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A Local Relativistic Boson Field with the $\lambda|\varphi|$ Interaction

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Abstract. We prove that the Heisenberg picture fields for a self interacting Boson field with the $\lambda|\varphi|$ interaction in four space time dimensions exists as weak limits of Heisenberg picture fields corresponding to the cut-off interactions.

1. Introductions

In an earlier paper [4], here after refered to as paper I, we studied self interacting Boson fields with interaction densities of the form $V(\varphi(x))$ in four space time dimensions, where V was a bounded continuous function with a bounded uniformly continuous first derivative. We proved in I that the Heisenberg picture fields existed as weak limits of the Heisenberg picture fields corresponding to the cut-off interactions.

The purpose of this paper is to show that similar methods as these used in I, may also be used to prove existence of Heisenberg picture fields for more singular interaction densities. For this reason we shall study the self interacting Boson field $\varphi(x)$ in four space time dimensions, with the formally local and relativistic invariant interaction

$$\lambda \int_{R3} |\varphi(x)|\, dx$$

where $|\varphi(x)|$ is the absolute value of $\varphi(x)$. As in I we introduce the cut-off interaction

$$V_{\varepsilon,r} = \lambda \int_{|x| \leqq r} |\varphi_\varepsilon(x)|\, dx\,,$$

where $\varphi_\varepsilon(x)$ is the momentum cut-off field, and prove that the Heisenberg picture fields corresponding to the cut-off interaction converges weakly as the cut-off is taken away.

2. The Cut-Off Interaction

We shall use the Fock space representation. The Fock space $\mathscr{F}$ is a Hilbert space where the elements are sequences of functions $f = \{f_0, f_1, \ldots\}$ where $f_n(p_1, \ldots, p_n)$ is a symmetric function of n-variables $p_1, \ldots, p_n$ with

p_i in R^3. The inner product in $\mathscr{F}$ is given by

$$(f, g) = \sum_{n=0}^{\infty} n! \int \cdots \int \bar{f}_n(p_1, \ldots, p_n)\, g_n(p_1, \ldots, p_n) \frac{dp_1}{\omega(p_1)} \cdots \frac{dp_n}{\omega(p_n)}$$

where $\omega(p) = (p^2 + m^2)^{\frac{1}{2}}$ and we shall assume that m, the mass of the free fields, is strictly positive. The annihilation operator $a(p)$ is defined by

$$(a(p) f)_n(p_1, \ldots, p_n) = (n+1)\, \omega(p)^{-\frac{1}{2}} f_{n+1}(p, p_1, \ldots, p_n).$$

The creation operator $a^*(p)$ is the formal adjoint of $a(p)$, and we have

$$[a(p), a^*(p')] = \delta(p - p').$$

The free energy operator H_0 is defined by

$$(H_0 f)_n(p_1, \ldots, p_n) = \sum_{i=1}^{n} \omega(p_i)\, f_n(p_1, \ldots, p_n).$$

H_0 is obviously self adjoint on its natural domain of definition D_0.

Let h be in $L_2(R^3)$. It is well known that $a(h) = \int a(p)\, h(p)\, dp$ and $a^*(h) = \int a^*(p)\, h(p)\, dp$ are closed operators with domains containing $D_{\frac{1}{2}}$, the domain of $H_0^{\frac{1}{2}}$, and on $D_{\frac{1}{2}}$ we have the estimate

$$\|a^\#(h)\| \leqq m^{-\frac{1}{2}} \|h\|_2 \, \|(H_0 + 1)^{\frac{1}{2}} \psi\|, \tag{2.1}$$

where $a^\#$ stands for a^* or a, and m is the mass of the free field.

Moreover $a^*(h)$ and $a(\bar{h})$ have the same domain of definition and are adjoints of each other, and on this domain $a^*(h) + a(\bar{h})$ is self adjoint. $a^*(h) + a(\bar{h})$ is also essentially self adjoint on D_0. We have also for h_1 and h_2 in $L_2(R^3)$ that $a^\#(h_2)$ maps D_0 into the domain of $a^\#(h_1)$, and on D_0 we have the following estimate

$$\|a^\#(h_1)\, a^\#(h_2)\| \leqq m^{-1} \|h_1\|_2 \, \|h_2\|_2 \, \|(H_0 + 1)\, \psi\|. \tag{2.2}$$

For the proofs of these statements and more details on the creation-annihilation operators see Ref. [3].

The Boson field $\varphi(x)$ is defined by

$$\varphi(x) = 2^{-\frac{1}{2}} (2\pi)^{-\frac{3}{2}} \int (e^{ipx} a(p) + e^{-ipx} a^*(p))\, \omega(p)^{-\frac{1}{2}}\, dp.$$

As for the creation-annihilation operators $a^\#(p)$, we have also that $\varphi(x)$ is a improper operator valued function, or if we like operator valued distribution; and only after integrating with smooth enough test functions do we get operators. From what is said above about the creation-annihilation operators, we see that

$$\varphi(h) = \int h(x)\, \varphi(x)\, dx$$

is a closed operator if $\omega^{-\frac{1}{2}}\hat{h}$ is in L_2, where $\hat{h}$ is the Fourier transform of h. We also see that if h is real then $\varphi(h)$ is self adjoint and also essentially self adjoint on D_0.

Let g be a positive C^∞-function on R^3, with support contained in the open unite ball $B=\{x; |x|<1\}$, and such that $\int g(x)\,dx=1$. Define $g_\varepsilon(x)=\frac{1}{\varepsilon}g\left(\frac{x}{\varepsilon}\right)$, then $g_\varepsilon(x)$ has support in $B_\varepsilon=\{x; |x|<\varepsilon\}$ and $g_\varepsilon(x)$ converge to the Dirac δ-distribution as ε tends to zero. As in paper I we define the momentum cut-off field $\varphi_\varepsilon(x)$ by

$$\varphi_\varepsilon(x)=\int g_\varepsilon(x-y)\,\varphi(y)\,dy\,.$$

From what is said above about the field $\varphi(x)$, we see that the momentum cutt-off field $\varphi_\varepsilon(x)$ is a self adjoint operator which is essentially self adjoint on D_0 for all ε and all x.

Let $U(x)$ be the unitary group of space translations on $\mathscr{F}$. That is $U(x)$ is the strongly continuous group on $\mathscr{F}$ defined by

$$(U(x)\,f)_n(p_1,\ldots,p_n)=e^{i\sum_{j=1}^{n}xp_j}f_n(p_1,\ldots,p_n)\,.$$

Since H_0 commutes with $U(x)$, we see that $U(x)$ leaves D_0 as well as $D_{\frac{1}{2}}$ invariant. By (2.1) $D_{\frac{1}{2}}$ is contained in the domain of $\varphi_\varepsilon(x)$ for all ε and all x. Let $|\varphi_\varepsilon(x)|$ be the absolute value of $\varphi_\varepsilon(x)$, i.e. $|\varphi_\varepsilon(x)|=(\varphi_\varepsilon(x)^2)^{\frac{1}{2}}$. Since $\varphi_\varepsilon(x)$ and $|\varphi_\varepsilon(x)|$ have the same domains of definition, we get that $D_{\frac{1}{2}}$ is in the domain of $|\varphi_\varepsilon(x)|$ for all ε and all x, and (2.1) gives us the following estimation $D_{\frac{1}{2}}$

$$\|\,|\varphi_\varepsilon(x)|\,\psi\|\leqq C_\varepsilon\|(H_0+1)^{\frac{1}{2}}\psi\|\,, \tag{2.3}$$

where C_ε is a constant depending only on ε. From the definition of $\varphi(x)$ we get that

$$\varphi_\varepsilon(x)=U(-x)\,|\varphi_\varepsilon(0)|\,U(x)\,.$$

Let ψ be in $D_{\frac{1}{2}}$. Since $U(x)$ is strongly continuous and leaves $D_{\frac{1}{2}}$ invariant, we see that $U(x)\,\psi$ is a strongly continuous function contained in the domain of $|\varphi_\varepsilon(0)|$. Using now that $|\varphi_\varepsilon(0)|$ is closed we see that $|\varphi_\varepsilon(0)|\,U(x)\,\psi$ is strongly continuous, hence that $|\varphi_\varepsilon(x)|\,\psi$ is strongly continuous in x for all ψ in $D_{\frac{1}{2}}$.

Let ψ be in $D_{\frac{1}{2}}$, then we define the cut-off interaction $V_{\varepsilon,r}$ by

$$V_{\varepsilon,r}\psi=\lambda\int_{|x|\leqq r}|\varphi_\varepsilon(x)|\,\psi\,dx \tag{2.4}$$

where the integral is a strong integral. The strong integral in (2.4) exists since the integrand is strongly continuous and the domain of integration is compact.

From (2,3) we get the following estimate on $D_{\frac{1}{2}}$

$$\|V_{\varepsilon,r}\psi\| \leqq C\|(H_0+1)^{\frac{1}{2}}\psi\| \tag{2.5}$$

where C depends only on ε, r and λ. Since $(H_0+1)^{\frac{1}{2}}$ is infinitessimally small with respect to H_0, i.e. for any $a>0$ there exists a $b>0$ such that for all ψ in D_0

$$\|(H_0+1)^{\frac{1}{2}}\psi\| \leqq a\|H_0\psi\| + b\|\psi\|\,, \tag{2.6}$$

we see that $V_{\varepsilon,r}$ is infinitessimally small with respect to H_0 for all ε, r and λ. Therefore

$$H_{\varepsilon,r} = H_0 + V_{\varepsilon,r}$$

is self adjoint with the same domain of definition D_0 as H_0.

We now introduce the free Heisenberg picture fields

$$\varphi^t(h) = e^{-itH_0}\varphi(h)\,e^{itH_0}\,,$$

and in the same way we define

$$\varphi^t_\varepsilon(x) = e^{-itH_0}\varphi_\varepsilon(x)\,e^{itH_0}$$

and

$$V_{\varepsilon,r}(t) = e^{-itH_0}V_{\varepsilon,r}e^{itH_0}\,.$$

Since e^{itH_0} leaves $D_{\frac{1}{2}}$ invariant we see that $\varphi^t(h)$ and $\varphi^t_\varepsilon(x)$ are self adjoint operators with domain containing $D_{\frac{1}{2}}$. Since $V_{\varepsilon,r}$ is a symmetric operator with dense domain it is closable, and we will also write $V_{\varepsilon,r}$ for its closure. By (2.4) $D_{\frac{1}{2}}$ is contained in the domain of $V_{\varepsilon,r}$, hence $V_{\varepsilon,r}(t)$ is also a closed symmetric operator with domain containing $D_{\frac{1}{2}}$.

It follows from (2.2) that if h_1 and h_2 is in L_2, then $\varphi^t(h_2)$ maps D_0 into the domain of $\varphi^s(h_1)$. From the commutation relations the annihilation-creation operators we get that the following commutation relations are valied on D_0

$$[\varphi^s(h_1), \varphi^t(h_2)] = i\int\int dx\,dy\, h_1(x)\,h_2(y)\,\Delta(x-y, s-t)\,. \tag{2.7}$$

The function $\Delta(x,t)$ is given by $\Delta(x,0)=0$, and for $t\neq 0$

$$\begin{aligned}&\Delta(x,t) \qquad\qquad (2.8)\\ &= \operatorname{sgn}(t)\left[\frac{-1}{2\pi}\,\delta(x^2-t^2) + \frac{m}{4\pi}\,\theta(t^2-x^2)\cdot(t^2-x^2)^{-\frac{1}{2}}J_1\big(m(t^2-x^2)^{\frac{1}{2}}\big)\right].\end{aligned}$$

Since

$$V_{\varepsilon,r}(t) = \lambda \int\limits_{|x|\leqq r} |\varphi^t_\varepsilon(x)|\,dx \tag{2.9}$$

and $|\varphi_\varepsilon^t(x)|$ has the same domain as $\varphi_\varepsilon^t(x)$, we see that the domain of $V_{\varepsilon,r}(t)$ contains the intersection of the domains of $\varphi^t(h)$ with h in L_2. But since $\varphi^s(h_1)$ maps D_0 into the domain of $\varphi^t(h)$ if h and h_1 is in L_2, we get that for h in L_2, $\varphi^s(h)$ maps D_0 into the domain of $V_{\varepsilon,r}(t)$. We will also need the following lemma.

Lemma 1. *Let h be in L_2. Then $V_{\varepsilon,r}(t)$ maps D_0 into the domain of $\varphi^s(h)$, and $\varphi^s(h)$ maps D_0 into the domain of $V_{\varepsilon,r}(t)$. Moreover on D_0 the cummutator $[\varphi^s(h), V_{\varepsilon,r}(t)]$ is a bounded operator given by*

$$[\varphi^s(h), V_{\varepsilon,r}(t)] = i\lambda \int_{|x|\leq r} dx \int\int dy dz\, h(y)\, g_\varepsilon(x-z)\, \Delta(y-z, s-t)\, \mathrm{sgn}\,\varphi_\varepsilon^t(x)$$

where the integral over x is a strong integral.

Proof. We have already seen that $\varphi^s(h)$ maps D_0 into the domain of $V_{\varepsilon,r}(t)$. To prove that $V_{\varepsilon,r}(t)$ maps D_0 into the domain of $\varphi^s(h)$, we shall prove that $e^{-i\tau\varphi^s(h)} V_{\varepsilon,r}(t)\,\psi$ *si* strongly differentiable from the right at $\tau=0$ whenever ψ is in D_0. We have that

$$\begin{aligned}
&\frac{1}{\tau}(e^{-i\tau\varphi^s(h)} V_{\varepsilon,r}(t)\,\psi)\\
&\qquad = \frac{1}{\tau}(e^{-i\tau\varphi^s(h)} V_{\varepsilon,r}(t)\, e^{i\tau\varphi^s(h)} - V_{\varepsilon,r}(t))\, e^{-i\tau\varphi^s(h)}\psi\\
&\qquad\quad + V_{\varepsilon,r}(t)\frac{1}{\tau}(e^{-i\tau\varphi^s(h)}\psi - \psi)\,.
\end{aligned}$$

$\frac{1}{\tau}(e^{-i\tau\varphi^s(h)}\psi - \psi)$ converge strongly to $-i\varphi^s(h)\,\psi$, since D_0 is contained in the domain of $\varphi^s(h)$. We have already proved that $\varphi^s(h)$ maps D_0 into the domain of $V_{\varepsilon,r}(t)$. Using now that $V_{\varepsilon,r}(t)$ is a closed operator we see that $V_{\varepsilon,r}(t)\frac{1}{\tau}(e^{-i\tau\varphi^s(h)}\psi - \psi)$ converge strongly. To prove strong convergence of the first term, it is enough to prove that

$$\frac{1}{\tau}(e^{-i\tau\varphi^s(h)} V_{\varepsilon,r}(t)\, e^{i\tau\varphi^s(h)} - V_{\varepsilon,r}(t)) \tag{2.10}$$

is uniformly norm bounded and converge strongly, since $e^{-i\tau\varphi^s(h)}\psi$ is strongly continuous in τ.

From (2.9) we see that the expression above may be written as

$$\lambda \int_{|x|\leq r} dx \frac{1}{\tau}(|e^{-i\tau\varphi^s(h)}\varphi_\varepsilon^t(x)\, e^{i\tau\varphi^s(h)}| - |\varphi_\varepsilon^t(x)|)\,. \tag{2.11}$$

By using a variant of Lebesgues lemma on dominated convergence, we see that it is enough to prove that the integrand above is uniformly bounded in τ and x, and converge strongly for each fixed x. Using (2.7) we get

$$e^{-i\tau\varphi^s(h)}\varphi_\varepsilon^t(x)\,e^{i\tau\varphi^s(h)} = \varphi_\varepsilon^t(x) - i\tau[\varphi^s(h), \varphi_\varepsilon^t(x)] = \varphi_\varepsilon^t(x) + \tau\beta(x)$$

where

$$\beta(x) = \int\int dy dz\, h(y)\, g_\varepsilon(x-z)\, \Delta(y-z, s-t).$$

We may therefore write the integrand in (2.11) as

$$\frac{1}{\tau}\left(|\varphi_\varepsilon^t(x) + \tau\beta(x)| - |\varphi_\varepsilon^t(x)|\right). \tag{2.12}$$

By its definition $\varphi_\varepsilon^t(x)$ is unitarily equivalent to $\varphi_\varepsilon(x)$, and by the definition of $\varphi_\varepsilon(x)$ we have $\varphi_\varepsilon(x) = a^*(h) + a(\bar{h})$ for an h in L_2. Set $P = (2i\|h\|_2^2)^{-1}(a^*(h) - a(\bar{h}))$. Then $\varphi_\varepsilon(x)$ as well as P are self adjoint operators, which satisfy the commutation relation $[P, \varphi_\varepsilon(x)] = i$. By the uniqueness of the representation of the commutation relation we get $\varphi_\varepsilon(x)$ has absolutely continuous spectrum with constant multiplicity on the whole real line. Therefore to prove that (2.12) is uniformly bounded in τ and in x for $|x| \leqq r$, and converge strongly as τ tends to zero, it is enough to prove that

$$\frac{1}{\tau}\left(|\omega + \tau\beta(x)| - |\omega|\right) \tag{2.13}$$

is bounded as a function of ω uniformly in τ and in x for $|x| \leqq r$, and converge strongly as an operator on $L_2(d\omega)$ as τ tends to zero. But

$$\frac{1}{\tau}\left(|\omega+\tau\beta(x)| - |\omega|\right) = \begin{cases} -\beta(x) & \text{for } \omega \leqq -\tau\beta(x) \\ -\beta(x) + \dfrac{1}{\tau}(\omega + \tau\beta(x)) & \text{for } -\tau\beta(x) \leqq \omega \leqq \tau\beta(x) \\ \beta(x) & \text{for } \omega \geqq \tau\beta(x) \end{cases}$$

if $\beta(x) \geqq 0$ and we get a similar expression for $\beta(x) \leqq 0$. It follows from the definition of $\beta(x)$, that it is uniformly bounded for $|x| \leqq r$. Hence (2.13) is uniformly bounded in τ and in x for $|x| \leqq r$, and it tends pointwise to $\beta(x)\,\mathrm{sgn}\,\omega$ as τ tends to zero. By Lebesques lemma on dominated convergence we then get also that (2.13) converges strongly as an operator on $L_2(d\omega)$ to $\beta(x)\,\mathrm{sgn}\,\omega$, and we have proved that $V_{\varepsilon,r}(t)$ maps D_0 into the domain of $\varphi^s(h)$.

That the commutator $[\varphi^s(h), V_{\varepsilon,r}(t)]$ is bounded on D_0, follows from the formula for the commutator given in the lemma. It is therefore enough to prove this formula. Using that $V_{\varepsilon,r}(t)$ is closed and that $\varphi^s(h)$ maps D_0 into the domain of $V_{\varepsilon,r}(t)$ and that $V_{\varepsilon,r}(t)$ maps D_0 into the domain of $\varphi^s(h)$, we get that

$$e^{-i\tau\varphi^s(h)}V_{\varepsilon,r}(t)\,e^{i\tau\varphi^s(h)}\psi \tag{2.14}$$

is differentiable for ψ in D_0, and with derivative at $\tau=0$ given by $-i[\varphi^s(h), V_{\varepsilon,r}(t)]\,\psi$.

On the other hand we see that (2.10) converge to the derivative of (2.14) as τ tends to zero. But we have already proved that (2.10) converge strongly to $\lambda \int_{|x|\leqq r} \beta(x)\,\mathrm{sgn}\,\varphi_\varepsilon^t(x)\,dx$. Consulting the definition of $\beta(x)$ we see that the formula of the lemma is proved.

This completes the proof of Lemma 1.

Corollary 1. *Let h be in $L_1 \cap L_2$. Then*

$$\|[\varphi^s(h), V_{\varepsilon,r}(t)]\| \leqq C|\lambda|\,|t-s|^2\,\|h\|_1\,,$$

where C is a constant that depends only on the mass m of the free field.

Proof. The norm estimate of this corollary follows by a direct norm estimate of the expression for the commutator given in Lemma 1, and by using (2.8) together with the asymptotic behavior of J_1.

3. The Heisenberg Picture Fields

We define the Heisenberg picture fields for the cut-off interaction by

$$\varphi_{\varepsilon,r,t}(h) = e^{-itH_{\varepsilon,r}}\varphi(h)\,e^{itH_{\varepsilon,r}}\,.$$

Since $e^{itH_{\varepsilon,r}}$ leaves D_0 invariant, we have that for h in L_2, $\varphi_{\varepsilon,r,t}(h)$ is a self adjoint operator with domain containing D_0.

Lemma 2. *Let h be in $L_1 \cap L_2$. Then $\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)$ is a bounded operator and*

$$\varphi_{\varepsilon,r,t}(h) - \varphi^t(h) = i\int_0^t e^{-isH_{\varepsilon,r}}[\varphi^{t-s}(h), V_{\varepsilon,r}]\,e^{isH_{\varepsilon,r}}ds\,.$$

where the integral is a strong integral.

Proof. Let ψ be in D_0, then $e^{i(t-s)H_0}e^{isH_{\varepsilon,r}}\psi$ is strongly differentiable with respect to s, with derivative

$$e^{i(t-s)H_0}iV_{\varepsilon,r}e^{isH_{\varepsilon,r}}\psi\,.$$

By Lemma 1 this is in the domain of $\varphi(h)$, and since $\varphi(h)$ is a closed operator we get that $\varphi(h)\,e^{i(t-s)H_0}e^{isH_{\varepsilon,r}}\psi$ is strongly differentiable with derivative

$$\varphi(h)\,e^{i(t-s)H_0}iV_{\varepsilon,r}e^{isH_{\varepsilon,r}}\,.$$

From this we get that if ψ_1 and ψ_2 are in D_0, then

$$(e^{i(t-s)H_0}e^{isH_{\varepsilon,r}}\psi_1,\, \varphi(h)\,e^{i(t-s)H_0}e^{isH_{\varepsilon,r}}\psi_2)$$

is a differentiable function of s with derivative equal to

$$(\psi_1, e^{-isH_{\varepsilon,r}}[\varphi^{t-s}(h), i V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}}\psi_2) \tag{3.1}$$

Integrating (3.1) from zero to t we therefore get

$$\begin{aligned}&(\psi_1, (\varphi_{\varepsilon,r,t}(h) - \varphi^t(h))\,\psi_2)\\ &\qquad = \int_0^t ds(\psi_1, e^{-isH_{\varepsilon,r}}[\varphi^{t-s}(h), i V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}}\psi_2).\end{aligned}$$

By Lemma 1 the integral above gives rise to a bounded operator, and using now that D_0 is dense in $\mathscr{F}$ we therefore get that $\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)$ is bounded and

$$\varphi_{\varepsilon,r,t}(h) - \varphi^t(h) = i\int_0^t e^{-isH_{\varepsilon,r}}[\varphi^{t-s}(h), V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}} ds,$$

where the integral is a weak integral. From the formula in Lemma 1 we see that $[\varphi^{t-s}(h), V_{\varepsilon,r}]$ is uniformly bounded and depend strongly continuous on s. Since also $e^{isH_{\varepsilon,r}}$ is uniformly bounded and depend strongly continuous on s, we see that the integrand above is bounded and depends strongly continuous on s. Hence it is strongly integrable, and using now that the weak integral is equal to the strong integral whenever the strong integral exists, we see that we may replace the weak integral by the strong integral in the formula above. This proves Lemma 2.

Lemma 3. *Let h be in $L_1 \cap L_2$. Then $\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)$ is norm continuous in t, and the norm continuity is uniform in ε and r. Moreover we have the following estimate for the norm*

$$\|\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)\| \leqq C|\lambda|\, |t|^3\, \|h\|_1 .$$

where C depends only on the mass m of the free field.

Proof. The moreover part of the lemma follows from Corollary 1 and Lemma 2. By Lemma 2 we have for $t_1 \leqq t_2$

$$\begin{aligned}&\varphi_{\varepsilon,r,t_2}(h) - \varphi^{t_2}(h) - \varphi_{\varepsilon,r,t_1}(h) + \varphi^{t_1}(h)\\ &\qquad = i\int_0^{t_1} e^{-isH_{\varepsilon,r}}[\varphi^{t_2-s}(h) - \varphi^{t_1-s}(h), V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}} ds\\ &\qquad\quad + i\int_{t_1}^{t_2} e^{-isH_{\varepsilon,r}}[\varphi^{t_2-s}(h), V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}} ds .\end{aligned}$$

From the formula in Lemma 1 we see that the first integral tends in norm to zero uniformly in ε, r as t_1 tends to t_2 or t_2 tends to t_1, if h is in C_0^∞. By Corollary 1 the second integral tends also in norm to zero uniformly in ε and r. Hence we have proved that $\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)$ is norm continuous in t uniformly in ε and r if h is in C_0^∞. Since C_0^∞ is dense in $L_1 \cap L_2$ in the strong L_1 topology, we then get by the norm estimate of Lemma 3 which we have already proved, that $\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)$ is norm continuous uniformly in ε and r for all h in $L_1 \cap L_2$. This proves Lemma 3.

Theorem 1. *There exists a sequence ε_n tending to zero and a sequence r_n tending to infinity, such that for all h in $L_1 \cap L_2$ and all t, $\varphi_{\varepsilon_n,r_n,t}(h) - \varphi^t(h)$ converge weakly to a bounded operator $\varphi(h,t) - \varphi^t(h)$. Moreover $\varphi(h,t) - \varphi^t(h)$ is norm continuous in t, and satisfy the following norm estimate*

$$\|\varphi(h,t) - \varphi^t(h)\| \leqq C|\lambda|\,|t|^3\,\|h\|_1\,,$$

where C depends only on the mass m of the free field.

Proof. Let h be in $L_1 \cap L_2$ and ψ_1 and ψ_2 in $\mathscr{F}$. By Lemma 3 $(\psi_1, (\varphi_{\varepsilon,r,t}(h) - \varphi^t(h))\,\psi_2)$ is a family of functions of t that is uniformly bounded in ε, r for t on compact intervals. By the same lemma it is also an equicontinuous family of functions. The Ascoli theorem then gives us that there exist sequences $\varepsilon_{n'}$ tending to zero and $r_{n'}$ tending to infinity such that the corresponding functions converge uniformly on compact intervals. By passing to subsequences ε_n and r_n we get that $(\psi_1, (\varphi_{\varepsilon_n,r_n,t}(h) - \varphi^t(h))\,\psi_2)$ converge uniformly on compact intervals in t for a countable dense set of ψ_1 and ψ_2, and a countable set of h that is dense in $L_1 \cap L_2$ with respect to the strong L_1 topology. The uniform norm estimate of Lemma 3 then gives us convergence uniformly on compact intervals in t for all ψ_1, ψ_2 and h. This proves weak convergence. To prove that the limit is norm continuous in t, we use again Lemma 3. This lemma gives us that for any $\varepsilon > 0$ there exists a $\delta > 0$ independent of n such that

$$\|\varphi_{\varepsilon_n,r_n,t}(h) - \varphi^t(h) - \varphi_{\varepsilon_n,r_n,t+\tau}(h) + \varphi^{t+\tau}(h)\| \leqq \varepsilon$$

as soon as $|\tau| \leqq \delta$. Using now that the set of operators with norm smaller or equal to ε is weakly closed, we get that

$$\|\varphi(h,t) - \varphi^t(h) - \varphi(h,t+\tau) + \varphi^{t+}\tau(h)\| \leqq \varepsilon$$

as soon as $|\tau| \leqq \delta$. This proves the norm continuity. The norm estimate of the theorem follows from the norm estimate of Lemma 3 by using that the set of operators with norm smaller or equal to $C|\lambda|\,|t|^3\,\|h\|_1$ is weakly closed. This proves the theorem.

Remark 1. We will like to point out that the method used in this paper to prove Theorem 1 for an interaction density of the form $\lambda|\varphi(x)|$, would work just as well for any interaction density $V(\varphi(x))$, where $V(\alpha)$ is any piecewise linear function. Combining the method of this paper with those of paper I we see that Theorem 1 may be proved for any interaction density $V(\varphi(x))$, where $V(\alpha)$ is a continuous function, with a piecewise uniformly continuous and bounded derivative $V'(\alpha)$. In paper I we get somewhat stronger results than in this paper, and a carefull reader will have observed that the stronger results depend on the boundedness of $V(\alpha)$ in addition to the boundedness of $V'(\alpha)$.

Remark 2. Having proved in this paper and in paper I, that the Heisenberg picture fields exists as weak limits of the Heisenberg picture fields corresponding to the cut-off interaction, for a rather general class of interaction densities; we would naturally ask if these Heisenberg picture fields are trivial or not. That is if the weak limit of $\varphi_{\varepsilon_n, r_n, t}(h) - \varphi^t(h)$, which we have proved exists, is zero or not. The methods used in this paper and in paper I, are of course not good enough to give an answer to this question. A partial answer to this question may however be found by considering the two interaction densities for which one is able to compute the Heisenberg picture fields. Namely the linear and the quadratic interaction. That is

$$V(\varphi(x)) = \lambda\varphi(x) \quad \text{and} \quad V(\varphi(x)) = \lambda\varphi(x)^2 .$$

If we for these two interactions form $\varphi_{\varepsilon, r, t}(h) - \varphi^t(h)$ we will see by explicit calculations that the weak limits exist, and are different from zero. The explicit calculations also gives us that the Heisenberg picture field $\varphi(h, t)$ we get by taking the limit is the correct Heisenberg picture field. Namely in the linear case we get the free Heisenberg picture field plus a scalar, and in the quadratic case we get a free field with mass $m+\lambda$. Sice the calculations are straight forward we will not give them here but leave it to the reader to verify these statements.

Of course these two trivial examples does not prove anything about what happens in the general case, but they may be used as an indication that the Heisenberg picture fields which we have proved to exist in this paper and in paper I are not trivial.

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