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Boson Fields Under a General Class of Local Relativistic Invariant Interactions

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Abstract. We consider a boson field $\varphi(x)$ under an interaction of the form $\int_{R^3} V(\varphi(x))\,dx$, where $V(\alpha)$ is a bounded continuous real function of a real variable α. If $V(\alpha)$ has a uniformly continuous and bounded first derivative, we prove that the Heisenberg picture field exists as weak limits of the Heisenberg picture fields corresponding to the cut-off interaction.

1. Introduction

The object of this paper is to study a general class of quantum fields with a local relativistic invariant interaction in four space time dimensions. The fields will be self interacting boson fields, with energy operator of the form

$$H = H_0 + \int_{R^3} V(\varphi(x))\,dx\,.$$

H_0 is the free energy operator of a free boson field $\varphi(x)$ of strictly positive mass m. $V(\alpha)$ is a real function of a real variable α, such that $V(\alpha)$ is bounded, continuous and with a bounded and uniformly continuous first derivative.

In two space time dimensions Glimm [1] has investigated the case where $V(\alpha)$ is a polynomial containing only terms of even degree and a positive leading coefficient. For this case he proves that, after renormalization of the interaction by introducing the Wick product, the total energy with a space cut off interaction becomes a semi bounded symmetric operator on the Fock space. The case $V(\alpha) = \lambda\alpha^4$ and still in two space time dimensions, can be treated more thoroughly, as shown by Glimm and Jaffé [4]. Glimm was also able to treat the case $V(\alpha) = \lambda\alpha^4$ in three space time dimensions [2]. The author's reason for studying interactions given by bounded continuous functions instead of polynomials, is strictly that of mathematical convenience, and he hopes that may be in this way enough experience can be gained, so that later on one may be able to treat more realistic models.

The advantage from a mathematical point of view in studying interactions coming from bounded functions $V(\alpha)$, was also to some extent demonstrated in a previous paper [6], where the author was able to prove existence of the asymptotic fields for the cut-off interactions.

The idea behind this paper is first to introduce the cut-off interactions, and to study the Heisenberg picture fields for the cut-off interactions. Then we use compactness arguments to prove that the weak limits of these fields exists as the cut-off is removed. The idea of studying the weak limits of Heisenberg picture fields was also used by Glimm to remove the space cut-off for the Yukawa interaction with a momentum cut-off [3].

2. The Heisenberg Picture Fields

We shall use the Fock space representation. The Fock space $\mathscr{F}$ is a Hilbert space where the elements are sequences of functions $f=\{f_0, f_1, \ldots\}$ with $f_n = f_n(p_1, p_2, \ldots, p_n)$ a symmetrical function of n variables $p_1, \ldots, p_n$; $p_i \in R^3$. The inner product in $\mathscr{F}$ is given by

$$(f, g) = \sum_{n=0}^{\infty} n! \int \cdots \int \bar{f}_n(p_1, \ldots, p_n)\, g_n(p_1, \ldots, p_n) \frac{dp_1}{\omega(p_1)} \cdots \frac{dp_n}{\omega(p_n)}$$

where $\omega(p) = (p^2 + m^2)^{\frac{1}{2}}$ and $m > 0$. The annihilation operator $a(p)$ is defined by

$$(a(p) f)_n (p_1, \ldots, p_n) = (n+1)\, \omega(p)^{-\frac{1}{2}} f_{n+1}(p, p_1, \ldots, p_n)\,.$$

The creation operator $a^*(p)$ is the formal adjoint of $a(p)$, and we have

$$[a(p), a^*(p')] = \delta(p - p')\,.$$

The free energy operator H_0 is defined by

$$(H_0 f)_n (p_1, \ldots, p_n) = \sum_{i=1}^{n} \omega(p_i) f_n(p_1, \ldots, p_n)\,.$$

H_0 is obviously self adjoint on its natural domain of definition D_0.

For $h \in L_2(R^3)$, it is well known that $a(h) = \int a(p)\, h(p)\, dp$ and $a^*(h) = \int a^*(p)\, h(p)\, dp$ are closed operators with domains containing D_0, and that $a^*(\bar{h})$ is the adjoint of $a(h)$. Moreover $a^*(h) + a(\bar{h})$ is a self adjoint operator that is essentially self adjoint on D_0.

The field operators $\varphi(x)$ are given in terms of the annihilation-creation operators by

$$\varphi(x) = 2^{-\frac{1}{2}} (2\pi)^{-\frac{3}{2}} \int_{R^3} (e^{ipx} a(p) + e^{-ipx} a^*(p))\, \omega(p)^{-\frac{1}{2}}\, dp\,.$$

$\varphi(x)$ is of course just as $a(p)$ and $a^*(p)$ an improper operator function, or if we want an operator valued distribution; and only when $\varphi(x)$ is integrated against sufficiently smooth test functions, do we get operators. For real h we set

$$\varphi(h) = \int_{R^3} \varphi(x)\, h(x)\, dx .$$

From what is said above, about the annihilation-creation operators, we see that if $\hat{h}\omega^{-\frac{1}{2}}$ is in L_2, where $\hat{h}$ is the Fourier transform of h, then $\varphi(h)$ is a self adjoint operator which is essentially self adjoint on D_0.

Let g be in $C_0^\infty(R^3)$, such that $g \geqq 0$, $g(x) = g(-x)$, $\int g(x)\, dx = 1$ and g has support in the open sphere of radius 1 and center at the origin in R^3. Set $g_\varepsilon(x) = \varepsilon^{-1}\, g\left(\frac{x}{\varepsilon}\right)$, then g_ε has support in the sphere of radius ε, and g_ε tends to Dirac's δ-distribution as ε tends to zero. The cut-off field operators are defined by

$$\varphi_\varepsilon(x) = \int g_\varepsilon(x-y)\, \varphi(y)\, dy .$$

By what is said above about the field operators we see that $\varphi_\varepsilon(x)$ are self adjoint operators which are all essentially self adjoint on D_0.

Let $V(\alpha)$ be a bounded continuous real function. Then $V(\varphi_\varepsilon(x))$ is a bounded self adjoint operator, such that $\|V(\varphi_\varepsilon(x))\| \leqq \|V\|_\infty = \sup_\alpha |V(\alpha)|$.

Since

$$V(\varphi_\varepsilon(x)) = U(-x)\, V(\varphi_\varepsilon(0))\, U(x) ,$$

where $U(x)$ is the strongly continuous unitary group defined by

$$(U(x)f)_n\,(p_1, \ldots, p_n) = e^{i \sum_{n=1}^{n} x p_j} f_n(p_1, \ldots, p_n) ,$$

we see that $V(\varphi_\varepsilon(x))$ is strongly continuous in x.

We now define the cut-off interaction energy by

$$V_{\varepsilon,r} = \int_{|x| \leqq r} V(\varphi_\varepsilon(x))\, dx$$

where the integral is a strong integral. We see that $V_{\varepsilon,r}$ is a bounded self adjoint operator, and we have the following ε-independent estimate for its norm

$$\|V_{\varepsilon,r}\| \leqq \frac{4\pi}{3} r^3 \|V\|_\infty . \tag{2.1}$$

The cut-off energy operator is defined by

$$H_{\varepsilon,r} = H_0 + V_{\varepsilon,r} .$$

Since $V_{\varepsilon,r}$ is a bounded self adjoint operator we get that $H_{\varepsilon,r}$ is a self adjoint operator with the same domain D_0 as H_0.

Let h be real and in L_2, we then define

$$\varphi^t(h) = e^{-itH_0}\,\varphi(h)\,e^{itH_0}\,,$$
$$\varphi^t_\varepsilon(x) = e^{-itH_0}\,\varphi_\varepsilon(x)\,e^{itH_0}\,,$$
$$\varphi_{\varepsilon,r,t}(h) = e^{-itH_{\varepsilon,r}}\,\varphi(h)\,e^{itH_{\varepsilon,r}}\,.$$

We see that all operators defined above are self adjoint operators with domain containing D_0.

We define also

$$V_{\varepsilon,r}(t) = e^{-itH_0}\,V_{\varepsilon,r}\,e^{itH_0}\,.$$

It is well known that the product $\varphi^{t_1}(h_1)\,\varphi^{t_2}(h_2)$ is defined on the domain of H_0 and that on this domain

$$[\varphi^{t_1}(h_1),\ \varphi^{t_2}(h_2)] = \iint dx\,dy\,\Delta(x-y,\ t_1-t_2)\,h_1(x)\,h_2(y) \tag{2.2}$$

where

$$\Delta(x,t) = -i\varepsilon(t)\left[\frac{1}{2\pi}\delta(t^2-x^2) - \frac{m}{4\pi}\theta(t^2-x^2)\cdot(t^2-x^2)^{-\frac{1}{2}}J_1\big(m(t^2-x^2)^{\frac{1}{2}}\big)\right]$$

for $t \neq 0$ and $\Delta(x,0)=0$. (2.3)

From now on we shall assume that $V(\alpha)$ has a bounded uniformly continuous derivative $V'(\alpha)$.

Lemma 1. *Let h be in L_2 and assume that $V'(\alpha)$ is bounded and uniformly continuous. Then $V_{\varepsilon,r}(s)$ leaves the domain of $\varphi^t(h)$ invariant, the commutator of these two operators is a bounded operator given by*

$$[\varphi^t(h),\,V_{\varepsilon,r}(s)] = \int_{|x|\leqq r} dx \iint dz\,dy\,h(y)\,g_\varepsilon(x-z)\,\Delta(y-z,t-s)\,V'(\varphi^s_\varepsilon(x))\,.$$

Proof. We start with proving that $V_{\varepsilon,r}(s)$ leaves the domain of $\varphi^t(h)$ invariant. Let ψ be in the domain of $\varphi^t(h)$, then we shall prove that $e^{i\tau\varphi^t(h)}\,V_{\varepsilon,r}(s)\,\psi$ is strongly differentiable with respect to τ, and hence $V_{\varepsilon,r}(s)\psi$ is in the domain of $\varphi^t(h)$.

$$e^{i\tau\varphi^t(h)}\,V_{\varepsilon,r}(s)\,\psi = e^{i\tau\varphi^t(h)}\,V_{\varepsilon,r}(s)\,e^{-i\tau\varphi^t(h)}\,e^{i\tau\varphi^t(h)}\,\psi\,.$$

But $e^{i\tau\varphi^t(h)}\psi$ is strongly differentiable with respect to τ, hence it is enough

$$e^{i\tau\varphi^t(h)}\,V_{\varepsilon,r}(s)\,e^{-i\tau\varphi^t(h)} = \int_{|x|\leqq r} dx\,V\big(e^{i\tau\varphi^t(h)}\,\varphi^s_\varepsilon(x)\,e^{-i\tau\varphi^t(h)}\big)$$

is strongly differentiable with respect to τ.

Using (2.2) we get that this is equal to

$$\int_{|x|\leqq r} V\Big(\varphi^s_\varepsilon(x) + i\tau \iint dy\,dz\,h(y)\,g_\varepsilon(x-z)\,\Delta(y-z,t-s)\Big)\,.$$

Since $V(\alpha)$ has a bounded uniformly continuous derivative $V'(\alpha)$, we have that $\frac{1}{\tau}(V(\alpha+c\tau)-V(\alpha))$ tends to $cV'(\alpha)$ uniformly in α. Hence for any self adjoint operator A, $\frac{1}{\tau}(V(A+c\tau)-V(A))$ tends to $cV'(A)$ in norm, and the convergence is uniform in A where A range over the set of all self adjoint operators. Hence

$$V(\varphi_\varepsilon(x)+i\tau\iint dy\,dz\,h(y)\,g_\varepsilon(x-z)\,\Delta(y-z,t-s))$$

is norm differentiable in τ and the convergence of the difference to the derivative is uniform in x. Hence

$$\int_{|x|\leqq r} V(\varphi_\varepsilon(x)+i\tau\iint dy\,dz\,h(y)\,g_\varepsilon(x-z)\,\Delta(y-z,t-s))$$

is norm differentiable with respect to τ, and with derivative given by

$$i\int_{|x|\leqq r} dx\iint dy\,dz\,h(y)\,g_\varepsilon(x-z)\,\Delta(y-z,t-s)\,V'(\varphi_\varepsilon^s(x))\,.$$

This proves that $V_{\varepsilon,r}(s)\psi$ is in the domain of $\varphi^t(h)$. By differentiating the first formula of this proof we see that

$$\varphi^t(h)\,V_{\varepsilon,r}(s)\psi=V_{\varepsilon,r}(s)\,\varphi^t(h)\psi + \int_{|x|\leqq r} dx\iint dy\,dz\,h(y)\,g_\varepsilon(x-z)\,\Delta(y-z,t-s)\,V'(\varphi_\varepsilon^s(x))\,.$$

This gives us the formula for the commutator as given in the lemma. Using this formula together with the assumption that $V'(\alpha)$ is bounded, we get that the commutator is bounded. This proves the lemma.

Corollary 1. $\|[\varphi^t(h),V_{\varepsilon,r}(s)]\|\leqq C|t-s|^2\,\|V'\|_\infty\,\|h\|_1$ *where C depends only on the mass m of the free field.*

Proof. From Lemma 1 we get that

$$\|[\varphi^t(h),V_{\varepsilon,r}(s)]\|\leqq\|V'\|_\infty\int dz\,|\int h(y)\,\Delta(y-z,t-s)|$$

and the estimate in the corollary follows then from (2.3) and the asymptotic behaviour of J_1. This proves the corollary.

The Heisenberg picture fields corresponding to the cut-off interaction is given by $\varphi_{\varepsilon,r,t}(h)$. We intend to use the fact that the estimate of Corollary 1 is uniform in ε to select a sequence ε_n tending to zero such that $\varphi_{\varepsilon_n,r,t}(h)$ converge weakly for all t and r, and all h in $L_2\cap L_1$. The way we do this is by showing that Corollary 1 implies that $\varphi_{\varepsilon,r,t}(h)$ is equicontinuous as functions of r,t and h with respect to ε. Then we use the Ascoli theorem to pick out the sequence ε_n.

Let ψ be in D_0. Since D_0 is the domain of H_0 as well as $H_{\varepsilon,r}$, and also therefore $e^{isH_{\varepsilon,r}}$ leaves D_0 invariant, we get that $e^{-i(s-t)H_0}\,e^{isH_{\varepsilon,r}}\psi$

is strongly differentiable with respect to s, and the strong derivative is given by

$$e^{-i(s-t)H_0}\, i V_{\varepsilon,r}\, e^{isH_{\varepsilon,r}}\psi = i V_{\varepsilon,r}(s-t)\, e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi .$$

Let h be in L_2. Since D_0 is contained in the domain of $\varphi(h)$, we get from Lemma 1 that the derivative is in the domain of $\varphi(h)$. Using now that $\varphi(h)$ is closed we see that

$$\frac{d}{ds}\varphi(h)\, e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi = \varphi(h)\, i V_{\varepsilon,r}(s-t)\, e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi .$$

where the derivative is taken in the strong sense. From this we get that if ψ_1 and ψ_2 are in D_0 then

$$\begin{aligned}&(\psi_1, e^{-isH_{\varepsilon,r}}\, e^{i(s-t)H_0}\, \varphi(h)\, e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi_2)\\ &\qquad = (e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi_1,\ \varphi(h)\, e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi_2)\end{aligned}$$

is differentiable with respect to s and the derivative is given by

$$\begin{aligned}&(e^{-i(s-t)H_0}\, i V_{\varepsilon,r}\, e^{isH_{\varepsilon,r}}\psi_1,\ \varphi(h)\, e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi_2)\\ &\qquad + (e^{-i(s-t)H_0}\, e^{isH_{\varepsilon,r}}\psi_1,\ \varphi(h)\, e^{-i(s-t)H_0}\, i V_{\varepsilon,r}\, e^{isH_{\varepsilon,r}}\psi_2)\\ &\qquad = (\psi_1,\ e^{-isH_{\varepsilon,r}}[\varphi^{t-s}(h), i V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}}\psi_2) .\end{aligned}$$

By integrating the derivative we then get

$$(\psi_1, (\varphi_{\varepsilon,r,t}(h) - \varphi^t(h))\psi_2) = \int_0^t ds (\psi_1, e^{-isH_{\varepsilon,r}}[\varphi^{t-s}(h), i V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}}\psi_2)$$

Since the right hand side is a bounded operator by Lemma 1 and D_0 is dense in $\mathscr{F}$ we get

$$\varphi_{\varepsilon,r,t}(h) - \varphi^t(h) = \int_0^t ds\, e^{-isH_{\varepsilon,r}}[\varphi^{t-s}(h), i V_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}} , \tag{2.4}$$

where the integral on the right hand side is a weak integral.

Lemma 2. *Let h be in $L_1 \cap L_2$, then*

$$[\varphi^t(h), i V_{\varepsilon,r}(s)]$$

is norm equicontinuous in t with respect to ε and r.

Proof. Let h be in C_0^∞. Then we get from Lemma 1 that

$$\begin{aligned}&\|[\varphi^{t_1}(h), i V_{\varepsilon,r}(s)] - [\varphi^{t_2}(h), i V_{\varepsilon,r}(s)]\|\\ &\qquad \leqq \|V'\|_\infty \int dz \left|\int h(y)\,(\Delta(y-z, t_1-s) - \Delta(y-z, t_2-s))\, dy\right|\end{aligned}$$

We see that the right hand side is independent of ε and r. Moreover as t_1 tends to t_2 $\int h(y)\,\Delta(y-z, t_1-s)\, dy$ tends uniformly in z to

$\int h(y)\,\Delta(y-z, t_2-s)\,dy$, and as the functions are zero outside a fixed compact we see that the right hand side tends to zero. Hence we have proved that for h in C_0^∞ $[\varphi^t(h), iV_{\varepsilon,r}(s)]$ is norm equicontinuous. But C_0^∞ is dense in L_1 and by linearity in h and corollary 1 we get norm equicontinuity for all h in L_1. This proves the lemma.

Lemma 3. *Let h be in $L_1 \cap L_2$, then*

$$\varphi_{\varepsilon,r,t}(h) - \varphi^t(h) = \int_0^t ds\, e^{-isH_{\varepsilon,r}} [\varphi^{t-s}(h), iV_{\varepsilon,r}]\, e^{isH_{\varepsilon,r}},$$

where the integral is a strong integral, and the integrand is strongly continuous.

Proof. By Lemma 2 $[\varphi^{t-s}(h), iV_{\varepsilon,r}]$ is a norm continuous function of s. Hence the integrand is strongly continuous and therefore strongly integrable. But for strongly integrable functions the strong and the weak integral coinside and the lemma is therefore proved by formula 2.4.

Corollary 2. *For h in $L_1 \cap L_2$, we have that*

$$\|\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)\| \leqq C|t|^3 \, \|V'\|_\infty \, \|h\|_1 ,$$

where C depends only on the mass m of the free field.

Proof. We use the norm estimate of Corollary 1 to estimate the integrand of Lemma 3. This gives us the estimate of Corollary 2, and this proves the corollary.

Lemma 4. *Let h be in $L_1 \cap L_2$, then $\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)$ are norm equicontinuous functions of t with respect to ε and r.*

Proof. By Lemma 3 we get for $t_1 \leqq t_2$

$$\begin{aligned} &\|\varphi_{\varepsilon,r,t_1}(h) - \varphi^{t_1}(h) - \varphi_{\varepsilon,r,t_2}(h) + \varphi^{t_2}(h)\| \\ &\qquad \leqq \int_0^{t_1} ds \{\| [\varphi^{t_1-s}(h), V_{\varepsilon,r}] - [\varphi^{t_2-s}(h), V_{\varepsilon,r}] \|\} \\ &\qquad + \int_{t_1}^{t_2} ds \|[\varphi^{t_2-s}(h), V_{\varepsilon,r}]\| . \end{aligned}$$

By Corollary 1 the integrand of the first integral is uniformly bounded, and by Lemma 2 it tends pointwise to zero as t_1 tends to t_2 or t_2 tends to t_1, uniformly in ε and r. Hence by Lebesques lemma on dominated convergence the first integral tends to zero as t_1, tends to t_2, or t_2 tends to t_1, and the convergence is uniform in ε and r. By Corollary 1 the second integral is dominated by $C|t_1-t_2|\, t_2^2 \, \|V'\|_\infty \, \|h\|_1$ which obviously tends to zero uniformly in ε and r. This proves the lemma.

Lemma 5. *$e^{itH_{\varepsilon,r}}$ is norm continuous in r, and the normcontinuity in r is uniform in ε and on compact intervals in t. We have the estimate*

$$\|e^{itH_{\varepsilon,r_1}} - e^{itH_{\varepsilon,r_2}}\| \leqq \frac{4\pi}{3} |r_1^3 - r_2^3| \, |t| \, \|V\|_\infty .$$

Proof. Since the domain of $H_{\varepsilon,r}$ is equal to D_0 for all r, we have that $e^{isH_{\varepsilon,r}}$ leaves D_0 invariant for all r, and therefore $e^{i(t-s)H_{\varepsilon,r_1}} e^{isH_{\varepsilon,r_2}}$ is strongly differentiable in s on D_0 with derivative

$$\frac{d}{ds} e^{i(t-s)H_{\varepsilon,r_1}} e^{isH_{\varepsilon,r_2}} = e^{i(t-s)H_{\varepsilon,r_1}} i(V_{\varepsilon,r_2} - V_{\varepsilon,r_1}) e^{isH_{\varepsilon,r_2}} .$$

Integrating both sides, we get on D_0

$$e^{itH_{\varepsilon,r_2}} - e^{itH_{\varepsilon,r_1}} = \int_0^t ds \, e^{i(t-s)H_{\varepsilon,r_1}} i(V_{\varepsilon,r_2} - V_{\varepsilon,r_1}) e^{isH_{\varepsilon,r_2}} .$$

Since D_0 is dense in $\mathscr{F}$ the estimate in the lemma follows from a direct estimate of the norm of the integral above. This proves the lemma.

Lemma 6. *Let h be in $L_1 \cap L_2$ then $\varphi_{\varepsilon,r,t}(h) - \varphi^t(h)$ is norm continuous in r, and the normcontinuity in r is uniform in ε and in t on compact intervals.*

Proof. From Lemma 3 we get

$$\begin{aligned}
\varphi_{\varepsilon,r_1,t}(h) - \varphi_{\varepsilon,r_2,t}(h) &= i \int_0^t ds \, e^{-isH_{\varepsilon,r_1}} [\varphi^{t-s}(h), V_{\varepsilon,r_1}] \, e^{isH_{\varepsilon,r_1}} \\
&\quad - i \int_0^t ds \, e^{-isH_{\varepsilon,r_2}} [\varphi^{t-s}(h), V_{\varepsilon,r_2}] \, e^{isH_{\varepsilon,r_2}} \\
&= i \int_0^t ds \{ (e^{-isH_{\varepsilon,r_1}} - e^{-isH_{\varepsilon,r_2}}) [\varphi^{t-s}(h), V_{\varepsilon,r_1}] e^{isH_{\varepsilon,r_1}} \\
&\quad + e^{-isH_{\varepsilon,r_2}} [\varphi^{t-s}(h), V_{\varepsilon,r_1} - V_{\varepsilon,r_2}] \, e^{isH_{\varepsilon,r_1}} \\
&\quad + e^{-isH_{\varepsilon,r_2}} [\varphi^{t-s}(h), V_{\varepsilon,r_2}] \, (e^{isH_{\varepsilon,r_1}} - e^{isH_{\varepsilon,r_2}}) \} .
\end{aligned}$$

The first and the last term of the integrand is estimated uniformly in ε and in t on compact intervals by Lemma 5. The second term in the integrand is estimated by $C|r_1^3 - r_2^3|$ where C is independent on ε and on t if t is on a compact interval, by the formula of Lemma 1. This proves the lemma.

Theorem 1. *There exists a sequence ε_n tending to zero, such that for all h in $L_1 \cap L_2$ and all t and r, $\varphi_{\varepsilon_n,r,t}(h) - \varphi^t(h)$ converge weakly to a limit $\varphi_r(h,t) - \varphi^t(h)$. The convergence is uniform for r and t on compact intervals, and the limit is normcontinuous in r and t. Relative to the strong L_1 topology,* *the limit is also normcontinuous in h and*

$$\|\varphi_r(h,t)-\varphi^t(h)\| \leqq C|t|^3 \, \|V'\|_\infty \, \|h\|_1 ,$$

where C depends only on the mass m of the free field.

Proof. Let ψ_1 and ψ_2 be in $\mathscr{F}$ and h be in $L_1 \cap L_2$. $(\psi_1, \varphi_{\varepsilon,r,t}(\mathrm{h}) - \varphi^t(\mathrm{h})\psi_2)$ is then by Corollary 2 a uniformly bounded family of functions of t and r, depending on a parameter ε. By Lemma 4 and Lemma 6 it is also an equicontinuous family of functions of r and t. The Ascoli theorem then gives the existence of a sequence ε_n', tending to zero, such that the corresponding sequence of functions converge uniformly for t and r on compact intervals. By passing to a subsequence ε_n we get uniform convergence for a countable dense set of ψ_1 and ψ_2, and a countable set of h that is dense in $L_1 \cap L_2$ in the strong L_1 topology. The norm estimate of Corollary 2 then gives us convergence, uniformly for t and r on compact intervals, for all ψ_1 and ψ_2 in $\mathscr{F}$ and all h in $L_1 \cap L_2$. This proves the weak convergence. To see that the limit is normcontinuous in t, we use Lemma 4 which gives us that for all $\varepsilon > 0$ there exists a $\delta > 0$ independent of n such that

$$\|\varphi_{\varepsilon_n,r,t+\tau}(h) - \varphi^{t+\tau}(h) - \varphi_{\varepsilon_n,r,t}(h) + \varphi^t(h)\| \leqq \varepsilon$$

as soon as $|\tau| < \delta$. Now we use that the set of operators with norm smaller or equal to ε is weakly closed to get the same estimate for the limit. This proves normcontinuity in t, and normcontinuity in r is proved in the same way by using Lemma 6. Normcontinuity in h follows from the norm estimate of the theorem. Hence it is enough to prove this estimate. But using again that a closed ball of operators is weakly closed, we see that this estimate follows directly from Corollary 2. This proves the theorem.

Lemma 7. *Let H_0 and H be two self adjoint operators on a Hilbert space, such that $V = H - H_0$ is bounded. Let A be a bounded operator and define $A_t = e^{-itH} e^{itH_0} A e^{-itH_0} e^{itH}$. Then we have for $t \geqq 0$,*

$$A_t = \sum_{n=0}^{\infty} i^n \int \cdots \int_{t \geqq t_1 \geqq t_2 \cdots \geqq t_n \geqq 0} [\ldots[A, V(t_1)] \ldots, V(t_n)] \, dt_1 \ldots dt_n$$

where $V(t) = e^{-itH_0} V(t) e^{itH_0}$. The integrals are strong integrals and the sum is norm convergent.

Proof. Since V is bounded, H and H_0 have the same domain and e^{itH} and e^{itH_0} leaves this domain invariant. Let ψ_1 and ψ_2 be in this domain, then we see that $(\psi_1, A_t \psi_2)$ is differentiable with respect to t and

$$\frac{d}{dt}(\psi_1, A_t \psi_2) = (\psi_1, e^{-itH}[e^{itH_0} A e^{-itH_0}, iV] \, e^{itH} \psi_2) .$$

By integration we get that

$$(\psi_1, A_t \psi_2) = (\psi_1, A\psi_2) + \int_0^t ds (\psi_1, e^{-isH} e^{isH_0} [A, iV(s)]\, e^{isH_0} e^{isH} \psi_2)\,.$$

Since the domain of H_0 is dense and A_t, A and $V(t)$ are bounded the identity above holds for all ψ_1 and all ψ_2. Hence

$$A_t = A + i \int ds\, e^{-isH} e^{isH_0} [A, V(s)]\, e^{-isH_0} e^{isH}\,,$$

where the integral is a weak integral. But we see that the integrand is bounded and strongly continuous, hence strongly integrable, and therefore the weak integral coincides with the strong integral. By iterating the formula above we get the formula of Lemma 5. The norm convergence follows from a direct norm estimate. This proves the lemma.

Lemma 8. *Let h be in $L_1 \cap L_2$, then for $t \geqq 0$*

$$\varphi_{\varepsilon,r,t}(h) - \varphi^t(h) = \sum_{n=1}^{\infty} i^n \int_{t \geqq t_1 \cdots \geqq t_n \geqq 0} \cdots \int [\ldots[\varphi^t(h), V_{\varepsilon,r}(t_1)] \ldots, V_{\varepsilon,r}(t_n)]\, dt_1 \ldots dt_n$$

where the integrals are strong integrals and the sum is normconvergent uniformly in ε.

Proof. The formula of Lemma 3 may be written

$$\varphi_{\varepsilon,r,t}(h) - \varphi^t(h) = i \int ds\, e^{-isH_{\varepsilon,r}} e^{isH_0} [\varphi^t(h), V_{\varepsilon,r}(s)]\, e^{-is} e^{isH_{\varepsilon,r}}\,.$$

By Corollary 1 $[\varphi^t(h), V_{\varepsilon,r}(s)]$ is bounded uniformly in ε. Hence we may apply the formula of Lemma 7 to the integrand, and this gives us the formula of Lemma 8. To prove that the normconvergence is uniform in ε, we estimate the norm of the n'th term in the sum by

$$C \|V'\|_\infty \|h\|_1 \frac{t^{n+1}}{n!} \|V_{\varepsilon,r}\|^{n-1}$$

using Corollary 1. But since

$$\|V_{\varepsilon,r}\| \leqq \frac{4\pi}{3} r^3 \|V\|_\infty$$

we see that the normconvergence is uniform in ε. This proves the lemma.

Definition. For any subset S of R^3 we define $S_t = S + B_t$ where $B_t = \{x; |x| \leqq t\}$. S_t is then the set of points in R^3 which is causally dependent on S up to the time t, or in other words S_t is the set of points that can be reached, in a time less or equal to t, by a light signal emitted from S. We shall also use the expression that to sets S_1 and S_2 are causally independent up to the time t if $S_{1,t} \cap S_2 = \emptyset$.

Let Λ be any compact set in R^3. Then we define

$$V_{\varepsilon,\Lambda} = \int V(\varphi_\varepsilon(x))\,dx\,,\quad H_{\varepsilon,\Lambda} = H_{\varepsilon,\Lambda}$$

and

$$\varphi_{\varepsilon,\Lambda,t}(h) = e^{-itH_{\varepsilon,\Lambda}}\,\varphi(h)\,e^{itH_{\varepsilon,\Lambda}}\,.$$

Theorem 2. *Let h be in $L_1 \cap L_2$, then $\varphi_r(h,t) - \varphi^t(h)$ converge in norm in norm to $\varphi(h,t) - \varphi^t(h)$. The convergence is uniform for t on compact intervals, and the limit is norm continuous in t. Relative to the strong L_1 topology, the limit is also normcontinuous in h and*

$$\|\varphi(h,t) - \varphi^t(h)\| \leqq C\,|t|^3\,\|V'\|_\infty\,\|h\|_1$$

where C depends only on the mass m of the free field.

Proof. We shall assume first that h is in L_2 and have compact support S. It follows from the expression for the function Δ that $\varphi^s(h_1)$ and $\varphi^t(h_2)$ commute if the support of h_1 and the support of h_2 are causally independent up to the time $|t-s|$. This gives us for any two bounded continuous functions F_1 and F_2 that $F_1(\varphi^s_\varepsilon(x))$ and $F_2(\varphi^t_\varepsilon(y))$ commute if $|x-y| \geqq |t-s| + 2\varepsilon$. From Lemma 1 we get for $t \geqq t_1 \geqq 0$

$$\begin{aligned}[\varphi^t(h), V_{\varepsilon,r}(t_1)] &= [\varphi^t(h), V_{\varepsilon,S_{t-t_1+\varepsilon}}(t_1)]\\ &= [\varphi^t(h), V_{\varepsilon,\Lambda}(t_1)]\end{aligned}$$

if $S_{t-t_1+\varepsilon} \subset B_r$ and $S_{t-t_1+\varepsilon} \subset \Lambda$.

Using now that $F_1(\varphi^s_\varepsilon(x))$ and $F_2(\varphi^t_\varepsilon(y))$ commute if $|x-y| \geqq |t-s| + 2\varepsilon$, we get for $t \geqq t_1 \geqq t_2 \geqq 0$.

$$\begin{aligned}&[[\varphi^t(h), V_{\varepsilon,r}(t_1)], V_{\varepsilon,r}(t_2)]\\ &\quad= [[\varphi^t(h), V_{\varepsilon,S_{t-t_1+\varepsilon}}(t_1)], V_{\varepsilon,S_{t-t_2+3\varepsilon}}(t_2)]\\ &\quad= [[\varphi^t(h), V_{\varepsilon,\Lambda}(t_1)], V_{\varepsilon,\Lambda}(t_2)]\end{aligned}$$

if $S_{t+3\varepsilon} \subset B_r$ and $S_{t+3\varepsilon} \subset \Lambda$.

In the same way we get for $t \geqq t_1 \geqq t_2 \ldots \geqq t_n \geqq 0$

$$\begin{aligned}&[\ldots[\varphi^t(h), V_{\varepsilon,r}(t_1)]\ldots, V_{\varepsilon,r}(t_n)]\\ &\quad= [\ldots[\varphi^t(h), V_{\varepsilon,S_{t-t_1+\varepsilon}}(t_1)]\ldots, V_{\varepsilon,S_{t-t_n+(2n+1)\varepsilon}}(t_n)] \qquad (2.5)\\ &\quad= [\ldots[\varphi^t(h), V_{\varepsilon,\Lambda}(t_1)]\ldots, V_{\varepsilon,\Lambda}(t_n)]\end{aligned}$$

if $S_{t+(2n+1)\varepsilon} \subset B_r$ and $S_{t+(2n+1)\varepsilon} \subset \Lambda$.

But this proves that the n first terms in the sum in Lemma 8 is independent of r as soon as $S_{t+(2n+1)\varepsilon} \subset B_r$. Assume now that S_t is contained in the interior of B_{r_1} and that $r_1 \leqq r_2$. Since the sum in Lemma 8 is norm-convergent uniformly in ε we then get that $\varphi_{\varepsilon_n,r_2,t}(h) - \varphi_{\varepsilon_n,r_1,t}(h)$ tends to zero in norm as ε_n tends to zero. This proves that the weak limit $\varphi_r(h,t) - \varphi^t(h)$ is independent of r as soon as S_t is contained in the interior of B_r. This proves the normconvergence uniformly for t on compact intervals, if h has compact support. But since functions with compact support is dense in $L_1 \cap L_2$ relative to the strong topology in L_1, the estimate of Theorem 1 gives us normconvergence uniformly for t on compact intervals for all h in $L_1 \cap L_2$. That the limit is normcontinuous in t follows from uniform convergence. The estimate of the theorem follows immediately from the estimate of Theorem 1, and this gives also the normcontinuity in h. This proves the theorem.

Lemma 9. *Let h be in $L_1 \cap L_2$, then for $t \geqq 0$*

$$\varphi_{\varepsilon,\Lambda,t}(h) - \varphi^t(h) = \sum_{n=1}^{\infty} i^n \int_{t \geqq t_1 \cdots \geqq t_n \geqq 0} \cdots \int [\ldots[\varphi^t(h), V_{\varepsilon,\Lambda}(t_1)] \ldots, V_{\varepsilon,\Lambda}(t_n)]\, dt_1 \ldots dt_n$$

where the integrals are strong integrals and the sum is normconvergent uniformly in ε and uniformly in Λ, with $|\Lambda| < C$. $|\Lambda|$ is the total volume of Λ.

Proof. This lemma is proved in the same way as Lemma 8, and only trivial modifications of the proof of Lemma 8 is needed to prove Lemma 9.

We will now prove two theorems which express that the interaction is local.

Theorem 3. *Let ε_n be the sequence of Theorem 1.*

Let h be in L_2 with compact support S. Let Λ be any compact such that S_t is contained in the interior of Λ. Then $\varphi_{\varepsilon_n,\Lambda,t}(h) - \varphi^t(h)$ converge weakly to $\varphi(h,t) - \varphi^t(h)$ as n tends to infinity.

Proof. From the proof of Theorem 2 we see that the n first terms in the sums in Lemma 8 and Lemma 9 is the same if $S_{t+(2n+1)\varepsilon} \subset \Lambda$ and $S_{t+(2n+1)\varepsilon} \subset B_r$. Hence $\varphi_{\varepsilon_n,\Lambda,t}(h) - \varphi_{\varepsilon_n,r,t}(h)$ converge to zero in norm by Lemma 8 and Lemma 9, if S_t is contained in the interior of Λ and of B_r. But from the proof of Theorem 2 we see that $\varphi_{\varepsilon_n,r,t}(h) - \varphi^t(h)$ converge weakly to $\varphi(h,t) - \varphi^t(h)$. This proves the theorem.

Theorem 4. *Let h_1 and h_2 be in L_2 with compact supports S_1 and S_2. If S_1 and S_2 are causally independent up to time t, then $\varphi(h_1,t)$ and $\varphi(h_2)$ commute.*

Proof. Since $\varphi^t(h_1)$ and $\varphi(h_2)$ commute, it is enough to prove that $\varphi(h_1,t) - \varphi^t(h_1)$ and $\varphi(h_2)$ commute. But since the set of operators that commute with $\varphi(h_2)$ is weakly closed it is enough to prove that $\varphi_{\varepsilon,r,t}(h_1) - \varphi^t(h_1)$ commutes with $\varphi(h_2)$ if ε is small and r such that $S_{1,t}$ is contained in the interior of B_r. Using once more that the commutator is weakly closed and Lemma 8, we see that it is enough to prove that for $t \geqq t_1 \geqq \cdots \geqq t_n \geqq 0$

$$[\ldots[\varphi^t(h_1), V_{\varepsilon,r}(t_1)] \ldots, V_{\varepsilon,r}(t_n)] \tag{2.6}$$

commute with $\varphi(h_2)$ for ε small enough. But by (2.5) we have that (2.6) is equal to

$$[\ldots[\varphi^t(h_1), V_{\varepsilon, S_{1,t-t_1+\varepsilon}}(t_1)]\ldots, V_{\varepsilon, S_{1,t-t_n+(2n+1)\varepsilon}}(t_n)]. \tag{2.7}$$

Using Lemma 1 we see that $\varphi(h_2)$ commutes with $[\varphi^t(h_1), V_{\varepsilon,S_{1,t-t_1+\varepsilon}}(t_1)]$ if S_1 and S_2 are causally independent up to time $t+\varepsilon$. For $j=2, 3, \ldots, n$ we see that $\varphi(h_2)$ commutes with $V_{\varepsilon,S_{1,t-t_j+(2j+1)\varepsilon}}(t_j)$ if S_1 and S_2 are causally independent up to time $t+(2j+1)\varepsilon$. Therefore $\varphi(h_2)$ commutes with (2.7) if S_1 and S_2 are causally independent up to time $t+(2n+1)\varepsilon$. But since S_1 and S_2 are compacts, we see that if they are causally independent up to a time t, then there exists $\delta>0$ such that S_1 and S_2 are causally independent up to time $t+\delta$. Hence for $(2n+1)\varepsilon<\delta$ we find that $\varphi(h_2)$ commutes with (2.7). This proves the theorem.

For the free Heisenberg picture fields $\varphi^t(h)$ we have the following translation invariance

$$U(-x)\,\varphi^t(h)\,U(x)=\varphi^t(h_x),$$

where $U(x)$ is the unitary group of space translations introduced in the beginning of Section 2 and $h_x(y)=h(y-x)$. The next theorem states that the interacting Heisenberg picture fields are also translation invariant.

Theorem 5. *Let h be in $L_1\cap L_2$. Then for any x in R^3*

$$U(-x)\,\varphi(h,t)\,U(x)=\varphi(h_x,t)$$

where $U(x)$ is the unitary group of space translations and $h_x(y)=h(y-x)$.

Proof. Since $\varphi^t(h)$ is translationinvariant, it is enough to prove that $\varphi(h,t)-\varphi^t(h)$ is translation invariant. Due to the uniform norm estimate in Theorem 2, it is enough to prove that $\varphi(h,t)-\varphi^t(h)$ is translation invariant for h with compact support S. By weak convergence and Theorem 3 it is enough to prove that $\varphi_{\varepsilon_n,\Lambda,t}(h_x)-\varphi^t(h_x)$ and

$$U(-x)\,(\varphi_{\varepsilon_n,\Lambda,t}(h)-\varphi^t(h))\,U(x)=\varphi_{\varepsilon_n,\Lambda_x,t}(h_x)-\varphi^t(h_x)$$

have the same weak limits when S_t is contained in Λ. The identity above follows from the identity

$$U(-x)\,V_{\varepsilon,\Lambda}\,U(x)=V_{\varepsilon,\Lambda_x}.$$

But that $\varphi_{\varepsilon_n,\Lambda,t}(h_x)-\varphi^t(h_x)$ and $\varphi_{\varepsilon_n,\Lambda_x,t}(h_x)-\varphi^t(h_x)$ have the same limit follows immediately from Theorem 3. This proves the theorem.

Remark. To get rotational invariance of the Heisenberg picture fields, we have only to choose $g(x)$ rotational invariant. Since then all the cut-off fields $\varphi_{\varepsilon,r,t}(h)$ will be rotational invariant.

Even though the interaction is formally relativistic invariant, we can not at this stage at least prove that the Heisenberg picture fields are relativistic invariant or not. Already for the time translation we do not even know if it is implemented by a group of C^*-automorphisms or not. So probably stronger results then weak convergence of the cut-off fields would be needed to discuss relativistic invariance.

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