bi-commute* and write $\alpha_1 \uplus \alpha_2$.

The relations between generators in the crunode and cusp cases are worked out in the same way (again using Lemma 1.2.7). The results are

Table. *Relations between group elements in the neighbourhood of a singular point*

Singularity type	Identification relations		van Kampen relation
Crunode (Fig. 3)	$\alpha_1=\alpha_3$	$\alpha_2=\alpha_4$	$\alpha_1\alpha_2=\alpha_2\alpha_1$
Tacnode (Fig. 4)	$\alpha_3=\alpha_2^{-1}\alpha_1\alpha_2$	$\alpha_4=\alpha_1\alpha_2\alpha_1^{-1}$	$(\alpha_1\alpha_2)^2=(\alpha_2\alpha_1)^2$
Cusp (Fig. 5)	—		$\alpha_1\alpha_2\alpha_1=\alpha_2\alpha_1\alpha_2$

given in Table 1. Note that in the cuspidal case α_3, α_4 are elements defined by loops around complex points of L so we do not write down the identification relations in this case.

§ 2. Determination of the Monodromy Rings in the Generic Case

2.1. The Feynman Integrals Associated with the Graphs G_N (Fig. 9)

In a space-time of dimension m the Feynman integral associated with G_N in a theory in which all particles have spin 0 is given as a function of the energy s_0 and the masses s_i $1\leqq i\leqq N$ of the exchanged particles by the integral

$$I(s)=\int\prod_{i=1}^{N}\frac{d^m k_i}{k_i^2+s_i}\,\delta(\Sigma k_i-p),\qquad (p^2=s_0)\,. \tag{2.1.0}$$

(2.1.0) can be written in the parametric form

$$I(s)=\Gamma\left(-\frac{m}{2}(N-1)+N\right)\int_{\Delta}\frac{\delta(\Sigma\alpha_i-1)\,d\alpha_1\,\dots\,d\alpha_N}{[d(\alpha)]^{m/2}[D/d]^{-(m/2)(N-1)+N}}\,. \tag{2.1.1}$$

The integration region Δ is the simplex

$$\Delta=\left\{(\alpha):\alpha_i\geqq 0\ \forall i,\ \sum_{i=1}^{N}\alpha_i=1\right\}; \tag{2.1.2}$$

$d(\alpha)$, $D(s,\alpha)$ are the Symanzik functions

$$d(\alpha)=\sum_{i=1}^{N}\alpha_1\,\dots\,\alpha_{i-1}\alpha_{i+1}\,\dots\,\alpha_N\,, \tag{2.1.3}$$

$$D(s,\alpha)=s_0\alpha_1\,\dots\,\alpha_N-\left(\sum_{i=1}^{N}\alpha_i s_i\right)d(\alpha)\,. \tag{2.1.4}$$

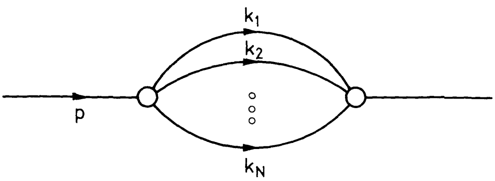


Fig. 9. The self-energy graph G_N

The integral (2.1.0) is convergent only in the cases

$$m=2, \qquad N \text{ arbitrary}, \tag{2.1.5}$$

$$m=3, \qquad N=2. \tag{2.1.6}$$

For the remaining values of $m(\geqq 2)$ and N, the integral (2.1.0) diverges and we must study the renormalized integral defined by a suitable renormalization procedure (see § 4).

(2.1.1) may be rewritten in a form in which the integrand is more symmetric. Define

$$\alpha_0 = -\frac{\alpha_1 \dots \alpha_N}{d(\alpha)}, \tag{2.1.7}$$

$$g(\alpha) = \sum_{i=0}^{N} \alpha_i^{-1}, \tag{2.1.8}$$

$$D'(s,\alpha) = \sum_{i=0}^{N} s_i \alpha_i. \tag{2.1.9}$$

Then up to a sign

$$I(s) = \Gamma\left(-\frac{m}{2}(N-1)+N\right) \cdot \int_{\Delta'} \frac{\delta\left(\sum_{i=1}^{N} \alpha_i - 1\right)\delta(g(\alpha))\,\alpha_0^{(m/2)-2}\alpha_1^{-m/2}\dots\alpha_N^{-m/2}\,d\alpha_0\dots d\alpha_N}{[D'(\alpha)]^{-(m/2)(N-1)+N}}, \tag{2.1.10}$$

where

$$\Delta' = \left\{(\alpha) : \alpha_i \geqq 0 \;\; 1 \leqq i \leqq N,\; \sum_{i=1}^{N} \alpha_i = 1,\; g(\alpha)=0\right\}. \tag{2.1.11}$$

The integrand in (2.1.10) has the form

$$\delta\left(\sum_{i=1}^{N} \alpha_i - 1\right) f(\alpha)\, d\alpha_0 \dots d\alpha_N,$$

where $f(\alpha)$ is homogeneous in α of degree

$$1+\left(\frac{m}{2}-2\right)-\frac{m}{2}N+\frac{m}{2}(N-1)-N=-(N+1)$$

so (2.1.10) can be rewritten as an integral in projective space [12] (again up to a sign)

$$I(s) = \Gamma\left(-\frac{m}{2}(N-1)+N\right)\int_{\Delta''} \frac{\delta(g(\alpha))\,\alpha_0^{(m/2)-2}\alpha_1^{-m/2}\dots\alpha_N^{-m/2}\,\eta}{[D'(\alpha)]^{-(m/2)(N-1)+N}}. \tag{2.1.12}$$

η is the fundamental projective form

$$\eta = \sum_{i=0}^{N} (-1)^i \alpha_i \, d\alpha_0 \dots d\alpha_{i-1} \, d\alpha_{i+1} \dots d\alpha_N \tag{2.1.13}$$

and

$$\Delta'' = \{(\alpha) : \alpha_i \geqq 0 \quad 1 \leqq i \leqq N\} \subset \mathbb{P}^N . \tag{2.1.14}$$

In order to study the monodromy ring of $I(s)$ we choose a reference point $B = (1, \varepsilon, \dots, \varepsilon)$ in the space

$$\mathbb{C}^{N+1} = \{(s) = (s_0, s_1, \dots, s_N)\} .$$

Here ε is a positive number, sufficiently small so that B lies above the normal threshold $\sqrt{s_0} = \sqrt{s_1} + \dots + \sqrt{s_N}$, i.e.

$$N^2 \varepsilon < 1 .$$

(2.1.12) is not an integral of standard form but it may be shown that the ambiant isotopy component of B for a suitable resolution of (2.1.12) into standard form is the complement in $\mathbb{C}^{N+1}$ of the set

$$L = \bigcup_{i=0}^{N+1} L_i , \tag{2.1.15}$$

where

$$L_i = \{(s)\} : s_i = 0\} \qquad 0 \leqq i \leqq N , \tag{2.1.16}$$

$$L_{N+1} = \{(s) : \pm\sqrt{s_0} \pm \sqrt{s_1} \pm \dots \pm \sqrt{s_N} = 0\} \tag{2.1.17}$$

Define $\mathcal{G}_N$ to be the fundamental group

$$\mathcal{G}_N = \pi_1(\mathbb{C}^{N+1} - L; B)$$

and V_N to be the vector space spanned by germs of $I(s)$ with centre B. Then we have a representation[4]

$$\mathcal{L} : \mathcal{G}_N \to L(V_N) = GL(d, \mathbb{C}) \qquad d = \dim V_N$$

defined by assigning to each loop on B the linear transformation of V_N induced by analytic continuation along this loop. $\mathcal{L}$ can evidently be extended to a representation of the group ring $\mathbb{C}(\mathcal{G}_N)$ of $\mathcal{G}_N$ over the complex field $\mathbb{C}$ into $L(V_N)$. We wish to study the monodromy ring

$$\mathcal{A}_N = \mathcal{L}(\mathbb{C}(\mathcal{G}_N)) \subset L(V_N) .$$

It is important to note that the ambiant isotopy component of B is completely determined by the location of the singularities of the integrand

[4] The fact that d *is* finite is established in the course of our investigation.

for $I(s)$. It is thus natural to introduce the more general integral

$$I(s,\lambda)=\Gamma(\Sigma\lambda_i+(N+1))\int_{\Delta''}\frac{\delta(g(\alpha))\prod_{i=0}^{N}\alpha_i^{\lambda_i}}{[D'(\alpha)]^{\Sigma\lambda_i+(N+2)}}\eta\,, \qquad (2.1.18)$$

which again defines an analytic function on $\mathbb{C}^{N+1}-L$. The exponent of $D'(\alpha)$ is determined by the requirement that the integrand be a projective form. We obtain for arbitrary complex λ_i, such that (2.1.18) is convergent, a representation $\mathscr{L}(\lambda)$ of the same group $\mathscr{G}_N$ and a monodromy ring

$$\mathscr{A}_N(\lambda)=\mathscr{L}(\lambda)\,(\mathbb{C}(\mathscr{G}_N))\,. \qquad (2.1.19)$$

We refer to the ring $\mathscr{A}_N(\lambda)$ associated with G_N for generic λ_i (i.e. λ_i not satisfying certain equations – in particular non-integral λ_i) as the generic monodromy ring for G_N. It turns out to have a very simple structure – it is a complete matrix ring over a vector space of dimension 2^N-1 (cf. § 2.4). Nevertheless the ring $\mathscr{A}_{N,m=2}$ of the original integral with $m=2$ can be obtained from $\mathscr{A}_N(\lambda)$ by specializing the λ_i to the values λ_i^0 appearing in (2.1.12). For $I(s,\lambda)$, regarded as a function defined on the universal covering space of $\mathbb{C}^{N+1}-L$, is continuous in λ in the nhd. of λ_i^0, uniformly for s in any compact set. The specialization is carried out in § 3.

2.2. The Fundamental Group $\mathscr{G}_N$

We construct a set of elements which generate $\mathscr{G}_N$ by choosing certain representative loops on B in the space $\mathbb{C}^{N+1}-L$. $\mathscr{G}_N$ is the quotient of the free group on these generators by the normal subgroup defined by the van Kampen relations for singular points of L, nodes or tacnodes, lying in or on the boundary of the region

$$\{(s): s_i\geqq 0\ \forall i\}\,.$$

We will need to write down explicitly only certain of these van Kampen relations, since these turn out to give sufficient information to construct the monodromy ring $\mathscr{A}_N(\lambda)$.

In constructing loops to define elements of $\mathscr{G}_N$ we follow the anti-clockwise convention of § 1.2. (Fig. 2). The following components of L_r-S are good with respect to the base point B chosen in § 2.1. (This is strictly true only in the limit $\varepsilon\to 0$. For small finite ε a set of points whose relative measure $\to 0$ as $\varepsilon\to 0$ must be deleted from the components. However, this does not affect the presentation we obtain for $\mathscr{G}_N$.)

$$L_i^{0)}=\Big\{(s): s_i=0\ \ s_j>0\ \ j\neq i\ \ \sqrt{s_0}>\sum_{j\neq 0}\sqrt{s_j}\Big\}\,. \qquad (2.2.1)$$

$$L_0^{i)}=\Big\{(s): s_0=0\ \ s_j>0\ \ j\neq 0\ \ \sqrt{s_i}>\sum_{j\neq i}\sqrt{s_j}\Big\}\,. \qquad (2.2.2)$$

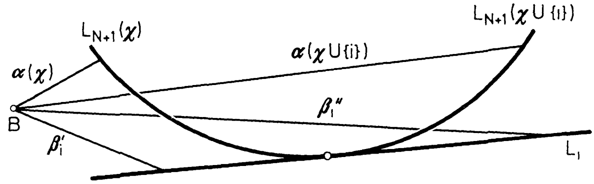


Fig. 10. Tacnodal contact of L_i and L_{N+1}

We write $\alpha(L_i^0)=\beta_i$ $1 \leqq i \leqq N$, $\alpha(L_0^{i)})=\beta_0^{i)}$ $1 \leqq i \leqq N$. Also for any proper subset χ of $\Omega=\{1, \ldots, N\}$ any component of $L_r - S$ contained in

$$L_{N+1}(\chi)=\left\{(s): s_i>0 \quad \text{for all } i \quad \sqrt{s_0}+\sum_{i \in \chi} \sqrt{s_i}=\sum_{i \notin \chi} \sqrt{s_i}\right\}. \qquad (2.2.3)$$

These components are separated only by transverse intersections so we may make immediate use of the identification relations for a transverse intersection (Fig. 3) and introduce one element $\alpha(\chi)$ of $\mathscr{G}$ defined by any one of them.

Any real nonsingular point of L in the component L_i for some i, $1 \leqq i \leqq N$, can be joined in L_i to a point in $L_i^{0)}$ by a real path which intersects the set of singular points of L only in nodal points or in tacnodal points of the kind shown in Fig. 10. [In Fig. 10 the loop corresponding to a point P of L_r is indicated by the line BP.] The identification relations for a tacnode are simplified by the position of the base point B (B is not in the symmetrical position considered in § 1.2) – the elements β_i', β_i'' of $\mathscr{G}_N$ defined by points of L_i on either side of the tacnode are the same. Any real nonsingular point of L in L_i thus defines the same element of $\mathscr{G}_N$ as a point in $L_i^{0)}$: $\beta_i'=\beta_i''=\beta_i$. The remaining identification relation obtained from the tacnode shown in Fig. 10 is

$$\alpha(\chi \cup \{i\})=\beta_i \alpha(\chi) \beta_i^{-1} \qquad (2.2.4)$$

and the van Kampen relation is

$$(\beta_i \alpha(\chi))^2=(\alpha(\chi) \beta_i)^2 . \qquad (2.2.5)$$

Since the components L_i of L intersect transversely $1 \leqq i \leqq N$

$$\beta_i \beta_j=\beta_j \beta_i \qquad 1 \leqq i, j \leqq N . \qquad (2.2.6)$$

Similarly we have

$$\beta_i \beta_0^{j)}=\beta_0^{j)} \beta_i \qquad i \neq j, 1 \leqq i, j \leqq N \qquad (2.2.7)$$

(but β_i, $\beta_0^{i)}$ do not commute since L_i does not intersect the region $L_0^{i)}$ used to define $\beta_0^{i)}$).

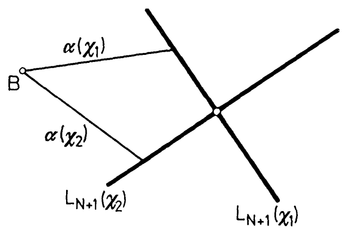


Fig. 11. Self-intersection of L_{N+1}

Next we consider the real self-intersections of L_{N+1} in the interior of the region $\{(s): s_i > 0 \ 0 \leqq i \leqq N\}$ (Fig. 11). These are nodal points. We have already used the corresponding identification relations to define the elements $\alpha(\chi)$. We now write down the van Kampen relations. The intersection

$$L_{N+1}(\chi_1) \cap L_{N+1}(\chi_2) = \Big\{(s): s_i > 0 \quad \text{for all } i \quad \sqrt{s_0} + \sum_{i \in \chi_1 \cap \chi_2} \sqrt{s_i} = \sum_{i \in \chi_1' \cap \chi_2'} \sqrt{s_i}, \sum_{i \in \chi_1 \cap \chi_2'} \sqrt{s_i} = \sum_{i \in \chi_1' \cap \chi_2} \sqrt{s_i}\Big\}$$

$(\chi' = \Omega - \chi)$ is non-empty unless $\chi_1 \subset \chi_2$ or $\chi_2 \subset \chi_1$ or $\chi_1 \cup \chi_2 = \Omega$. Thus we obtain the commutation relations

$$\left.\begin{array}{l} \alpha(\chi_1)\,\alpha(\chi_2) = \alpha(\chi_2)\,\alpha(\chi_1) \\ \text{for all proper subsets } \chi_1, \chi_2 \text{ of } \Omega \text{ such that} \\ \chi_1 \not\subset \chi_2, \chi_2 \not\subset \chi_1, \chi_1 \cup \chi_2 \neq \Omega\,. \end{array}\right\} \tag{2.2.8}$$

Note that we have not named the elements of $\mathscr{G}_N$ for regions of L_0 other than the $L_0^{i)}$ $(i = 1 \ldots N)$. Neither have we written down the van Kampen relations for tacnodes on L_0. We will show below that the elements $\alpha(\chi)$, β_i, and $\beta_0^{i)}$ generate $\mathscr{G}_N$. The additional van Kampen relations suffice to determine $\mathscr{G}_N$, but will not be required for the determination of $\mathscr{A}_N$.

Now let $Q^{(i)}$ be the point

$$Q^{(i)} = (\eta_0, \eta_1 \ldots \eta_{i-1}, 1, \eta_{i+1} \ldots \eta_N),$$

where the η_j are positive and less than ε, and consider the (complex) line ℓ_i given by

$$s = (1 - z)B + zQ^{(i)}.$$

This line intersects $L_i^{0)}$ at a real negative value z_i. It intersects $L(\chi)$, for $i \notin \chi$, at a real positive value $z(\chi)$; moreover, $z(\chi_1) < z(\chi_2)$ if $\chi_1 \subset \chi_2$. Finally, it intersects $L_0^{i)}$ and L_j $(j = 1, \ldots \hat{i} \ldots N)$ at real values z_0, z_j which are greater than any $z(\chi)$. The situation is illustrated in Fig. 12.

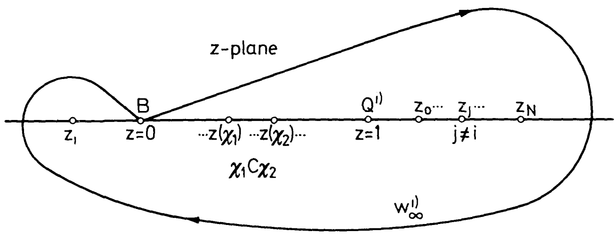


Fig. 12. The word at infinity in ℓ_i

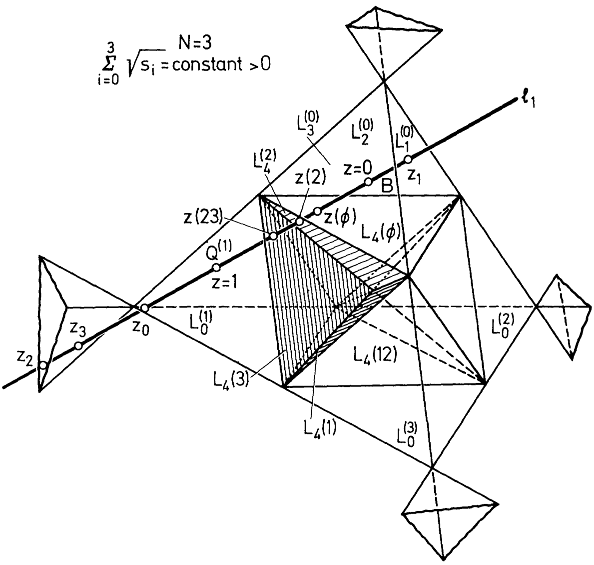


Fig. 13

Since each L_j ($j=0 \ldots N$) is of degree one in s, and since L_{N+1} is of degree 2^{N-1}, these real intersections are all intersections of ℓ_i with L. Therefore, by the Picard-Severi Theorem 1.2.2 the corresponding elements generate $\mathcal{G}_N$. For future reference we write down the word at infinity in ℓ_i, i.e., the anticlockwise loop around the point at infinity in ℓ_i (such a loop is shown in Fig. 12):

$$w_\infty^{i)^{-1}} = \beta_i \alpha(\phi) \prod_{j \neq i} \alpha(\{j\}) \ldots \alpha(\Omega_i) \beta_0^{i)} \prod_{j \neq i} \beta_j, \tag{2.2.10}$$

where $\Omega_i = \{1 \ldots \hat{i} \ldots N\}$. Note that the order of elements $\alpha(\chi)$ in (2.2.10) is sufficiently determined by the order relations on the $z(\chi)$ noted above, in view of the commutation relations (2.2.9).

We close this section by defining a homomorphism

$$j : \mathcal{G}_{N-1} \to \mathcal{G}_N$$

which will be needed in the isomorphism theorem of § 2.4. The parameter space for the graph G_{N-1} may be identified with the subspace L_N of the parameter space $\mathbb{C}^{N+1}$ for G_N. The Landau variety $L' = \bigcup_{i=0}^{N} L'_i$ of G_{N-1} then is identified with $\bigcup_{i=0}^{N-1} (L_i \cap L_N) \cup (L_{N+1} \cap L_N) \subset \mathbb{C}^{N+1}$. We choose for $\mathcal{G}_{N-1}$ the base point $B' = (1, \varepsilon, \varepsilon, \ldots, \varepsilon, 0) \in \mathbb{C}^{N+1}$. For sufficiently small η any element γ of $\mathcal{G}_{N-1}$ has a representative $\underline{\gamma}$ which does not come within a distance η of L'. We can then choose ε (depending only on η) sufficiently small that for all t, $0 < t \leqq \varepsilon$, the loop $\underline{\gamma}(t) = \{(s) : (s_0, \ldots, s_{N-1}) \in \underline{\gamma}\ s_N = t\}$ does not intersect L. We then define $j(\gamma) \in \mathcal{G}_N$ to be the element of $\mathcal{G}_N$ defined by $\underline{\gamma}(\varepsilon)$. If $\underline{\gamma}'$ is another representative of γ, we may choose the homotopy H between $\underline{\gamma}$ and $\underline{\gamma}'$ so that the image of H also does not come within η of L'; H then gives rise to a homotopy $H(\varepsilon)$ between $\underline{\gamma}(\varepsilon)$ and $\underline{\gamma}'(\varepsilon)$. Thus j is well defined on $\mathcal{G}_{N-1}$.

We remark that

$$\left.\begin{aligned} j(\beta'_k) &= \beta_k && (k = 1, \ldots N-1) \\ j(\beta_0^{k)\prime}) &= \beta_0^{k)} && (k = 1, \ldots N-1) \\ j[\alpha(\chi)'] &= \alpha(\chi)\alpha(\chi \cup N) && (\chi \text{ a proper subset of } \Omega_N)\,. \end{aligned}\right\} \tag{2.2.11}$$

2.3. *The Relations Obtained from the Integral Representation*

(i) Homogeneity.

The function $D'(s, \alpha)$ is homogeneous and linear in s. It follows that the function $I(s, \lambda)$ defined by (2.1.18) is homogeneous in s of degree

$$-\sum_{i=0}^{N} \lambda_i - (N+2) = \mu \text{ (say)}\,. \tag{2.3.1}$$

This homogeneity can be expressed as a condition on the representation $\mathcal{L}(\lambda)$. If w_∞ denotes the element of $\mathcal{G}_N$ defined by a loop circling the point at infinity in a generic line ℓ counterclockwise (i.e. circling all the points $L \cap \ell$ clockwise) we have

$$\mathcal{L}(w_\infty) = c_\infty 1 \tag{2.3.2}$$

where the constant c_∞ is given by

$$c_\infty = \exp[-2\pi i \mu]\,. \tag{2.3.3}$$

(ii) Consequences of the Picard-Lefschetz theorem.

In the group ring $\mathbb{C}(\mathscr{G}_N)$, write

$$\alpha(\phi) = 1 + a\,, \qquad (2.3.4)$$

$$\beta_i = 1 + b_i \quad 1 \leqq i \leqq N\,, \qquad (2.3.5)$$

$$\beta_0^{i)} = 1 + b_0^{i)} \quad 1 \leqq i \leqq N\,. \qquad (2.3.6)$$

It can be shown that in a standard form presentation of $I(s, \lambda)$ the points on L used to define the elements $\alpha = \alpha(\phi)$, β_i, $\beta_0^{i)}$ of $\mathscr{G}_N$ correspond to a quadratic pinch in the integration space; in the case of α this pinch is simple. We can therefore apply the Picard-Lefschetz theorem. For a simple quadratic pinch and integer exponents for the singularities of the integrand this is given in [8]. The extension to a non-simple quadratic pinch is given in [13], and to non-integral exponents for the singularities of the integrand in [4].

In the present situation we obtain from the theorem the relations

$$\mathscr{L}(a^2) = A\,\mathscr{L}(a)\,, \qquad (2.3.7)$$

$$\mathscr{L}(b_i^2) = B_i \mathscr{L}(b_i) \quad 1 \leqq i \leqq N \qquad (2.3.8)$$

$$\mathscr{L}[(b_0^{i)})^2] = B_0 \mathscr{L}(b_0^{i)}) \quad 1 \leqq i \leqq N\,, \qquad (2.3.9)$$

where the constants A, B_i $0 \leqq i \leqq N$ are given by

$$1 + A = (-1)^{N+1} \exp[2\pi i \mu]\,, \qquad (2.3.10)$$

$$1 + B_i = \exp[-2\pi i \lambda_i]\,. \qquad (2.3.11)$$

Thus

$$\prod_{i=0}^{N} (1 + B_i) = (-1)^{N+1}(1 + A) = c_\infty^{-1}\,. \qquad (2.3.12)$$

To express the further relations derived from the fact that α corresponds to a *simple* quadratic pinch we introduce the notion of a *Lefschetz element.*

Definition 2.3.13. An element e of an associative algebra A is *Lefschetz* if

$$e^2 = Ee$$

and for all $x \in A$

$$exe \quad \text{is a multiple of } e\,.$$

We then write

$$exe = (e \cdot x)e\,.$$

Remark. $e \cdot x$ is linear in x. If e, f are both Lefschetz

$$e \cdot f = f \cdot e\,.$$

Thus the dot product defines a symmetric bilinear product on the linear span of the Lefschetz elements.

Now we can state the further relations as:

$$\mathscr{L}(a) \text{ is a Lefschetz element of } \mathscr{A}_N(\lambda). \tag{2.3.14}$$

The linear functional $\mathscr{L}(a)\cdot x$, $x \in \mathscr{A}_N(\lambda)$, which appears in (2.3.13) is not given by the Picard-Lefschetz theorem but must be determined by exploiting the other relations satisfied by the representation (see § 2.4).

(iii) Localization conditions.

These conditions follow from

Theorem 2.3.15. *Suppose $L_1 \dots L_k$ are Landau varieties intersecting transversely at a point O of the parameter space, so that we may choose local coordinates u with O as origin such that*

$$L_i = \{(u) : u_i = 0\} \qquad 1 \leqq i \leqq k .$$

Choose a base point B in the neighbourhood of O, and denote by $\gamma_1 \dots \gamma_k$ the (mutually commuting) elements of the fundamental group corresponding to loops on B around $L_1 \dots L_k$, and by

$$T_i = \mathscr{L}(\gamma_i) - 1 \qquad 1 \leqq i \leqq k$$

the corresponding discontinuity operations in the monodromy ring. Suppose that $L_1 \dots L_k$ correspond to quadratic pinches in the integration space on the varieties $P_1 \dots P_k$, and that for $u = 0$ $P_1 \dots P_k$ are in general position. Suppose further that

$$P_1 \cap \cdots \cap P_k = \phi . \tag{2.3.16}$$

Then

$$T_1 \dots T_k = 0 . \tag{2.3.17}$$

Remark. For $k=2$ and P_1, P_2 single points, Theorem 2.3.15 follows immediately from the localization lemma used in the FFLP proof of the Picard-Lefschetz theorem [8]. We give a sketch of the proof in the general case:

Construct a metric in the integration space with respect to which $P_1, \dots, P_k$ are orthogonal, and denote by $N_i(\eta)$ the tubular neighbourhood of radius η constructed with the aid of this metric. Then for η sufficiently small

$$N_1(\eta) \cap \cdots \cap N_k(\eta) = \phi .$$

We can choose $\delta = \delta(\eta)$ sufficiently small, that for $u \in \sigma_i$

$$\sigma_i = \{(u) : u_j = \delta \quad j \neq i \quad |u_i| \leqq \delta\}$$

the poles of the integrand are in G.P. outside $N_i(\eta)$. Then we may choose

$$B = (\delta, \delta, \ldots, \delta)$$

as base point, and take $\gamma_i = \partial\sigma_i$ $1 \leqq i \leqq N$. The ambiant isotopy A_i for γ_i may then be chosen to be the identity outside $N_i(\eta)$ and inside $N_i(\eta)$ to commute with the projection P_i of the normal bundle $N_i(\eta)$ onto P_i.

Now let $\underline{h}$ be an arbitrary integration cycle defining a function element h. Then if we compute a representative cycle for $T_1 \ldots T_k h$ with the help of the ambiant isotopies A_i we obtain successively

$$\begin{aligned} & \operatorname{supp} A_k \underline{h} \subset N_k(\eta) \\ \ldots \quad & \operatorname{supp} A_j \ldots A_k \underline{h} \subset N_j(\eta) \cap \cdots \cap N_k(\eta) \\ \text{and so} \quad & \operatorname{supp} A_1 \ldots A_k \underline{h} \subset N_1(\eta) \cap \cdots \cap N_k(\eta) = \phi\,, \\ \text{so} \quad & T_1 \ldots T_k h = 0\,. \end{aligned}$$

h is arbitrary so $T_1 \ldots T_k = 0$.

We apply Theorem 2.3.15 to the transverse intersection of $L_1 \ldots L_N$. If $\lambda_0, \ldots, \lambda_N$ satisfy a certain linear relation we may rewrite (2.1.18) in the momentum space form, $m = 2$,

$$I(s, \lambda) = \int \prod_{i=1}^{N} \frac{d^2 k_i}{(s_i + k_i^2)^{\lambda_i + 2}} \, \delta\left(\sum_{i=1}^{N} k_i - p \right). \tag{2.3.18}$$

Now a point on L_i corresponds in the integration space to a quadratic pinch on

$$P_i = \left\{ (k) : k_i = 0, \sum_{j=1}^{N} k_i - p = 0 \right\}.$$

The P_i $1 \leqq i \leqq N$ are in general position, and for $s_0 \neq 0$, i.e. $p \neq 0$

$$P_i \cap \cdots \cap P_N = \phi\,.$$

Hence Theorem 2.3.15 applies, and we obtain the relation

$$\mathscr{L}(b_1 \ldots b_N) = 0\,. \tag{2.3.19}$$

It is also possible to derive (2.3.19) from the geometry of the pinches corresponding to the L_i in the α-space, so that (2.3.19) holds for generic λ. In view of the symmetry of the α-space representation (2.1.18) noted in § 2.1 we have also for the transverse intersection of L_i $i \neq j$, $L_0^{j)}$

$$\mathscr{L}(b_1 \ldots \hat{b}_j \ldots b_N b_0^{j)}) = 0\,. \tag{2.3.20}$$

Finally we apply Theorem 2.3.15 to the real self-intersections of L_{N+1} (cf. (2.2.9)) and obtain

$$\left.\begin{aligned} & \mathscr{L}[a(\chi_1) a(\chi_2)] = 0 \\ & \text{for all proper subsets } \chi_1, \chi_2 \text{ of } \Omega \text{ such that} \\ & \qquad \chi_1 \not\subset \chi_2,\ \chi_2 \not\subset \chi_1,\ \chi_1 \cup \chi_2 \neq \Omega \end{aligned}\right\} \tag{2.3.21}$$

where

$$a(\chi) = \alpha(\chi) - 1 .$$

(iv) Physical sheet conditions.

Denote by Ψ the vector in V_N defined by the contour of integration Δ'' in (2.1.18). Ψ is called the physical sheet. By examining whether the corresponding point P on L corresponds to a pinch of Δ'' we may decide whether Ψ is invariant under a given element of $\mathscr{G}_N$ (for generic λ there are no polar singularities so Ψ is invariant if it is nonsingular at P). This gives the physical sheet conditions

$$\mathscr{L}(a)\Psi \neq 0, \tag{2.3.22}$$

$$\mathscr{L}[a(\chi)]\Psi = 0 \qquad \chi \neq \phi, \tag{2.3.23}$$

$$\mathscr{L}(b_0^i)\Psi = 0 \qquad 1 \leqq i \leqq N. \tag{2.3.24}$$

Finally an element of $\mathscr{A}_N$ must be determined by its action on the vectors of $V_N = \mathscr{A}_N \Psi$, i.e.

$$y \in \mathscr{A}_N, y\mathscr{A}_N \Psi = 0 \Rightarrow y = 0 . \tag{2.3.25}$$

It turns out that the ring defined by all the relations except the physical sheet conditions is already isomorphic with a complete matrix ring. We do not therefore obtain any additional relations from (2.3.25). The role of the physical sheet conditions is just to give a matrix representation of the abstract ring by singling out the vector Ψ, and to guarantee that the tacnode points of Fig. 10 are effective intersections (see § 2.4), that is,

$$\mathscr{L}[ab_i] \neq \mathscr{L}[b_i a] \qquad (i = 1 \ldots N). \tag{2.3.26}$$

Eqs. (2.3.2), (2.3.7)-(2.3.9), (2.3.14), (2.3.19)-(2.3.25) are the relations which we use in § 2.4. Note that the relations listed in this section do not include the Cutkosky-Steinmann relations used in [11]. These relations depend on the vanishing of certain intersection numbers of cycles in the integration space, algebraically not geometrically (the cycles are not disjoint), and do not persist in the generic case.

2.4. *Derivation of the Monodromy Ring $\mathscr{A}_N$ in the Generic Case*

Let $\mathscr{M}$ denote the set of all proper subsets of Ω. For any $\chi \in \mathscr{M}$, define

$$b(\chi) = \prod_{i \in \chi} b_i$$

(with $b(\phi) = 1$).

Lemma 2.4.1. *For any* $\chi \in \mathcal{M}$

$$\mathcal{L}(ab(\chi)a) = M(\chi)\mathcal{L}(a) \tag{2.4.2}$$

where

$$M(\chi) = (-)^{|\chi|}(1+A) - \prod_{i \in \chi}(1+B_i) \tag{2.4.3}$$

(i.e., in the notation of Definition 2.3.13,

$$\mathcal{L}(a) \cdot \mathcal{L}[b(\chi)] = M(\chi)) .$$

Proof. For $|\chi| = 0$, this is an immediate consequence of (2.3.7). For $\chi = \{i\}$, we apply $\mathcal{L}$ to the bicommutation relation

$$(\alpha\beta_i)^2 = (\beta_i\alpha)^2$$

yielding

$$[\mathcal{L}(a) \cdot \mathcal{L}(b_i) - M(\{i\})] \, \mathcal{L}[ab_i - b_i a] = 0 .$$

(2.4.2) then follows from (2.3.26).

Suppose now that we have verified (2.4.2) for all $|\chi| \leqq m < N-2$, and suppose $|\eta| = m+1$, with $\eta = i \cup \chi$. Eq. (2.3.21) implies

$$\mathcal{L}[a\beta_i^{-1} \prod_{j \in \chi} \beta_j a] = 0 ;$$

applying the induction assumption to this formula gives (2.4.2).

Lemma 2.4.4 [Reduction of the word at infinity]:

$$\mathcal{L}(\beta_0^{i)^{-1}})$$
$$= c_\infty \mathcal{L}\left\{\beta_1 \ldots \beta_N + \beta_i \left[\sum_{\substack{\psi, \chi \subset \Omega_i \\ \psi \cap \chi = \phi}} \prod_{\chi} \beta_j a \prod_{\psi} \beta_j \prod_{\Omega_i - (\chi \cup \psi)} [-(2+B_j)]\right]\right\}. \tag{2.4.5}$$

Proof. From (2.2.10) we have

$$\beta_0^{i)^{-1}} = w_\infty^{(i)} \beta_i \alpha(\phi) \prod_{j \neq i} \alpha(\{j\}) \ldots \alpha(\Omega_i) \prod_{j \neq i} \beta_j . \tag{2.4.6}$$

We apply $\mathcal{L}$ to this and use (2.3.2). Write $\alpha(\chi) = 1 + a(\chi)$ and expand

$$\mathcal{L}\left[\alpha(\phi) \prod_{j \neq i} \alpha(\{j\}) \ldots \alpha(\Omega_i)\right] .$$

All products

$$\mathcal{L}[\alpha(\chi_1)\alpha(\chi_2) \ldots \alpha(\chi_k)]$$

vanish by (2.3.21) unless $\chi_1 \subset \chi_2 \subset \cdots \subset \chi_k$. Then using

$$a(\chi) = \prod_{\chi} \beta_j a \prod_{\chi} \beta_j^{-1}$$

(2.4.6) becomes

$$\mathscr{L}[\beta_0^{i)^{-1}}] \tag{2.4.7}$$
$$= c_\infty \left\{ \beta_1 \dots \beta_N + \beta_i \sum_{k=1}^{N} \sum_{\chi_1, \chi_2 \dots \chi_k} \prod_{\chi_1} \beta_j a \prod_{\chi_2} \beta_j \dots \prod_{\chi_k} \beta_j a \prod_{(\Omega_i - \chi_1 \cup \dots \cup \chi_k)} \beta_j \right\},$$

where $\chi_1 \dots \chi_k$ are disjoint subsets of Ω_i with $\chi_j \neq \phi$ for $j > 1$. Set $\chi = \chi_1$, $\psi = \Omega_i - (\chi_1 \cup \dots \cup \chi_k)$. From Lemma (2.4.2) we have

$$\mathscr{L}\left[a \prod_{\chi} \beta_j a\right] = - \prod_{\chi} (2 + B_j) \mathscr{L}(a), \qquad (\chi \in \mathscr{M})$$

so that

$$\mathscr{L}\left[a \prod_{\chi_2} \beta_j a \dots a \prod_{\chi_k} \beta_j a\right] = (-)^k \prod_{\Omega_i - (\chi \cup \psi)} (2 + B_j) \mathscr{L}(a).$$

But if η is any set of $m > 0$ elements, one has

$$\sum_P (-)^{|P|} = (-)^m,$$

where the sum runs over all ordered partitions P of η into $|P|$ sets. Thus we may do the sum over $\chi_2 \dots \chi_k$ in (2.4.7) for fixed χ, ψ; this yields (2.4.5).

We define $I_N \subset \mathbb{C}(\mathscr{G}_N)$ to be the two-sided ideal generated by the elements

$$\begin{array}{lll} a(\chi)a(\psi) & & (\chi \not\subset \psi, \psi \not\subset \chi, \chi \cup \psi \neq \Omega) \\ b_1 \dots b_N & & \chi, \psi \in \mathscr{M} \\ w_\infty^{i)} - c_\infty 1 & 1 \leqq i \leqq N & \\ b_i^2 - B_i b_i & 1 \leqq i \leqq N & \\ b_0^{i)2} - B_0 b_0^{i)} & & \\ a^2 - Aa & & \\ ab(\chi)a - M(\chi)a & & \chi \in \mathscr{M} \end{array} \tag{2.4.8}$$

and form the quotient ring

$$R_N = \mathbb{C}(\mathscr{G}_N)/I_N. \tag{2.4.9}$$

We denote by q_N the natural projection $\mathbb{C}(\mathscr{G}_N) \to R_N$ and indicate the image under q_N of an element of $\mathbb{C}(\mathscr{G}_N)$ by underlining. We have shown that the elements of I_N lie in the kernel of $\mathscr{L}: \mathbb{C}(\mathscr{G}_N) \to L(V_N)$, so there is a natural homomorphism $\underline{\mathscr{L}}: R_N \to L(V_N)$.

Lemma 2.4.10. *R_N is finite dimensional.*

Proof. $\mathbb{C}(\mathscr{G}_N)$ is spanned by all products of the elements 1, $\underline{b}_i$, $\underline{b}_0^{i)}$, $\underline{a}$. The reduction of the word at infinity (Lemma 2.4.4), which is also valid in R_N (since only the vanishing of the elements (2.4.8) was used in its derivation), enables us to eliminate the $\underline{b}_0^{i)}$. We then obtain the following finite set of elements spanning R_N

$$\underline{b}(\chi), \underline{b}(\chi_1)\underline{a}\underline{b}(\chi_2), \qquad \chi, \chi_1, \chi_2 \in \mathscr{M} . \tag{2.4.11}$$

Lemma 2.4.12. *$\underline{b}(\Omega_i)$ is Lefschetz.*

Proof. It is sufficient to prove that

$$\underline{b}(\Omega_i)\underline{x}\underline{b}(\Omega_i) - (\underline{b}(\Omega_i) \cdot \underline{x})\underline{b}(\Omega_i) = 0$$

for each $\underline{x}$ in the set (2.4.11) and some $\underline{b}(\Omega_i) \cdot \underline{x} \in \mathbb{C}$. This follows easily for all $\underline{x}$ in (2.4.11) once it is established for $\underline{x} = \underline{a}$. In this case the symmetry of the $\cdot$ product gives $\underline{b}(\Omega_i) \cdot \underline{a} = M(\Omega_i)$ so we must prove

$$\underline{y} = \underline{b}(\Omega_i)\underline{a}\underline{b}(\Omega_i) - M(\Omega_i)\underline{b}(\Omega_i) = 0 .$$

Now

$$\begin{aligned} \underline{b}_j\underline{y} &= B_j\underline{y} \qquad j = 1 \ldots \hat{i} \ldots N \\ \underline{b}_i\underline{y} &= 0 \\ \underline{a}\underline{y} &= 0 . \end{aligned}$$

From (2.4.4) we obtain

$$\underline{\beta}_0^{i)^{-1}} \underline{y} = c_\infty \prod_{j \neq i} (1 + B_j)\underline{y} .$$

But $\underline{\beta}_0^{i)^{-1}}\underline{y} = f\underline{y}$ implies $\underline{y} = 0$ or $f = (1 + B_0)^{-1}$ or 1. For generic λ $f = c_\infty \prod_{j \neq i} (1 + B_j) \neq (1 + B_0)^{-1}$ or 1 so $\underline{y} = 0$.

Lemma 2.4.13 *(reduction of the identity). The element $\underline{1} = \underline{b}(\phi) \in R_N$ is linearly dependent on the elements $\underline{b}(\chi)$ $(\chi \neq \phi)$ and $\underline{b}(\chi)\underline{a}\underline{b}(\psi)$. Specifically, if*

$$Y(\chi) = \prod_{\chi} \left(- \frac{1}{B_i} \right) \qquad (Y(\phi) = 1) , \tag{2.4.14}$$

$$D = - \sum_{\chi \in \mathscr{M}} M(\chi)\, Y(\chi) , \tag{2.4.15}$$

then

$$\underline{h} = \sum_{\chi \in \mathscr{M}} Y(\chi)\underline{b}(\chi) + \sum_{\chi, \psi} \frac{Y(\chi)\, Y(\psi)}{D} \underline{b}(\chi)\underline{a}\underline{b}(\psi) = 0 . \tag{2.4.16}$$

Proof. Direct calculation gives

$$\underline{a}\underline{h} = \underline{b}_i\underline{h} = 0 \qquad 1 \leqq i \leqq N .$$

Then from (2.4.4) $\underline{\beta}_0^{i)^{-1}}\underline{h} = c_\infty\underline{h}$. Since $c_\infty \neq (1 + B_0)^{-1}$ or 1 this implies $\underline{h} = 0$ (cf. proof of 2.4.12).

We now give the main result of this section.

Theorem 2.4.17 [*Determination of $\mathscr{A}_N$*]. *The dimension of V_N is $2^N - 1$, and $\mathscr{L} : R_N \to L(V_N)$ is an isomorphism onto. Thus $R_N \cong \mathscr{A}_N$, which is a complete matrix algebra of dimension $2^N - 1$.*

Proof. Recall that Ψ denotes the physical sheet. From Lemma 2.4.10, V_N is spanned by the elements

$$\underline{\mathscr{L}}[\underline{b}(\chi)]\,\Psi,\ \underline{\mathscr{L}}[\underline{b}(\chi_1)\underline{a}\underline{b}(\chi_2)]\,\Psi \qquad (\chi, \chi_1, \chi_2 \in \mathscr{M}).$$

The following Lemma shows that V_N is actually spanned by the $2^N - 1$ elements $\underline{\mathscr{L}}[\underline{b}(\chi)]\,\Psi$, $\chi \in \mathscr{M}$.

Lemma 2.4.18

$$\underline{\mathscr{L}}[\underline{a}\underline{b}(\chi)]\,\Psi = \prod_{\chi}(1+B_i)\,\underline{\mathscr{L}}(\underline{a})\,\Psi\,, \tag{2.4.19}$$

$$\underline{\mathscr{L}}[\underline{b}(\chi)\,(1+\underline{a})]\,\Psi = \sum_{\psi \supset \chi,\, \psi \in \mathscr{M}} \frac{(-)^{|\psi-\chi|}c_\infty^{-1}}{\prod\limits_{\psi}(1+B_i)}\,\underline{\mathscr{L}}[\underline{b}(\chi)]\,\Psi\,. \tag{2.4.20}$$

Proof. Eq. (2.3.23) implies

$$\underline{\mathscr{L}}\Big[\underline{a}\prod_{\eta}\underline{\beta}_i^{-1}\Big]\,\Psi = 0 \qquad (\eta \in \mathscr{M}, \eta \neq \phi)\,. \tag{2.4.21}$$

Now (2.4.19) is trivially true for $\chi = \phi$; it then follows from (2.4.21) by induction on $|\chi|$, using $\underline{\beta}_i^{-1} = 1 - \dfrac{\underline{b}_i}{1+B_i}$.

Since β_j $(j \neq i)$ commutes with $\beta_0^{i)}$, we may rewrite (2.4.6) as

$$\beta_0^{i)\,-1} = \prod_{j \neq i}\beta_j w_\infty^{i)}\beta_i \alpha(\phi)\ldots\alpha(\Omega_i)\,.$$

Using (2.3.23) and (2.3.24), we have

$$\underline{\mathscr{L}}\left(\prod_{i=1}^{N}\underline{\beta}_i\underline{\alpha}\right)\Psi = c_\infty^{-1}\,\Psi \tag{2.4.22}$$

and multiplying (2.4.22) by $\underline{\mathscr{L}}[\underline{b}(\chi)]$ gives

$$\prod_{\eta}(1+B_j)\,\underline{\mathscr{L}}\left\{\Big[\sum_{\psi \supset \eta,\, \psi \in \mathscr{M}}\underline{b}(\psi)\Big](1+\underline{a})\right\}\Psi = c_\infty^{-1}\,\Psi\,. \tag{2.4.23}$$

Taking $\eta = \Omega_i$ in (2.4.23) verifies (2.4.20) for $\chi = \Omega_i$. (2.4.20) follows for any χ_0 from (2.4.23) (taking $\eta = \chi_0$) if we assume, inductively, that (2.4.20) holds for any $\chi \supset \chi_0$.

We now return to the proof of the main theorem. We write $\Psi(\varrho) = \mathscr{L}(b(\varrho))\,\Psi$. From Lemma 2.4.18 we obtain the relation

$$\begin{aligned}
&\mathscr{L}(b(\psi)ab(\chi))\,\Psi(\varrho)\\
&= \begin{cases} 0 \quad \text{if} \quad \varrho \cup \chi = \Omega \\ \prod\limits_{\varrho \cap \chi} B_i \prod\limits_{\varrho \cup \chi}(1+B_i)\left[-b(\psi) + \sum\limits_{\eta \supset \psi}\dfrac{(-1)^{|\eta-\psi|}c_\infty^{-1}b(\eta)}{\prod\limits_{\eta}(1+B_i)}\right]\Psi \end{cases} \qquad (2.4.24)\\
&= v_\varrho^\chi \sum_{\eta \in \mathscr{M}} u_\eta^\psi\,\Psi(\eta)\,,
\end{aligned}$$

where the coefficients v_ϱ^χ, u_η^ψ are given by

$$v_\varrho^\chi = \begin{cases} 0 & \varrho \cup \chi = \Omega \\ \prod_{\varrho \cap \chi} B_i \prod_{\varrho \cup \chi} (1+B_i) & \text{otherwise} \end{cases} \tag{2.4.25}$$

$$u_\eta^\psi = \begin{cases} 0 \quad \text{unless} \quad \eta \supset \psi \\ -1 + \dfrac{c_\infty^{-1}}{\prod_\psi (1+B_i)} & \eta = \psi \\ (-1)^{|\eta - \psi|} \dfrac{c_\infty^{-1}}{\prod_\eta (1+B_i)} & \eta \supsetneqq \psi \,. \end{cases} \tag{2.4.26}$$

We may regard (2.4.25), (2.4.26) as defining two sets of $2^N - 1$ vectors $v(\chi)$ $\chi \in \mathcal{M}$, $u(\psi)$ $\psi \in \mathcal{M}$ in the vector space

$$\mathbb{C}^{2^N - 1} = \{(x) = (x_\varrho)\, \varrho \in \mathcal{M}\} \,.$$

We claim that for generic λ each of these sets is a basis for $\mathbb{C}^{2^N-1}$.

For the set $\{u(\psi)\}$ this follows immediately from (2.4.26) since the coefficient matrix $[u] = [u_\eta^\psi]$ is "triangular", i.e. $u_\eta^\psi = 0$ for $\eta \not\supset \psi$, $u_\eta^\eta \neq 0$. For the set $\{v(\chi)\}$ we argue as follows. It is sufficient to show that the vectors $w(\chi)$ defined by

$$w_\varrho^\chi = \begin{cases} 0 & \varrho \cup \chi = \Omega \\ \prod_{\varrho \cap \chi} B_i \prod_{\varrho - \chi} (1+B_i) & \text{otherwise} \end{cases}$$

are linearly independent since

$$w(\chi) = \prod_\chi (1+B_i)^{-1} v(\chi) \,,$$

i.e., to show that the corresponding matrix $[w] = [w_\varrho^\chi]$ has non-zero determinant. This determinant has degree

$$\leqq \sum_{\varrho \in \mathcal{M}} |\varrho| = \sum_{k=0}^{N-1} k^N \binom{N}{k} = N(2^{N-1} - 1)$$

in the B_i. It will suffice to show that there is a non-zero term of this degree. To do this we may specialize to the case in which all the B_i are equal. Then the leading term in B of $\det[w]$ is

$$B^{N(2^{N-1}-1)} \det[y]$$

where

$$y_\varrho^\chi = \begin{cases} 0 & \varrho \cup \chi = \Omega \\ 1 & \text{otherwise} \,. \end{cases}$$

From the vectors $y(\chi)$ we can form vectors

$$z(\chi) = \sum_{\eta \subset \chi} (-1)^{|\eta|} y(\eta)$$

which have components

$$z_\eta^\chi = \delta_{\chi\phi} - (-1)^{|(\Omega-\varrho)|} \delta_{\chi,\Omega-\varrho}$$

and are manifestly linearly independent.

Since the sets $\{u(\psi)\}$, $\{v(\psi)\}$ form bases for $\mathbb{C}^{2^N-1}$ for generic λ there exists coefficients $C_\eta^\chi(\lambda)$, $D_\eta^\chi(\lambda)$ such that

$$\sum_{\chi \in \mathcal{M}} C_\eta^\chi v_\varrho^\chi = \delta_{\eta\varrho},$$

$$\sum_{\chi \in \mathcal{M}} D_\eta^\chi u_\varrho^\chi = \delta_{\eta\varrho}.$$

Then

$$f_{\eta\varrho} = \sum_{\chi, \psi \in \mathcal{M}} \mathcal{L}[b(\psi)ab(\chi)] C_\eta^\chi D_\varrho^\psi \in \mathcal{A}_N(\lambda)$$

satisfies

$$f_{\eta\varrho}\Psi_\kappa = \delta_{\eta\kappa}\Psi_\varrho. \tag{2.4.27}$$

If for some coefficients $g_\varrho \in \mathbb{C}$ we have

$$\sum g_\varrho \Psi_\varrho = 0$$

we may apply $f_{\eta\phi}$ to this relation and use (2.4.27) to obtain

$$g_\eta \Psi = 0.$$

But $\Psi \neq 0$ so $g_\eta = 0$ for all η, i.e. the Ψ_ϱ are linearly independent. Moreover the $f_{\eta\varrho}$ are $(2^N-1)^2$ linearly independent elements of $\mathcal{A}_N(\lambda)$.

This completes the proof of the first part of the main theorem.

There remains to show only that $\mathcal{L}: R_N \to \mathcal{A}_N$ has zero kernel; we do this by showing that the dimension of R_N is at most $(2^N-1)^2$. In the course of the argument we will also prove the important isomorphism theorem mentioned in the Introduction. We proceed by induction on N: for $N=2$, Lemmas (2.4.10), (2.4.12), and (2.4.13) imply that R_N is spanned by the nine elements

$$\underline{b}(\chi_1)\underline{ab}(\chi_2) \qquad [\chi_1, \chi_2 = \phi, \{1\}, \text{ or } \{2\}],$$

thus $R_2 \cong \mathcal{A}_2$. We assume inductively that R_{N-1} is spanned by the elements

$$\underline{b}(\chi_1)\underline{ab}(\chi_2) \qquad (\chi_1, \chi_2 \in \mathcal{M}).$$

Define $P \in R_N$ by $P = \underline{b}_N/B_N$ (so that $P^2 = P$), and let $S_N \subset R_N$ denote the sub-algebra $S_N = PR_NP$. There is a natural map p: $R_N \to S_N$ given by

$p(\underline{x}) = P\underline{x}P$; p is a linear map but is not a ring homomorphism. However, if

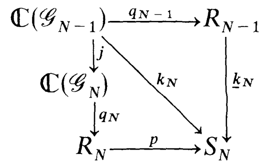
$$Y_N = \{\underline{x} \in R_N \mid \underline{x} \text{ commutes with } P\}$$

then $p \mid Y_N$ is a ring homomorphism.

Now recall (§ 2.2) the homomorphism $j: \mathscr{G}_{N-1} \to \mathscr{G}_N$; this extends directly to a ring homomorphism of $\mathbb{C}(\mathscr{G}_{N-1})$ into $\mathbb{C}(\mathscr{G}_N)$ which we also denote by j. From (2.2.11) and (2.2.5)–(2.2.8) we see that the image $j(\mathscr{G}_{N-1}) \subset \mathscr{G}_N$ commutes with β_N; therefore the image $q_N j[\mathbb{C}(\mathscr{G}_N)] \subset R_N$ is contained in Y_N and we may define a ring homomorphism

$$k_N : \mathbb{C}(\mathscr{G}_{N-1}) \to S_N$$

by $k_N = p q_N j$.



Lemma 2.4.28. *The map k_N is onto S_N. The ideal I_{N-1} of $\mathbb{C}(\mathscr{G}_{N-1})$ is contained in the kernel of k_N, so that k_N may be factored through a map*

$$\underline{k}_N : R_{N-1} \to S_N$$

with $k_N = \underline{k}_N q_{N-1}$. $\underline{k}_N$ is an isomorphism.

Proof. We denote by primes variables referring to the graph G_{N-1}. Note that

$$B'_i = B_i \qquad (i = 0 \ldots N-1)$$

but that

$$A' = -1 - \frac{1+A}{1+B_N}.$$

The verifications of the first two statements of the Lemma are straightforward. $\underline{k}_N$ is an isomorphism because, by induction, R_{N-1} is a simple ring, and S_N is not trivial.

Now our induction assumption and Lemma (2.4.28) imply that any element $\underline{b}(\chi) \in R_N$, with $N \in \chi$, may be expressed as a linear combination of the elements

$$\underline{b}(\chi_1)\underline{a}\underline{b}(\chi_2) \qquad (\chi_1, \chi_2 \in \mathscr{M}). \tag{2.4.29}$$

Clearly, this also holds for any $\underline{b}(\chi)$ with χ non-empty. An application of Lemma (2.4.13) then completes the proof.

It is interesting to note that if we require that the representation $\mathscr{L}$ of $\mathscr{G}_N$ satisfy all conditions of § 2.3, but consider the constants c_∞, A, and B_i of (2.3.2) and (2.3.7)–(2.3.9) as independent variables (not connected with the λ's), the relations imply that these constants must satisfy (2.3.12).

§ 3. Specialization of Parameters

3.1. Introduction

In this section we consider three specializations of the generic integral (2.1.18) studied in § 2. The first of these is the convergent integral (2.1.1) with the dimension of space-time m set equal to two. (The convergent case $m=3$, $N=2$ could be studied similarly.) The second is the integral obtained from (2.1.18) by setting all the masses equal; we also study (2.1.1) in this equal mass case. Finally, we study the integrals obtained from (2.1.18) and (2.1.1) by setting some of the masses equal to zero.

In § 3.2 we establish certain algebraic results which we will need in this analysis. These are applied in § 3.3–§ 3.5.

3.2. Rank of the Matrix v

In (2.4.25) we defined a $(2^N-1)\times(2^N-1)$ matrix $v=[v_\varrho^\chi]$ $(\chi, \varrho \in \mathscr{M})$. In this section we determine the row rank of this matrix when the λ_i's have values corresponding to (2.1.1) and the set of linear relations which hold between the row vectors v^χ. In this case $B_i=0$ $(i=1, \ldots N)$, and (2.2.25) reduces to

$$v_\varrho^\chi=\begin{cases}0 & \text{if} \quad \chi \cup \varrho=\Omega \quad \text{or} \quad \chi \cap \varrho \neq \phi, \\ 1 & \text{otherwise}.\end{cases} \tag{3.2.1}$$

Note that $v_\varrho^\chi=0$ if $|\chi|+|\varrho| \geqq N$.

Lemma 3.2.2. *For any $\chi, \varrho \in \mathscr{M}$ with $|\varrho|<N-|\chi|$, and any k with $|\varrho| \leqq k<N-|\chi|$,*

$$v_\varrho^\chi=\binom{N-|\chi|-|\varrho|}{k-|\varrho|} \sum_{\substack{\psi \supset \varrho \\ |\psi|=k}} v_\psi^\chi .$$

Proof. Clear.

The matrix v may be partitioned into submatrices

$$v_j^k=\{v_\varrho^\chi \mid |\chi|=k,|\varrho|=j\} .$$

We have already noted that $v_j^k=0$ if $k+j \geqq N$. We will be interested in particular in the submatrices v_j^k with $k+j=N-1$.

Lemma 3.2.3. *The row rank of v is equal to the sum of the row ranks of the matrices v_j^k with $k+j=N-1$.*

Proof. Suppose that

$$\sum_{|\chi|=k} a_\chi v_\varrho^\chi=0 \quad(|\varrho|=N-k-1) \tag{3.2.4}$$

is a linear relation on the rows of v^k_{N-k-1}. Then by Lemma 3.2.2, this relation holds for all ϱ; that is, it gives a relation on the rows of v. Conversely, let

$$\sum_{\chi \in \mathcal{M}} a_\chi v^\chi_\varrho = 0 \qquad (\varrho \in \mathcal{M}) \tag{3.2.5}$$

be a relation on the rows of v. Then for $|\varrho| = N-1$, this reduces to

$$\sum_{|\chi|=0} a_\chi v^\chi_\varrho = 0 .$$

Again from Lemma 3.2.2, this holds for all ϱ, so that (3.2.5) becomes

$$\sum_{|\chi|>0} a_\chi v^\chi_\varrho = 0 .$$

By induction we obtain

$$\sum_{|\chi|=k} a_\chi v^\chi_\varrho = 0$$

for all $k \leqq N-1$; that is, the relation (3.2.5) is the sum of relations on the rows of the matrices v^k_{N-k-1}.

Lemma 3.2.6. *The matrix* v^k_j $(k+j \leqq N-1)$ *has maximum possible rank, i.e.*

$$\operatorname{rank} v^k_j = \min\left[\binom{N}{k}, \binom{N}{j}\right].$$

Proof. Since v is symmetric, we may assume without loss of generality that $\binom{N}{j} \leqq \binom{N}{k}$. We proceed by induction on N; the result is clear for $N=2$. For the induction step we have two cases:

Case 1. $\binom{N}{j} < \binom{N}{k}$.

We partition $v^k_j(N)$ into 4 submatrices $w_1 \dots w_4$

$$w_1 = [v^\chi_\varrho]\ 1 \in \chi,\ 1 \notin \varrho, \qquad w_2 = [v^\chi_\varrho]\ 1 \in \chi,\ 1 \in \varrho ,$$
$$w_3 = [v^\chi_\varrho]\ 1 \notin \chi,\ 1 \notin \varrho, \qquad w_4 = [v^\chi_\varrho]\ 1 \notin \chi,\ 1 \in \varrho .$$

Note that $w_2 = 0$, and $w_1 = v^{k-1}_j(N-1)$, $w_4 = v^k_{j-1}(N-1)$.

By the induction assumption both w_1 and w_4 have maximum possible rank. The assumption of case 1 $\binom{N}{j} < \binom{N}{k}$ implies

$$\binom{N-1}{j} \leqq \binom{N-1}{k-1} \quad \text{and} \quad \binom{N-1}{j-1} \leqq \binom{N-1}{k}$$

so

$$\operatorname{rank} w_1 = \binom{N-1}{j} \qquad \operatorname{rank} w_4 = \binom{N-1}{j-1}.$$

This gives rank $v_j^k(N) = \binom{N-1}{j-1} + \binom{N-1}{j} = \binom{N}{j}$, which completes the induction step.

Case 2. $\binom{N}{j} = \binom{N}{k}\left[\text{i.e., } j = k \leqq \left[\frac{N}{2}\right]\right]$.

We partition $v_j^k(N)$ into 4 submatrices $w_1 \dots w_4$ as in case 1, and note that w_3 is nonsingular, since $w_3 = v_j^k(N-1)$ unless N odd and $k = \left[\frac{N}{2}\right]$, when

$$v_\varrho^\chi = \delta_\varrho^{\Omega_1 - \chi} \quad \text{for} \quad 1 \notin \chi,\ 1 \notin \varrho \quad |\chi| = |\varrho| = \left[\frac{N}{2}\right].$$

We now define a new matrix x by $x_\varrho^\chi = v_\varrho^\chi$ for $1 \notin \chi$ and

$$x_\varrho^\chi = (N-2k)\, v_\varrho^\chi - \sum_{j \notin \chi} v_\varrho^{\chi - \{1\} \cup \{j\}} \quad \text{for} \quad 1 \in \chi.$$

Then if $1 \in \chi$, $x_\varrho^\chi = 0$ for $1 \notin \varrho$, and for $1 \in \varrho$

$$x_\varrho^\chi = -(N-2k+1)\, v_{\varrho - \{1\}}^{\chi - \{1\}}.$$

We thus obtain a partitioned matrix x equivalent to v_j^k whose blocks $x_1 \dots x_4$ satisfy

$$x_1 = 0, \qquad x_2 = -(N-2k+1)\, v_{j-1}^{k-1}(N-1),$$
$$x_3 \text{ is nonsingular}.$$

By the induction assumption the square matrix x_2 has maximum possible rank. This gives rank $v_j^k(N) = \binom{N-1}{k} + \binom{N-1}{k-1} = \binom{N}{k}$.

Lemma 3.2.6 implies

$$\operatorname{rank} v = 2^N - \binom{N}{[N/2]}. \tag{3.2.7}$$

Note that for $\binom{N}{k} \leqq \binom{N}{N-1-k}$ the rows of the matrix v_{N-1-k}^k are linearly independent. For $\binom{N}{k} > \binom{N}{N-1-k}$ they satisfy $\left|\binom{N}{N-1-k} - \binom{N}{k}\right|$ independent linear relations, and by Lemma 3.2.3 these relations also hold as relations on the complete rows v^χ with $|\chi| = k$.

We now write down a set of linear relations on the rows of v and show that any linear relation on the rows of v is a consequence of these.

Lemma 3.2.8. *For any integer* $p \leqq \left[\frac{N}{2}\right]$ *and any subset of* $2p$ *distinct indices chosen from* $\{1, \dots, N\}$ $j(i,s)$ $1 \leqq i \leqq p$, $s = 0$ *or* 1

$$\sum_{s_1 \dots s_p = 0, 1} (-1)^{\Sigma s_i}\, v^{\Omega - \{j(1,s_1), \dots, j(p,s_p)\}} = 0. \tag{3.2.9}$$

Before proceeding to the proof we write down examples of (3.2.9) *for* $p = 1, 2$.

$$\begin{aligned} p = 1: &\quad v^{\Omega_i} = v^{\Omega_j} \qquad \forall\, i, j \\ p = 2: &\quad v^{\Omega_{ij}} + v^{\Omega_{kl}} = v^{\Omega_{il}} + v^{\Omega_{kj}} \end{aligned} \tag{3.2.10}$$

where $\Omega_{rs} = \Omega - \{r\} - \{s\}$.

Proof. We consider the ϱ component of (3.2.9). There are 3 cases.

1. ϱ contains $j(i, 0)$ and $j(i, 1)$ for some i, $1 \leqq i \leqq p$.

Then each term v_ϱ^χ in $(3.2.9)_\varrho$ has $\chi \cap \varrho \neq \phi$ and vanishes.

2. ϱ does not contain either $j(i, 0)$ or $j(i, 1)$ for some i, $1 \leqq i \leqq p$.

Then two terms in the sum $(3.2.9)_\varrho$ which differ only in the value of s_i are numerically equal but have opposite sign so the sum gives zero.

3. For each i, $1 \leqq i \leqq p$, ϱ contains just one of $j(i, 0)$, $j(i, 1)$.

Then $|\varrho| \geqq p$ so each term in $(3.2.9)_\varrho$ is zero by the remark preceding Lemma 3.2.2.

Lemma 3.2.11. *There are at least* $\binom{N}{p} - \binom{N}{p-1}$ *independent relations in the set* (3.2.9).

Proof. The relations (3.2.9) span a vector space $R(N, p)$; we must show $\dim R(N, p) \geqq \binom{N}{p} - \binom{N}{p-1}$. The proof is by induction on N; the proof is obvious for $N = 2$ from (3.2.10). For general N we define a map $f : R(N-1, p) \to R(N, p)$ by

$$f[\Sigma a_\chi v^\chi] = \sum a_\chi v^{\chi \cup \{N\}}$$

and a map $g : R(N, p) \to R(N-1, p-1)$ by

$$g[\Sigma a_\chi v^\chi] = \sum_{\chi \ni N} a_\chi v^\chi .$$

Then f is an injection, g is onto, and $gf = 0$, so that

$$\begin{aligned} \dim R(N, p) &\geqq \dim R(N-1, p) + \dim R(N-1, p-1) \\ &\geqq \binom{N-1}{p} - \binom{N-1}{p-1} + \binom{N-1}{p-1} - \binom{N-1}{p-2} \\ &= \binom{N}{p} - \binom{N}{p-1} . \end{aligned}$$

From Lemma 3.2.11 and the remark following Lemma 3.2.6 we see that the relations of Lemma 3.2.8 are a complete set of relations on the rows of v. This implies that the sequence

$$0 \to R(N-1, p) \xrightarrow{f} R(N, p) \xrightarrow{g} R(N-1, p-1) \to 0$$

is actually exact.

3.3. Dimension of V_N

We now discuss the algebra $\mathscr{A}_N$ in the case in which the λ_i have values corresponding to (2.1.1). Care must be taken in this specialization; in particular, the elements (2.4.29) (or more precisely their images under $\mathscr{L}$) no longer span $\mathscr{A}_N$. However, the formulae for the matrix representation

$$\mathscr{L}(x)\psi_\varrho = \sum_{\chi \in \mathscr{M}} \mathscr{L}(x, \lambda)_{\chi\varrho}\psi_\chi \qquad \varrho \in \mathscr{M},\ x \in \mathbb{C}(\mathscr{G}_N) \tag{3.3.1}$$

continue to hold when we specialize the values of the λ_i. (The matrix elements $\mathscr{L}(x, \lambda)_{\chi\varrho}$ are continuous in λ.) But note that the vectors $\{\psi_\varrho\}$ do not remain linearly independent when we specialize the λ_i. We obtain this reduction in the dimension of V_N by application of the *separation principle* (Lemma 3.3.3).

Suppose that f is a homogeneous function of n complex variables, of degree d, which satisfies

(1) f is holomorphic in the universal covering of the complement of an algebraic variety

$$L \subset \mathbb{C}^n,$$

(2) if z_0 is a point not in L, the germs of f with center z_0 span a finite dimensional vector space $V(z_0)$,

(3) if z_1 is a regular point of L, $\ell(z)=0$ a local equation of L and U a neighbourhood of z_1 sufficiently small that the cut region

$$U' = \{(z) : z \in U, \quad \arg \ell(z) \neq 0\}$$

is simply connected, there is an integer p such that for any branch f' of $f \mid U'$, $[\ell(z)]^p f'(z)$ is bounded in U'.

Remark. Conditions (1)-(3) are essentially the conditions defining a function of Nilsson class [14], except that we have imposed a weaker form of the growth condition (3).

Lemma 3.3.2. *Under the above assumptions f' has a decomposition*

$$f'(z) = \sum_{k,\varrho} [\ell(z)]^\varrho [\log \ell(z)]^k A_{k,\varrho}(z). \tag{3.3.2}$$

In (3.3.2) the summation is over a finite set of pairs (k, ϱ), k a non-negative integer and ϱ a complex number and the functions $A_{k,\varrho}(z)$ are holomorphic in U.

Proof. $\pi_1(U-L)$ is infinite cyclic with generator α (say) and acts on the vector space $V(z_2)$ where z_2 is a base point in $U-L$. (3.3.2) is obtained by reducing the matrix $\mathscr{L}(\alpha)$ to Jordan canonical form (cf. [1]).

As in § 2.1 we may consider the representation $\mathscr{L}$ of $\pi_1(\mathbb{C}^n - L; z_0)$ defined on $V(z_0)$ by analytic continuation.

Lemma 3.3.3 *(separation principle). If in addition to conditions* (1)-(3) *formulated above, the homogeneous function f satisfies*

(4) *no exponent ϱ in any local decomposition* (3.3.2) *of a branch of f about a regular point of L is a negative integer, then the relation*

$$\mathscr{L}(\alpha)\Phi = \Phi \qquad \forall\, \alpha \in \pi_1(\mathbb{C}^n - L; z_0)$$

implies that the function $\Phi \in V(z_0)$ is either zero, if d is not a non-negative integer, or a homogeneous polynomial of degree d, if d is a non-negative integer.

Proof. Φ evidently defines a single valued function on $\mathbb{C}^n - L$. Lemma 3.3.3 then follows from the growth condition (3) which forces $\Phi(z)$ to be rational and (4) which rules out the possibility of polar singularities on L.

We have already remarked that our integral (2.1.18) satisfies (1) and (2). When the λ's are specialized, conditions (3) and (4) follow from the Picard-Lefschetz theorem.

Lemma 3.3.4. *If $\sum_{\chi \in \mathscr{M}} d_\chi v^\chi = 0$ is a linear relation on the row vectors of the matrix v of* § 3.2, *the corresponding relation $\sum_{\chi \in \mathscr{M}} d_\chi \psi_\chi = 0$ holds in V_N.*

Proof. It is sufficient to consider the case in which the linear relation is one of those obtained in Lemma 3.2.8. From the separation principle it is enough to show

$$a\Big(\sum_{\chi \in \mathscr{M}} d_\chi \psi_\chi\Big) = 0, \qquad b_i\Big(\sum_{\chi \in \mathscr{M}} d_\chi \psi_\chi\Big) = 0 \qquad 1 \leqq i \leqq N.$$

These relations may be directly verified using the explicit form of the coefficients d_χ corresponding to (3.2.9).

We now show that there are no more relations among the vectors Ψ_χ. Let W denote the vector space $\mathbb{C}^{2^N-1}$, and let W' be the subspace

$$W' = \Big\{ d_\chi \,\Big|\, \sum_{\chi \in \mathscr{M}} d_\chi v^\chi = 0 \Big\}.$$

Then the vectors v^χ span the dual space $(W/W')^* \subset W^*$. Let $f: W \to V_N$ be defined by

$$f(\{d_\chi\}) = \sum_{\chi \in \mathscr{M}} d_\chi \Psi_\chi.$$

f is onto. Then (2.4.24) becomes

$$\mathscr{L}[b(\psi)ab(\chi)]\, f(w) = (v^\chi \cdot w)\, f(u^\psi). \tag{3.3.5}$$

Lemma 3.3.6. *The relations of Lemma 3.3.4 exhaust the relations among the vectors Ψ_ϱ.*

Proof. This is precisely the statement that the kernel of f is W' (we know $\ker f \supset W'$ by Lemma 3.3.4). Suppose then $w \in \ker f$, but $w \notin W'$. Since the v^χ's span $(W/W')^*$, there is some $\varrho \in \mathcal{M}$ with $(v^\varrho \cdot w) \neq 0$; applying (3.3.5) with $\chi = \varrho$, $\psi = \Omega_{ij}$ gives

$$0 = (v^\varrho \cdot w)\, f(u^{\Omega_{ij}}),$$

so that

$$f(u^{\Omega_{ij}}) = [\psi_{\Omega_i} + \psi_{\Omega_j}] = 0 .$$

But again from (3.3.5) (with $\psi = \phi$)

$$\mathcal{L}(a)\,\Psi_{\Omega_i} = f(u^\phi) = \mathcal{L}(a)\,\Psi .$$

Thus $\mathcal{L}(a)\Psi = 0$, contradicting (2.3.22).

Corollary 3.3.7. *The dimension of* V_N *is* $2^N - \binom{N}{[N/2]}$.

Proof. This follows from Lemmas (3.3.4) and (3.3.7), and Eq. (3.2.7).

As we have seen in § 2 the representation $\mathcal{L}(\lambda)$ of $\mathcal{G}$ is irreducible for generic λ. However, in the specialized case under discussion the representation is reducible. The reducibility of the representation may be anticipated by considering the integral (2.1.0). Each discontinuity of $I(s)$ is given by integrating the differential form which appears in (2.1.0) over a suitable cycle. The cycles which arise in this way are of a particular form – they are coboundaries of cycles which lie on the intersection of one or more of the pole varieties

$$S_i = \{(k) : k_i^2 + s_i = 0\} .$$

The Leray residue calculus may thus be used to express the discontinuities as integrals in which one or more of the propagator poles $(k_i^2 + s_i)^{-1}$ in (2.1.0) is replaced by a δ-function [8]. We may define a sequence

$$V_N = W^0 \supset W^1 \supset \cdots \supset W^N \supset \phi \tag{3.3.8}$$

of subspaces of V_N:

$W^k = \{\mathcal{I} \in V_N : \mathcal{I}$ has a representation as a linear combination of discontinuities, each one of which can be written as an integral in momentum space in which k of the propagator poles are replaced by δ-functions$\}$. The subspaces W^k of (3.3.8) are invariant under $\mathcal{L}$ so $\mathcal{L}$ is reducible.

We now define (from a purely algebraic point of view) a sequence V^k of subspaces of V_N, which are invariant under $\mathcal{L}$. Presumably this sequence can be identified with the sequence (3.3.8).

Lemma 3.3.9. *There is a unique minimal subspace of* V_N *invariant under* $\mathcal{L}$.

Proof. Let V be an invariant subspace of V_N, and x a non-zero vector in V. Then we may find a vector v^χ such that $v^\chi \cdot f^{-1}(x) \neq 0$ (where $f: W \to V_N$ is defined above Lemma 3.3.6), and hence

$$\mathscr{L}[b(\psi)ab(\chi)]x = [v^\chi \cdot f^{-1}(x)]\, f(u^\psi) \neq 0 \tag{3.3.10}$$

for all $\psi \in \mathscr{M}$. V thus contains all the vectors $f(u^\psi)$ and hence their span. It is easy to check that this subspace $V_{\min}$ of V is invariant.

For each $\chi \in \mathscr{M}$ we can write

$$\begin{aligned} \mathscr{L}(a(\chi))\psi_\varrho &= \mathscr{L}(\beta(\chi)a\beta(\chi)^{-1})\psi_\varrho \\ &= \ell^\chi_\varrho(\Sigma m^\chi_\mu \psi_\mu)\,, \end{aligned} \tag{3.3.11}$$

where

$$\ell^\chi = \sum_{\psi \subset \chi} (-1)^{|\chi|} v^\psi\,, \tag{3.3.12}$$

and

$$m^\chi = \sum_{\psi \subset \chi} u^\psi\,. \tag{3.3.13}$$

From (3.3.13) it follows that $V_{\min} = \operatorname{span}\{f(m^\chi),\ \chi \in \mathscr{M}\}$, i.e. the span of the discontinuities for the leading Landau variety. Thus certainly

$$V_{\min} \subset W^N$$

which motivates the definition $V^N = V_{\min}$. For k, $0 \leqq k \leqq N-1$ we define V^k to be the minimal invariant subspace of V_N containing all vectors ψ_χ with $|\chi| = k$. Clearly $V^k \subset W^k$.

Lemma 3.3.14. *For*

$$k \geqq \left[\frac{N+1}{2}\right] \quad V^k = V_{\min}\,.$$

For

$$k < \left[\frac{N+1}{2}\right] \quad \dim(V^k - V^{k+1}) = \binom{N}{k} - \binom{N}{k-1}.$$

$$\dim V_{\min} = 2^N - \binom{N}{[(N+1)/2]} - \binom{N}{[(N+1)/2]}.$$

This Lemma is readily proved with the help of Lemma 3.2.6.

On W^N, and hence on $V_{\min} \subset W^N$, there is a natural scalar product defined by

$g_1 \cdot g_2 =$ Kronecker index of a pair of cycles $\underline{g}_1, \underline{g}_2$ on

$S_1 \cap \dots \cap S_N$ defining the discontinuities g_1, g_2 of $I(s)$.

The scalar product is symmetric for N even, antisymmetric for N odd. This scalar product on $V_{\min}$ may be recovered from our algebraic results. We may regard the scalar product as defining a linear map τ of $V_{\min}$ into its dual space. From the Picard-Lefschetz theorem we know that for each $\chi \in \mathcal{M}$

$$\tau(f(m^\chi)) = c_\chi \ell^\chi, \tag{3.3.15}$$

where ℓ^χ is to be regarded as defining an element of $V^*_{\min}$ via

$$\ell^\chi \cdot x = \ell^\chi \cdot f^{-1}(x) \qquad x \in V_{\min}$$

and c_χ is some constant to be determined. The symmetry property of the scalar product noted above gives

$$\tau(f(m^{\chi_1})) \cdot f(m^{\chi_2}) = (-1)^N \tau(f(m^{\chi_2})) \cdot f(m^{\chi_1}).$$

But a direct computation gives

$$\ell^{\chi_1} \cdot m^{\chi_2} = (-1)^N \ell^{\chi_2} \cdot m^{\chi_1} \tag{3.3.16}$$

so if we normalize τ by taking $c_\phi = 1$ we have $c_\chi = 1$ for all χ. If for some constants a_χ

$$\Sigma a_\chi f(m^\chi) = 0$$

(3.3.16) shows that for any ψ

$$\begin{aligned} \Sigma a_\chi \ell^\chi \cdot f(m^\psi) &= \Sigma a_\chi \ell^\chi \cdot m^\psi \\ &= (-1)^N \Sigma a_\chi \ell^\psi \cdot m^\chi = 0, \end{aligned}$$

i.e. that $\Sigma a_\chi \ell^\chi$ defines the element $0 \in V^*_{\min}$. The map τ defined by (3.3.15) with $c_\chi = 1$ may thus be extended to a linear map $\tau: V_{\min} \to V^*_{\min}$ as required by homological considerations.

3.4. Equal Mass Case

We now consider the integrals (2.1.18) and (2.1.1) in the case where all internal lines have the same mass:

$$s' = s_1 = s_2 = \cdots = s_N. \tag{3.4.1}$$

Let $\mathbb{C}^2$ denote the subspace of $\mathbb{C}^{N+1}$ specified by (3.4.1); we thus wish to find the representation $\mathcal{L}_0$ of $\pi_1(\mathbb{C}^2 - \mathbb{C}^2 \cap L)$ generated by $I(s, \lambda) | \mathbb{C}^2$ [see (2.1.18)] or $I(s) | \mathbb{C}^2$ [see (2.1.1)]. Our base point B for $\pi_1(\mathbb{C}^{N+1} - L) = \mathcal{G}_N$ was chosen to lie in $\mathbb{C}^2$; thus if we also use B as a base point for $\pi_1(\mathbb{C}^2 - \mathbb{C}^2 \cap L)$ there is a natural map

$$e: \pi_1(\mathbb{C}^2 - \mathbb{C}^2 \cap L) \to \mathcal{G}_N.$$

We clearly have

$$\mathcal{L}_0 = \mathcal{L} e,$$

so that finding the representation $\mathscr{L}_0$ is equivalent to finding e. Note that even if our sole objective were to construct the monodromy group in the equal mass case, it would still be useful to consider the case of unequal masses. For the fact that the Feynman integral regarded as a function in s, s' is a restriction of a function analytic in s, s_i $1 \leqq i \leqq N$ gives us additional information on its monodromy ring which is expressed by the factorization of $\mathscr{L}_0$ through e.

Now $\mathbb{C}^2 \cap L$ is given by

$$s' = 0$$
$$s_0 = k^2 s' \qquad (k = N, N-2, \ldots; k \geqq 0)$$
$$s_0 = 0 .$$

We let α_k $(k > 0)$ denote the generator of $\pi_1(\mathbb{C}^2 - \mathbb{C}^2 \cap L)$ corresponding to the line $s_0 = k^2 s'$ (see § 2.2), and β denote the generator for $s' = 0$; we do not need to consider the generator for $s_0 = 0$ since its image under $\mathscr{L}_0$ may be obtained from β, the α_k, and the loop at infinity (which is again represented by c_∞ times the identity). The line $s_0 = k^2 s'$ $(k > 0)$ is the intersection of all surfaces $L_{N+1}(\chi)$ (2.2.3) with $|\chi| = \dfrac{N-k}{2}$. Since the generators $\alpha(\chi)$ for these surfaces commute, we have

$$e(\alpha_k) = \prod_{|\chi| = (N-k)/2} \alpha(\chi) . \tag{3.4.2}$$

Similarly,

$$e(\beta) = \beta_1 \ldots \beta_N . \tag{3.4.3}$$

We now turn to the question of the dimension of the representation $\mathscr{L}_0$, that is, the dimension of the subspace V_N^0 of V_N spanned by vectors

$$\{\mathscr{L}_0(\gamma)\Psi \mid \gamma \in \pi_1(\mathbb{C}^2 - \mathbb{C}^2 \cap L)\} .$$

Theorem 3.4.4. *a) For generic* λ, $V_N^0 = V_N$, *so that* $\dim V_N^0 = 2^N - 1$. *b) For the* λ*'s specialized to correspond to* (2.1.1), V_N^0 *is spanned by the linearly independent vectors*

$$\Psi_k = \sum_{|\varrho| = k} \Psi_\varrho \qquad (k = 0, 1, \ldots N-1) \tag{3.4.5}$$

so that $\dim V_N^0 = N$.

We remark that the conclusion of b) is obtained if the λ's are only specialized to satisfy

$$\lambda_1 = \lambda_2 = \cdots = \lambda_N .$$

Proof. a) We will show that the orbit of Ψ under the action of β spans V_N. For $n \geqq 1$,

$$\mathscr{L}_0(\beta^n)\Psi = \sum_{\varrho \in \mathscr{M}} A(\varrho, n)\Psi_\varrho \tag{3.4.6}$$

where

$$A(\varrho, n) = \prod_{i \in \varrho} \left[\frac{(1+B_i)^n - 1}{B_i} \right]$$

as may be easily verified by induction. Thus it suffices to prove that the matrix $\{A(\varrho, n)\}$ $(\varrho \in \mathscr{M}, n = 1, \ldots, 2^N - 1)$ is nonsingular. But $A(\varrho, n)$ is equivalent to the matrix

$$\begin{aligned} A'(\varrho, n) &= \sum_{\chi \subset \varrho} \left(\prod_{\chi} B_i \right) A(\chi, n) \\ &= \left[\prod_{\varrho} (1 + B_i) \right]^n \end{aligned}$$

The determinant of A' is a Vandermonde determinant and is non-zero as long as

$$\prod_{\varrho} (1 + B_i) \neq \prod_{\varrho'} (1 + B_i)$$

for any distinct ϱ and ϱ' in $\mathscr{M}$.

b) In this case

$$B_1 = B_2 = \cdots = B_N = 0 \tag{3.4.7}$$

so that (3.4.6) becomes

$$\mathscr{L}_0(\beta^n)\Psi = \sum_{k=0}^{n=1} n^k \Psi_k \qquad (n \geqq 1).$$

The matrix

$$\{n^k\} \qquad (n = 1, \ldots N; k = 0, \ldots N-1)$$

has non-zero determinant, so the vectors Ψ_k all lie in V_N^0. It follows from Lemma 3.3.6 that these vectors are linearly independent. Finally, (3.4.7) implies that the operators $\mathscr{L}(\beta)$ and $\mathscr{L}(\alpha_k)$ are completely symmetric in the indices $1, \ldots N$. The vector space spanned by the Ψ_k is therefore invariant under these operators, and must coincide with V_N^0.

3.5. *The Zero Mass Case*

We now study the representations generated when the mass of one line in the graph G_N is set equal to zero. Since the amplitude $I(s, \lambda)$ is singular on the surface $\mathbb{C}^N = \{s \mid s_N = 0\}$, some care is needed in this discussion. For generic λ, the Picard-Lefschetz theorem implies that in a neighbourhood of any point of $\mathbb{C}^N - \mathbb{C}^N \cap L$ we may write

$$I(s, \lambda) = s_N^{-(1+\lambda_N)} R(s, \lambda) + S(s, \lambda), \tag{3.5.1}$$

where R and S are analytic in this neighbourhood. It is natural to study the analytic properties of both $R(s,\lambda)|\mathbb{C}^N$, a sort of residue of $I(s,\lambda)$, and $S(s,\lambda)|\mathbb{C}^N$, which is indeed the restriction of $I(s,\lambda)$ to $\mathbb{C}^N$ whenever $\operatorname{Re}\lambda_N < -1$.

Recall that Ψ is the germ of the physical sheet of $I(s,\lambda)$ defined at the base point $B=(1,\varepsilon,\dots\varepsilon)$. From (3.5.1) we see that the formulae

$$\Phi = \lim_{s_N\to 0} s_N^{(1+\lambda_N)} \frac{b_N}{B_N}\Psi, \tag{3.5.2}$$

$$\Theta = \lim_{s_N\to 0}\left(1-\frac{b_N}{B_N}\right)\Psi \tag{3.5.3}$$

give well defined functions at the base point $B=(1,\varepsilon,\dots\varepsilon,0)$ of $\mathbb{C}^N-\mathbb{C}^N\cap L$, which may be analytically continued throughout $\mathbb{C}^N-\mathbb{C}^N\cap L$. Φ and Θ generate representations

$$\mathscr{L}_1:\mathbb{C}(\mathscr{G}_{N-1})\to L(V_N^{(1)}),$$
$$\mathscr{L}_2:\mathbb{C}(\mathscr{G}_{N-1})\to L(V_N^{(2)}),$$

respectively.

Theorem 3.5.4. *There are natural isomorphisms*

$$f_1: V_N^{(1)}\to b_N V_N,$$
$$f_2: V_N^{(2)}\to\left(1-\frac{b_N}{B_N}\right)V_N,$$

and for $i=1,2$, *and* $y\in\mathscr{G}_{N-1}$,

$$\mathscr{L}_i(y)=f_i^{-1}\{\mathscr{L}j(y)\}f_i. \tag{3.5.5}$$

Here $j:\mathscr{G}_{N-1}\to\mathscr{G}_N$ *is the map of* § 2.2.

Proof. For $y\in\mathscr{G}_{N-1}$ we define

$$f_1[\mathscr{L}_1(y)\Phi]=[\mathscr{L}j](y)b_N\Psi,$$
$$f_2[\mathscr{L}_2(y)\Theta]=[\mathscr{L}j](y)\left[1-\frac{b_N}{B_N}\right]\Psi.$$

The above properties are then easily verified, using the fact that b_N commutes with $j(\mathscr{G}_{N-1})$. Note that (3.5.5) states essentially that $\mathscr{L}_i$ is given by restricting $\mathscr{L}$ to $b_N V_N$ or $\left(1-\frac{b_N}{B_N}\right)V_N$, for $i=1,2$ respectively.

We will not discuss these representations in detail, but will point out several immediate consequences of Theorem 3.5.4. Define

$$\Phi^\varrho=\mathscr{L}_1\left(\prod_\varrho b_i'\right)\Phi \quad (\varrho\subset\Omega_N),$$
$$\Theta_\varrho=\mathscr{L}_2\left(\prod_\varrho b_i'\right)\Theta$$

where again we have used primes to denote elements of $\mathbb{C}(\mathcal{G}_{N-1})$. The vector space $V_N^{(1)}$ has dimension $2^{N-1}-1$, and we may take as basis $\{\Phi_\varrho | \varrho \neq \Omega_N\}$ (note $\Phi_{\Omega_N}=0$). In fact, it follows from Lemma 2.4.28 that $\mathcal{L}_1$ is isomorphic to the standard representation associated with the graph G_{N-1}. The vector space $V_N^{(2)}$ has dimension 2^{N-1} with basis $\{\Theta_\varrho\}$. It may be shown that $\mathcal{L}_i[\mathbb{C}(\mathcal{G}_{N-1})]$ is a complete matrix ring for $i=1,2$, of dimension $2^{N-1}-1$ and 2^{N-1}, respectively.

We will not discuss the modifications of this behavior which occur when the λ's are specialized to correspond to (2.1.1). Such a discussion could easily be obtained by the techniques of this section, § 3.3, and § 4.

§ 4. The Renormalized Integrals

4.1. Introduction

In this section we discuss the analytic structure of the renormalized integral in the case in which the dimension m of space-time is set equal to 4. A similar discussion could be given for other values of m.

To define the renormalized integral for the graph G_N we use the method of analytic renormalization developed in [15]. We denote by $\mathcal{I} = \{\mathcal{I}_N\}$ an arbitrary generalized evaluator in the sense of [15][5], and by $\mathcal{I}^0$ the particular evaluator defined by

$$\mathcal{I}_N^0 F(\mu) = \sum_{\sigma \in S^N} \int\limits_{|\mu_1| = R_{\sigma(1)}} \cdots \int\limits_{|\mu_N| = R_{\sigma(N)}} \frac{F(\mu)\, d\mu_1 \ldots d\mu_N}{\mu_1 \ldots \mu_N}. \tag{4.1.1}$$

In (4.1.1) $F(\mu)$ is a function such that for some integer m

$$F(\mu)\Big(\prod_{\chi \in \mathcal{M} \cup \{\Omega\}} \Big(\sum_{i \in \chi} \mu_i\Big)\Big)^m \tag{4.1.2}$$

is holomorphic in the neighbourhood of $\mu = 0$. The summation is over all permutations of $1, \ldots, N$ and $R_1, \ldots, R_N$ are small positive real numbers satisfying

$$R_i > R_1 + \cdots + R_{i-1} \quad \text{for} \quad 2 \leqq i \leqq N. \tag{4.1.3}$$

The integral (2.1.18) is convergent for

$$\sum_{i=0}^{N} \operatorname{Re} \lambda_i + (N+2) > 0 \tag{4.1.4}$$

---

[5] The evaluator $\mathcal{I}$ must satisfy the following additional condition. Let $F(\mu)$ be a function as in (4.1.1), and let $F_z(\mu_1, \ldots, \mu_N) = F(z\mu_1, \ldots, z\mu_N)$, for any $z \in \mathbb{C}$. Then $\mathcal{I} F_z = \mathcal{I} F$.

and defines for s in the neighbourhood of the base point B a function holomorphic in λ in this region. However, the value of λ corresponding to $m=4$

$$\lambda_0 = 0 \quad \lambda_i = -2 \qquad 1 \leqq i \leqq N \tag{4.1.5}$$

does not lie in the region of convergence. In [15] the function

$$J(s,\lambda) = I(s,\lambda)|_{\lambda_0=0} \tag{4.1.6}$$

is considered, and it is shown that for s in the neighbourhood of B this function is the restriction to the region

$$\sum_{i=1}^{N} \operatorname{Re}\lambda_i + (N+2) > 0 \tag{4.1.7}$$

of a function $E(s,\lambda)$ meromorphic in the entire space $\mathbb{C}^N$ of λ. We will need the following more precise statement about the behaviour of $E(s,\lambda)$ in the neighbourhood of the point (4.1.5): if $\mu_i = 2\pi i(\lambda_i + 2)$, $E(s,\lambda)$ admits a representation

$$E(s,\lambda) = \sum_{\sigma \in A^N} \frac{E_\sigma(s,\mu)}{(\mu_{\sigma(1)} + \cdots + \mu_{\sigma(N)}) \ldots (\mu_{\sigma(1)} + \mu_{\sigma(2)})} . \tag{4.1.8}$$

Here the summation is over all even permutations of $\{1, \ldots, N\}$ and $E_\sigma(s,\mu)$ is holomorphic in μ in the neighbourhood of $\mu = 0$. The physical sheet of the renormalized amplitude is then defined to be

$$\mathscr{I} E(s,\lambda) = F(s) . \tag{4.1.9}$$

The extension construction can be applied also to the integrals which define other sheets of $J(s,\lambda)$. If $\gamma \in \mathscr{G}_N$ we obtain in this way the commutative diagram

$$\begin{array}{ccc}
J(s,\lambda) & \xrightarrow{T(\gamma)} & T(\gamma)J(s,\lambda) \\
\Big\downarrow {\scriptstyle \text{analytic continuation in } \lambda} & & \Big\downarrow {\scriptstyle \text{analytic continuation in } \lambda} \\
E(s,\lambda) & \xrightarrow{T(\gamma)} & T(\gamma)E(s,\lambda) \\
\Big\downarrow {\scriptstyle \mathscr{I}} & & \Big\downarrow {\scriptstyle \mathscr{I}} \\
F(s) & \xrightarrow{T(\gamma)} & T(\gamma)F(s)
\end{array} \tag{4.1.10}$$

$T(\gamma)$ denotes the operation of analytic continuation along some representative loop for γ. As in § 2.1 we may introduce the vector space $V_{N,\mathscr{I}}$ spanned by germs of $F_N(s)$ with center B and consider the representation

$$\mathscr{L}_{\mathscr{I}} : \mathscr{G}_N \to L(V_{N,\mathscr{I}}) = GL(d,\mathbb{C}) \qquad d = \dim V_{N,\mathscr{I}}$$

defined by analytic continuation. (We show below that $\dim V_{N,\mathscr{I}}$ is finite.)

To determine $\mathscr{L}_{\mathscr{I}}$ we start from the results established in § 2 for the generic case, which according to the upper square of (4.1.10) are valid also for generic λ in the neighbourhood of $\mu=0$. The vectors ψ_χ $\chi\in\mathscr{M}$ span $V_N(\lambda)$, but it does not follow that the vectors $\mathscr{I}\psi_\chi$ $\chi\in\mathscr{M}$ span $V_{N,\mathscr{I}}$. For when we express an analytic continuation of $E(s,\lambda)$ along some path as a linear combination of ψ_χ the coefficients c_χ depend on the parameters λ and

$$\mathscr{I}(c_\chi(\mu)\psi_\chi) \text{ may not be equal to } c_\chi(0)\mathscr{I}\psi_\chi .$$

A simple example should make this point clear and illustrate the technique of constructing $\mathscr{L}_{\mathscr{I}}$. Consider the function

$$\frac{z^\mu}{\mu}. \tag{4.1.11}$$

For $\mu \neq 0$ this function is holomorphic in $\mathbb{C}-\{0\}$. Its germs over a nonsingular point span a one-dimensional space on which the infinite cyclic group $\pi_1(\mathbb{C}-\{0\})$ acts. If γ denotes the generator of this group the representation is given by

$$\mathscr{L}(\gamma)\psi = (\exp(2\pi i\mu))\psi . \tag{4.1.12}$$

The operation $\mathscr{I}$ applied to (4.1.11) gives

$$\log z . \tag{4.1.13}$$

The germs of this function span a vector space of dimension 2, with basis

$$\psi_1 = \mathscr{I}\psi = \log z \quad\text{and}\quad \psi_2 = \mathscr{I}(\mu\psi) = 1 . \tag{4.1.14}$$

The action of $\pi_1(\mathbb{C}-\{0\})$ on ψ_1, ψ_2 may be deduced from (4.1.12)

$$\begin{aligned}\mathscr{L}_{\mathscr{I}}(\gamma)\psi_1 &= \mathscr{I}(\mathscr{L}(\gamma)\psi) = \mathscr{I}(\exp(2\pi i\mu)\psi)\\ &= \psi_1 + 2\pi i\psi_2 ,\\ \mathscr{L}_{\mathscr{I}}(\gamma)\psi_2 &= \mathscr{I}(\mu\mathscr{L}(\gamma)\psi) = \mathscr{I}(\mu\exp(2\pi i\mu)\psi)\\ &= \psi_2\end{aligned}$$

since $\mathscr{I}(\mu^k\psi)=0$ for $k\geqq 2$.

Returning to the discussion of the Feynman amplitudes, we denote by $M(\chi)$ the set of all monomials

$$\mu^m = \prod_{i\in\chi}\mu_i^{m_i} \tag{4.1.15}$$

in the variables $\{\mu_i\, i \in \chi\}$. Here m_i, $i \in \chi$ are non-negative integers and we use the customary multi-index notation $m = (m_i, i \in \chi)$

$$|m| = \Sigma m_i \qquad m! = \prod_i m_i! \tag{4.1.16}$$

Set

$$A(\chi, m) = \mathscr{I}(\mu^m \psi_\chi)\,. \tag{4.1.17}$$

From our results for the generic case it is clear that these vectors span $V_{\mathscr{I},N}$ – if we take our expression for $\mathscr{L}(a)$, $\mathscr{L}(b_i)$ applied to ψ_χ as a sum of vectors $\Sigma c_\varrho(\mu)\psi_\varrho$, multiply by μ^m and apply $\mathscr{I}$ we obtain corresponding expressions for $\mathscr{L}_\mathscr{I}(a)$, $\mathscr{L}_\mathscr{I}(b_i)$ applied to $A(\chi, m)$ as a sum of vectors $A(\chi', m')$. (The explicit formulae are given in § 4.2.) $V_{\mathscr{I},N}$ has finite dimension in view of Lemma 4.1.18.

Lemma 4.1.18. *$A(\chi, m) = 0$ unless*
i) $|\chi| + |m| < N$,
ii) $m \in M(\chi')$.

Proof. For $\chi = \phi$ this follows immediately from (4.1.8). For $\chi \neq \phi$ we need a corresponding decomposition of ψ_χ. In momentum space ψ_χ may be written as a repeated integral over the momentum vectors k_i $i \in \chi'$ and then over the k_i $i \in \chi$. The first integration gives the physical sheet of the graph $G_{N-|\chi|}$ and the second integration is over a compact region. If we substitute for the first integral its decomposition (4.1.8) we obtain for ψ_χ the decomposition

$$\psi_\chi(s, \mu) = \sum_{\sigma \in A^{N-|\chi|}} \frac{E^\chi_\varrho(s, \mu)}{(\mu_{\sigma(1)} + \cdots + \mu_{\sigma(N-|\chi|)}) \cdots (\mu_{\sigma(1)} + \mu_{\sigma(2)})}\,, \tag{4.1.19}$$

where the summation is over even permutations of the indices in χ'. The statements of Lemma 4.1.18 follow immediately from 4.1.19.

The only essential point in the determination of $\mathscr{L}_\mathscr{I}$ remaining is to decide what linear relations hold between the vectors $A(\chi, m)$. We exhibit in § 4.2 a number of these linear relations but we are not able by our algebraic methods to determine the dimension of $V_{N,\mathscr{I}}$ because there are a number of vectors in $V_{N,\mathscr{I}}$ representing single valued functions and we cannot decide how many of these may be zero.

A comparison between the method of analytic renormalization and the method of subtraction yields the following

Lemma 4.1.20. *For $|m| \geqq 1$ $A(\chi, m)$ is a polynomial in s_0.*

Proof. For $\chi = \phi$ we note that the proof of the equivalent of the method of subtraction and that of analytic renormalization gives

$$\psi = E(s_0, \lambda) = \sum_{j=0}^{\omega(G)} \frac{s_0^j}{j!} \left(\frac{d^j}{ds_0^j} E(s, \lambda) \right)\Bigg|_{s_0 = 0} + \mathscr{R}E(s, \lambda)\,, \tag{4.1.21}$$

where $\mathscr{R}E(s,\lambda)$ is holomorphic in λ for $\mu=0$. Multiplying (4.1.21) by μ^m and applying $\mathscr{I}$ we obtain

$$A(\phi, m) = \sum_{j=0}^{\omega(G)} \frac{s_0^j}{j!}\left(\left(\frac{d^j}{ds_0^j} E(s,\lambda)\right)\bigg|_{s_0=0}\right) \tag{4.1.22}$$

a polynomial in s_0 of degree $\leqq \omega(G)$.

The proof for $\chi \neq \phi$ is reduced to the preceding case by the argument used in the proof of Lemma 4.1.18.

We will obtain more precise information on the functional nature of the $A(\chi, m)$ in § 4.3.

4.2. The Linear Relations on the $A(\chi, m)$

Lemma 4.2.1. *For all $\psi \in \mathscr{M}$ and multi-indices m, $0<|m|<N-1-|\psi|$ we have*

$$\begin{aligned}\sum_{\substack{m' \in M(\psi') \\ 1 \leqq |m'| \leqq N-1-|\psi|-|m|}} & (-1)^{|m'|} \frac{1}{m'!} A(\psi, m+m') \\ + \sum_{\eta \supsetneqq \psi} \sum_{\substack{m' \in M(\eta') \\ 0 \leqq |m'| \leqq N-1-|\eta|-|m|}} & (-1)^{|m'|+|\eta-\psi|} \frac{1}{m'!} A(\eta, m+m') = 0\,.\end{aligned} \tag{4.2.2}$$

Proof. For all $\psi \in \mathscr{M}$ the vector $\mathscr{L}(b(\psi)a)\Psi \in V_N(\lambda)$ can be represented as an integral over a compact contour in momentum space, or in α-space over a contour which does not bound onto any $\alpha_i=0$ plane. It therefore has no divergences as $\mu \to 0$, i.e. it defines a function holomorphic in μ in the neighbourhood of $\mu=0$. Thus for all m with $|m|>0$ we have from (2.4.24)

$$\mathscr{I}\left\{\mu^m \sum_{\eta \in \mathscr{M}} u_\eta^\psi \psi_\eta\right\} = 0\,. \tag{4.2.3}$$

Substituting the explicit form of the coefficients as functions of μ given by (2.4.26) and (2.3.11), (2.3.12) we obtain (4.2.2).

Examples. For $N=2$ Lemma 4.2.1 gives us no relation.

For $N=3$ we obtain 3 relations ($m=\{i\}$, $\psi=\phi$)

$$-\sum_{j \neq i} \mathscr{I}(\mu_i \mu_j \psi) = \sum_j \mathscr{I}(\mu_i \mu_j \psi)\,, \tag{4.2.4}$$

We have now expressed in the above Lemma the fact that discontinuities of Ψ across the leading Landau singularity have no divergences, and in (4.1.29) the fact that discontinuities Ψ_χ across one or more of the internal mass singularities L_i have only divergences corresponding to certain subgraphs of G_N. It remains to decide what divergences may appear in a discontinuity taken across the second-type singularity L_0.

In the case $m=2$ $\mathscr{L}(b_0^j)V_N \subset V_{\min}$ the subspace spanned by discontinuities across the leading singularity. This does not mean, however, that the functions $\mathscr{L}(b_0^j)\psi$ have no divergences in the case $m=4$. In fact the pinch corresponding to the second-type singularity $s_0=0$ is given by

$$d(\alpha)=0 \qquad \sum_{i=1}^{N} \alpha_i s_i = 0\,. \tag{4.2.5}$$

For $N=2,3$ this does not intersect the $\alpha_i=0$ planes so $\mathscr{L}(b_0^j)\psi$ has no divergences, i.e. is holomorphic in μ for $\mu=0$. But for $N>3$ it has non-zero intersection with the sets

$$\{(\alpha): \alpha_i = 0 \ \ \forall i \in \chi\} \tag{4.2.6}$$

provided $|\chi| \leqq N-3$. $\mathscr{L}(b_0^j)\psi$ thus has a corresponding decomposition

$$\mathscr{L}(b_0^j)\psi = \Sigma \frac{H_{i_1 \dots i_{N-3}}(s,\mu)}{(\Sigma_{i\in\Omega_{i_1}}\mu_i)\ \cdots\ (\Sigma_{i\in\Omega_{i_1\dots i_{N-3}}}\mu_i)}\,, \tag{4.2.7}$$

where the summation is over all ordered subsets containing $N-3$ indices, and the functions H are holomorphic in μ at $\mu=0$. (4.2.7) can be rewritten in an equivalent form by using (2.4.5) and noting that all terms involving a define functions holomorphic in μ and may be absorbed into the functions H. The resulting formula no longer contains j. A similar discussion may be given also for $\mathscr{L}(b_0^j)\psi_\chi$ $\chi \neq \phi$.

We conclude this section by showing how the decomposition formulae (4.1.8), (4.1.29), (4.2.7) may be used to obtain further linear relations on the $A(\chi, m)$.

Consider for definiteness (4.1.8). We obtain a linear relation

$$\Sigma c_m A(\phi, m) = 0 \tag{4.2.8}$$

if we can find a polynomial

$$\Sigma c_m \mu^m = P(\mu) \tag{4.2.9}$$

of degree $N-1$ such that for all $\sigma \in A^N$

$$\mathscr{I}\left(\frac{P(\mu)}{(\mu_{\sigma(1)}+\cdots+\mu_{\sigma(N)})\dots(\mu_{\sigma(1)}+\mu_{\sigma(2)})}\right)=0\,. \tag{4.2.10}$$

For

$$\mathscr{I}\left(\frac{P(\mu)\,E_\sigma(s,\mu)}{(\mu_{\sigma(1)}+\cdots+\mu_{\sigma(N)})\dots(\mu_{\sigma(1)}+\mu_{\sigma(2)})}\right)$$
$$= E_\sigma(s,0)\,\mathscr{I}\left(\frac{P(\mu)}{(\mu_{\sigma(1)}+\cdots+\mu_{\sigma(N)})\dots(\mu_{\sigma(1)}+\mu_{\sigma(2)})}\right).$$

(4.2.10) gives a set of linear equations on the coefficients c_m. A first sight there are too many equations at least for large N and one would expect only trivial solutions. But by looking for solutions with some symmetry and exploiting the symmetry properties of $\mathscr{I}$ we can show that there are non-trivial solutions for any N and any choice of $\mathscr{I}$. It appears difficult to determine how many solutions exist in general. Note that the coefficients in the linear equations for the c_m depend on $\mathscr{I}$ so that to obtain the linear relations (4.2.8) explicitly in a particular case we must choose a particular $\mathscr{I}$, say $\mathscr{I} = \mathscr{I}^0$. For $N = 2, 3$ we obtain $1, 3$ relations (4.2.8) respectively.

4.3. The Functional Form of the $A(\chi, m)$

We distinguish four types of functions:

A. homogeneous polynomials in s of degree $\dfrac{\omega(G_N)}{2}$,

B. multilinear functions of $\ln s_i$ $1 \leqq i \leqq N$ with coefficients of type A

$$f = \sum_{\chi} c(\chi) \prod_{i \in \chi} (\ln s_i) \qquad c(\chi) \in A .$$

We call $\max|\chi|$ the logarithmic degree of f.

C. Polynomials in s of degree $\dfrac{\omega(G_N)}{2} + 1$, divided by s_0,

D. multilinear functions of $\ln s_i$ $1 \leqq i \leqq N$ with coefficients of type C.

It will follow from the results in this section that $V_{N,\mathscr{I}}$ has an invariant subspace K such that the induced representation of $\mathscr{G}_N$ on $V_{N,\mathscr{I}}/K$ is isomorphic with the representation of $\mathscr{G}_N$ obtained in the case $m = 2$ (§ 3.3).

Lemma 4.3.1. *If* $|\chi| + |m| = N - 1$ *and* $|m| \geqq 1$, $A(\chi, m)$ *is of type A.*

Proof. From (4.1.28) we have

$$b_i A(\chi, m) = A(\chi \cup \{i\}, m) = 0 \qquad 1 \leqq i \leqq N .$$

Also $a A(\chi, m) = 0$ since $A(\chi, m)$ is polynomial in s_0 (4.1.30). $A(\chi, m)$ is thus a single-valued function. The only pole of the renormalized amplitude is the second-type singularity $s_0 = 0$ which is a simple pole. But this pole cannot appear in $A(\chi, m)$. To complete the proof of the lemma we must show that $A(\chi, m)$ is homogeneous of degree $\dfrac{\omega(G_N)}{2}$ in s. Note the renormalized amplitude is not homogeneous in s. However, if we write down the equation which expresses the homogeneity of $\mu^m \psi_\eta$ for generic μ in the neighbourhood of $\mu = 0$ and apply $\mathscr{I}$ to both sides we find that

$A(\eta, m)$ is homogeneous modulo terms in $A(\eta, m')$ with $|m'| > |m|$

$$\mu^m \psi_\eta(cs) = c^{\omega(G_N)/2 - \Sigma_i \mu_i/2\pi i}\, \psi_\eta(s)\,,$$

$$A(\eta, m)(cs) = c^{\omega(G_N)/2} A(\eta, m)(s) \tag{4.3.2}$$

$$+ c^{-\Sigma_i \mu_i/2\pi i} \sum_{\substack{m' \\ 1 \leqq |m'| \leqq N-1-|\eta|-|m|}} \frac{(-1)^{|m'|}}{(2\pi i)^{|m'|}} \frac{(\log c)^{|m'|}}{m'!} A(\eta, m+m')(s)\,.$$

In the present case $\eta = \chi$ these additional terms vanish so $A(\chi, m)$ is homogeneous.

We remark that (4.3.2) shows that the sheets $A(\eta, m)(s)$ of the renormalized amplitude satisfy

$$A(\eta, m)(cs) = O\big(|c|^{\omega(G_N)/2} (\log |c|)^{N-1-|\eta|-|m|}\big)\,. \tag{4.3.3}$$

Lemma 4.3.4. *If $N-1-|\chi|-|m| = j$ and $|m| \geqq 1$ $A(\chi, m)$ is of type B with logarithmic degree j in the variables $\ln s_i$ $i \in \chi'$.*

The proof is by induction on j starting from the case $j=0$ (Lemma 4.3.1). Consider

$$B(\chi, m) = \sum_{\psi \subset \chi'} (-1)^{|\psi|} \prod_{i \in \psi} \left(\frac{1}{2\pi i} \ln s_i\right) A(\chi \cup \psi, m)\,.$$

For $1 \leqq i \leqq N$ we have $b_i B(\chi, m) = 0$. Since $B(\chi, m)$ is polynomial in s_0 it follows that it is single valued and nonsingular for all s. By (4.3.3) it is polynomially bounded, and hence a polynomial. By (4.3.2) this polynomial is homogeneous of degree $\omega(G_N)/2$. By the induction assumption $A(\chi \cup \psi, m)$ is of type B with logarithmic degree $j - |\psi|$ in the variables $\ln s_k$ $k \in (\chi \cup \psi)'$.

It is interesting to note that if we use (4.3.2) to examine the homogeneity of $B(\chi, m)$ we obtain for $B(\chi, m)(cs) - c^{\omega(G_N)/2} B(\chi, m)(s)$ a polynomial in $\log c$ multiplied by $c^{\omega(G_N)/2}$ which must vanish identically in $\log c$. Equating to zero the coefficients of $(\log c)^\ell$ $1 \leqq \ell \leqq j$ we obtain a further set of identities on the $A(\chi, m)$.

We now consider the question: do the identities on the vectors ψ_χ which we established in the case $m=2$ ((3.2.8), (3.3.4), (3.3.6)) persist in the renormalized case? This is answered by Lemma 4.3.6.

Lemma 4.3.6 *If $\sum_{\substack{\chi \in \mathcal{M} \\ |\chi| = j}} d_\chi \psi_\chi = 0$ is a linear relation on the vectors $\psi_\chi \in V_N$ in the case $m = 2$, $N - \left[\frac{N}{2}\right] \leqq j \leqq N-1$*

$$\sum_{\chi \in \mathcal{M}, |\chi| = j} d_\chi A(\chi, \phi) = D$$

is a function of type D of logarithmic degree $N-1-j$ in the $\ln s_i$.

The proof is by induction on $k = N-1-j$. For $k=0$ we have $b_i D = 0$ $1 \leqq i \leqq N$. Also $aD = 0$ in view of 4.2.1. D is therefore a single-valued function in s, and can at worst have the $s_0=0$ pole. Thus $s_0 D$ is everywhere nonsingular. From (4.3.2), (4.3.3) it follows that it is a homogeneous polynomial of degree $\omega(G_N)+1$. For $k>0$ we note again that $aD=0$ in view of 4.2.1. Also from the explicit form of the identities in the case $m=2$ we see that for all ψ $|\psi|\geqq 1$ $b(\psi)D$ is a sum of the same kind with $k'<k$ and hence by the induction assumption a function of type D of logarithmic degree $N-1-j-|\psi|$. We consider

$$B=D+\sum_{\psi:|\psi|\geqq 1}(-1)^{|\psi|}\prod_{i\in\psi}\left(\frac{1}{2\pi i}\ln s_i\right)b(\psi)D\,.$$

$b_i B=aB=0$ so B is a single-valued function. As in the case $k=0$ it follows that $s_0 B$ is a homogeneous polynomial of degree $\omega(G_N)+1$.

We remark that 4.3.6 cannot be improved to give D a polynomial in s_0. For an explicit calculation in the case $N=2$ gives

$$A(1,0)-A(2,0)=\frac{s_1-s_2}{s_0}\,.$$

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