Commun. math. Phys. 14, 165—170 (1969)

Causal Groups of Space-Time

CEZAR GHEORGHE and ELEONORA MIHUL
Institute for Atomic Physics, Bucharest, Laboratory of Theoretical Physics

Received June 25, 1969

Abstract. The present paper is concerned with a finite dimensional space, considered as a real ordered linear space, directed with respect to a partial ordering relation (causal relation) in which a given reflexion (called temporal inversion) is antiisotone and the positive cone is closed in the euclidean topology.

A generalized Zeeman theorem [1] is obtained, which states that the causal group relative to the causal relation is a subgroup of the affine group of M.

Let M be a N-dimensional linear real space ($N > 2$) with the vectors represented by $(x^0, \overline{x})$, where $x^0 \in R$ (R is the real field) and $\overline{x} \in \overline{M}$ ($\overline{M}$ is a $N-1$-dimensional linear real space).

A *causal relation* of M is a partial ordering relation $\geqq$ of M with regard to which M is directed [2] (for any $x, y \in M$ there is $z \in M$ so that $z \geqq x$, $z \geqq y$). The *causal group* G relative to the causal relation is the group of permutations $f: M \to M$ which leave invariant the relation $\geqq$ ($f(x) \geqq f(y)$ if and only if $x \geqq y$).

One defines $T: M \to M$ by $T(x) = (-x^0, \overline{x})$. Let $0 = (0, \overline{0})$ be the origin of M.

Theorem. *If the causal relation of M is compatible with linearity, T is anti-isotone (if $x \geqq y$, then $T(y) \geqq T(x)$) and the positive cone $C = \{x \mid x \in M; x \geqq 0\}$ is closed in euclidean topology of M, then the causal group relative to the causal relation is a subgroup of the affine group of M.*

The compatibility of $\geqq$ with linearity is equivalent to the translations and dilations of M belonging to G (if $x \geqq y$, $a \in M$, $\lambda \in R_+$, then $\lambda x + a \geqq \lambda y + a$; R_+ is the set of strictly positive natural numbers), and M is called an ordered linear space [3]. If M is an ordered linear space, then M is directed if and only if C generates M (the Clifford's theorem [2]).

In the particular case for which C is the Minkowski cone, the theorem has been proved by Zeeman [1]: G is generated by translations, dilations and orthochronous Lorentz transformations of M. There are also some generalizations of this theorem [4–6].

Let Q and $\operatorname{int} C$ denote the boundary and the interior of C.

Lemma 1. *There is a norm* $\| \ \|$ *in* $\bar{M}$ *so that:*

$$Q = \{x \mid x \in M; \varepsilon x^0 = \|\bar{x}\|\}, \qquad \operatorname{int} C = \{x \mid x \in M; \varepsilon x^0 > \|\bar{x}\|\}, \tag{1}$$

where $\varepsilon = 1$ *if* $(-1, \bar{0}) \notin C$ *and* $\varepsilon = -1$ *if* $(1, \bar{0}) \notin C$.

Proof. Since M is generated by C, for any $x \in M$ there are the vectors $y, z \in C$ so that $x = y - z$. We define $u = y - T(z)$, which belongs to C and $\bar{u} = \bar{x}$. Then for every $\bar{x} \in \bar{M}$ there exists $\inf \varepsilon y^0$, which we denote by $\|\bar{x}\|$, for $y \in C$ with $\bar{y} = \bar{x}$.

We have defined the map $\| \ \|: \bar{M} \to R$ such that $(\varepsilon \|\bar{x}\|, \bar{x}) \in Q$.

The following properties can be written:

$$C + C = C; \quad \lambda C = C, \ (\lambda \in R_+); \quad C \cap (-C) = \{0\};$$
$$T(C) = -C; \qquad C = Q \cup \operatorname{int} C, \ [3].$$

We show now that $\| \ \|$ is a norm.

If $x \in Q$, then $-T(x) \in C$ and $x - T(x) = (2\varepsilon \|\bar{x}\|, \bar{0}) \in C$. But $0 \in Q$, then $\|\bar{x}\| \geqq 0$ and $\|\bar{0}\| = 0$. If $\|\bar{x}\| = 0$, then $(0, \bar{x}) \in Q$ implies $(0, \bar{x}) \in T(C)$. As $C \cap T(C) = \{0\}$ we have $\bar{x} = \bar{0}$. It follows that $\|\bar{x}\| = 0$ if and only if $\bar{x} = \bar{0}$.

One obtains $-T(x) = (\varepsilon \|\bar{x}\|, -\bar{x}) \in C$ if $x \in Q$. But $(\varepsilon \| -\bar{x}\|, -\bar{x}) \in Q$ consequently $\varepsilon \|\bar{x}\| \geqq \varepsilon \| -\bar{x}\|$. Similarly $\varepsilon \| -\bar{x}\| \geqq \varepsilon \|\bar{x}\|$, whence $\|\bar{x}\| = \| -\bar{x}\|$, and according to the definition of $\| \ \|$: $\|\lambda \bar{x}\| = \lambda \|\bar{x}\|$, $\lambda \in R_+$, then $\|\lambda \bar{x}\| = |\lambda| \, \|\bar{x}\|$, $\lambda \in R$.

Now $(\varepsilon \|\bar{x}\|, \bar{x})$, $(\varepsilon \|\bar{y}\|, \bar{y}) \in Q$ implies $(\varepsilon \|\bar{x}\| + \varepsilon \|\bar{y}\|, \bar{x} + \bar{y}) \in C$. But $(\varepsilon \|\bar{x} + \bar{y}\|, \bar{x} + \bar{y}) \in Q$ and consequently $\|\bar{x}\| + \|\bar{y}\| \geqq \|\bar{x} + \bar{y}\|$. Taking into account the definition of $\| \ \|$ it follows that if $x \in C$ then $\varepsilon x^0 \geqq \|\bar{x}\|$ and the Lemma 1 is proved. ┘

There exists only one norm of $\bar{M}$ so that Lemma 1 is true. Hence there is a one-to-one correspondence between causal relations of M and norms $\| \ \|$ of $\bar{M}$ so that $x \geqq y$ is equivalent to $x^0 - y^0 \geqq \|\bar{x} - \bar{y}\|$. If $\| \ \|$ is the euclidean norm of $\bar{M}$, then C is the Minkowski cone of M.

Now we can define two topologies: the $\| \ \|$-topology in M with the norm defined in Lemma 1 as the topology for which to any $x \in M$ corresponds a fundamental system of neighborhoods $\{S_\varepsilon(x)\}_{\varepsilon \in R_+}$ where:

$$S_\varepsilon(x) = \{y \mid y \in M; [(y^0 - x^0)^2 + \|\bar{y} - \bar{x}\|^2]^{1/2} < \varepsilon\}, \tag{2}$$

and the causal topology of M for which the fundamental neighborhoods are the open ordered sets [2] $V(a, b)$, for any $a, b \in M$ with $b - a \in \operatorname{int} C$, where:

$$V(a, b) = \{y \mid y \in M; b - y, y - a \in \operatorname{int} C\}. \tag{3}$$

Lemma 2. *The causal topology of* M *is equivalent to the euclidean topology.*

Proof. For any $S_\varepsilon(x)$ given by (2) one chooses $a, b \in M$ with

$$a^0 = x^0 - \frac{1}{\sqrt{2}}\varepsilon, \quad b^0 = x^0 + \frac{1}{\sqrt{2}}\varepsilon, \quad \bar{a} = \bar{b} = \bar{x}$$

and using the Lemma 1 it follows that $V(a,b) \subset S_\varepsilon(x)$. Conversely for any $V(a,b)$ given by (3) one obtains $S_\varepsilon(x) \subset V(a,b)$ for $x \in V(a,\mathrm{b})$ and:

$$0 < \varepsilon < \frac{1}{2}\min\{|b^0 - x^0| - \|\bar{b} - \bar{x}\|, |x^0 - a^0| - \|\bar{x} - \bar{a}\|\}.$$

Then the causal and $\|\ \|$-topology are equivalent. Since M is finite dimensional then its euclidean topology is equivalent to $\|\ \|$-topology [7], and consequently the causal topology is equivalent to the euclidean topology. ⌟

For the particular case when $\|\ \|$ is the euclidean norm Lemma 2 has been proved in [5].

Taking into account that any $f \in G$ is a homeomorphism in causal topology, from Lemma 2 it follows:

Consequence 1. *If $f \in G$, then f is a homeomorphism of M in euclidean topology.*

Demonstration of the Theorem. Let F be the set of vectors $x \in Q \backslash \{0\}$ for which there are no $y, z \in Q$ linearly independent and $x = y + z$.

The maximal simplex which contains $u \in Q$ and is contained in $\{x \mid x \in Q, x^0 = u^0\}$ has the vertices in F. Then $Q \backslash \{0\} \subset \tilde{F}$ ($\tilde{A}$ is the convex covering of A). But $\tilde{F} \subset C \backslash \{0\}$ and $\tilde{Q} \backslash \{0\} = C \backslash \{0\}$. Hence F has the following properties:

a) $\tilde{F} = C \backslash \{0\}$.

b) $\lambda F = F$ for any $\lambda \in R_+$.

c) If $x_1, x_2, \ldots, x_n \in F$ and $x_1 + x_2 + \cdots + x_n \in F$, then $x_1, x_2, \ldots, x_n$ are two by two linearly dependent.

We denote $\Delta(u) = C \cap (u - C)$, where $u \in C$. It follows from the definition of F that $\Delta(u)$ is one-dimensional if and only if $u \in F$. Indeed if $u \in C \backslash F$ there exist $u_1, u_2 \in Q$ linearly independent with $u = u_1 + u_2$ and $u_1, u_2 \in \Delta(u)$. Then it follows that the dimension of $\Delta(u)$ is strictly greater then one. Let $u \in F$ and $x \in \Delta(u)$, $x \neq 0, u$; $x = u - v$, then $x = x_1 + x_2 + \cdots + x_m$ and $v = v_1 + v_2 + \cdots + v_n$, with $x_1, x_2, \ldots, x_m, v_1, v_2, \ldots, v_n \in F$. Now from the property c) of F the linear dependence of each x_i upon u is obtained. Hence $\Delta(u) = \{x \mid x = \lambda u, \lambda \in [0,1]\}$. It follows that $\Delta(u)$ is one-dimensional if and only if $u \in F$.

It is easy to see that the relation $f(\Delta(u)) = f(0) + \Delta(f(u) - f(0))$ holds and since f is bicontinuous it does not change the dimension of $\Delta(u)$. Then it follows that $u \in F$ if and only if $f(u) - f(0) \in F$ and we have $f(F) = f(0) + F$.

We introduce now:

$$\begin{aligned} F_x &= \{z \mid z = x + u;\, u \in F \cup (-F) \cup \{0\}\}\,, \\ D(x, y) &= \{z \mid z = x + \lambda y;\, \lambda \in R\}\,, \end{aligned} \tag{4}$$

where $x, y \in M$. It can be seen that for any $x \in M$, $u \in F$, $f \in G$:

$$\begin{aligned} D(x, u) = F_x \cap F_{x+u};\, f(F_x) = F_{f(x)}; \\ f(D(x, u)) = D(f(x), f(x+u) - f(x))\,. \end{aligned} \tag{5}$$

Let $u, v \in F$ be linearly independent. One considers $v' \neq 0$, v, $v' \in D(0, v)$ and $f \in G$. But $x + v' \in D(x, v)$, $x + u + v' \in D(x + u, v)$ and it follows from (5) that there are $\mu, \nu \in R$ so that:

$$\begin{aligned} f(x+v') &= \mu f(x) + (1-\mu) f(x+v)\,, \\ f(x+u+v') &= \nu f(x+u) + (1-\nu) f(x+u+v)\,, \\ f(x+u+v') - f(x+v') &= \nu [f(x+u) - f(x)] \\ &\quad + (1-\nu)[f(x+u+v) - f(x+v)] + (\mu - \nu)[f(x+v) - f(x)]\,. \end{aligned} \tag{6}$$

Let us suppose $f(x+u) - f(x)$, $f(x+v) - f(x)$ and $f(x+u+v) - f(x+v)$ linearly independent and $\mu \neq \nu$.

In this case there is $u' \in D(0, u)$ and $w \in F$ so that:

$$\begin{aligned} &D(f(x+v), f(x+u+v) - f(x+v)) \cap D(f(x+u'), f(x+u+v') - f(x+v')) \\ &\quad = D(f(x+v), f(x+u+v) - f(x+v)) \cap f(D(x+u, w)) \end{aligned}$$

is not empty (for $w = f^{-1}(f(x+u+v) - f(x+v) + f(x+u')) - x - u'$). Then $D(x+v, u) \cap D(x+u, w)$ is not empty. It follows from the definition of F that u and v are linearly dependent and this contradicts our assumption about u and v. Hence $f(x+u+v) - f(x+v)$ and $f(x+u) - f(x)$ are linearly dependent. Similarly one obtains that $f(x+u+v) - f(x+u)$ and $f(x+v) - f(x)$ are linearly dependent. Consequently:

$$f(x+u+v) + f(x) = f(x+u) + f(x+v)\,. \tag{7}$$

2. We consider now:

$$G_0 = \{f \mid f \in G;\, f(0) = 0\}\,. \tag{8}$$

Let $f \in G_0$. It follows from (7) that:

$$f(u_1 + u_2 + \cdots + u_n) = f(u_1) + f(u_2) + \cdots + f(u_n)\,, \tag{9}$$

for any linearly independent vectors $u_1, u_2, \ldots, u_n \in F$.

For any $x \in F$ we define $\varphi_x : R_+ \cup \{0\} \to R_+ \cup \{0\}$ through $f(\lambda x) = \varphi_x(\lambda) f(x)$, $\lambda \in R_+ \cup \{0\}$. It follows from the consequence 1 that φ_x is a continuous function strictly increasing (obviously $\varphi(0) = 0$ and $\varphi(1) = 1$).

Since M is generated by F there is a basis $\{y_i\}_{i=1,2,\ldots,N} \subset F$.

There exists $y \in F$ linearly independent of each y_i because otherwise $C \cap \{x \mid x \in M; |x^0| = 1\}$ would be not a simplex symmetric relative to $(\varepsilon, \bar{0})$ according to Lemma 1 and T would be not anti-isoton. (A $N-1$-dimensional simplex with N vertices is not symmetrically relative to any its point if $N > 2$.)

We consider now all the elements $x \in C \setminus \{0\}$ for which there are $\lambda_1, \lambda_2, \ldots, \lambda_N \in R_+$ such that:

$$x = \lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_N y_N + \lambda y; \lambda_i + \alpha_i \lambda \geqq 0 \quad (i = 1, 2, \ldots, N), \tag{10}$$

where $y = \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_N y_N; \alpha_1, \alpha_2, \ldots, \alpha_N \in R$.

From (9) and (10) one obtains:

$$\varphi_{y_i}(\lambda_i + \alpha_i \lambda) = \varphi_{y_i}(\lambda_i) + \beta_i \varphi_{y_i}(\lambda) \quad (i = 1, 2, \ldots, N), \tag{11}$$

where $f(y) = \beta_1 f(y_1) + \beta_2 f(y_2) + \cdots + \beta_N f(y_N)$; $\beta_1, \beta_2, \ldots, \beta_N \in R$.

The equation $h(\mu + v) = h(\mu) + h(v)$, where $\mu, v \in R_+ \cup \{0\}$ and $h: R_+ \cup \{0\} \to R_+ \cup \{0\}$ is a continuous function with $h(0) = 0$ and $h(1) = 1$, has a unique solution for h, namely the identical map.

One obtains from (11) $\varphi_y(\lambda) = \lambda$, $\lambda \in R_+$. Taking into account also (9) and the properties a) and b) of F it follows that the restriction of f to C is linear and (7) is true for any $x \in M$, $u, v \in C$ and $f \in G_0$ (because $g_x \in G_0$, where $g_x(y) = f(x+y) - f(x)$, $y \in M$).

It follows from (8) and from the property of continuity of $f \in G_0$ that $f(u) = -f(-u)$ for $x \to -u$ and $v \to u$, and $f(u-v) = f(u) - f(v)$ for $x \to -v$. Using these indications and the fact that M is generated by C it follows that f is additive being also continuous according to Lemma 2 f is linear. Then G is a subgroup of the affine group of M, because it is generated by the translations of M and by the linear transformations belonging to G_0. ⊣

Remark 1. For $N = 2$ the former proof of the theorem is no more applicable since $F = \{(\varepsilon \|\lambda\|, \pm\lambda); \lambda \in R_+\}$ and does not exist $y \in F$ linearly independent of the vectors of the basis $\{y_1 = (\varepsilon \|1\|, 1)$, $y_2 = (\varepsilon \|1\|, -1)\}$, consequently there is not $y \in F$ which verifies (10). In this case the non affine transformation will also belong to G and $f \in G_0$ has the following form:

$$f(x) = g(\xi) y_1 + h(\eta) y_2, \quad \text{or} \quad f(x) = g(\eta) y_1 + h(\xi) y_2,$$

with $x = \xi y_1 + \eta y_2$, where $\xi, \eta \in R$ and $g: R \to R$ and $h: R \to R$ are the continuous monotonous increasing functions [1, 3]. ⊣

Remark 2. We define $G_0' = G_0 \cap SL(N, R)$ where $SL(N, R)$ is the special linear group of M. For example for $\|\bar{x}\| = \left[\sum_{i=1}^{N-1} |x^i|^p\right]^{\frac{1}{p}}$

$\bar{x}=(x^1, x^2, \ldots, x^{N-1}) \in \bar{M}$, G'_0 is the orthochronous Lorentz group for $p=2$ and it is the discrete group of the permutations and the symmetries relative to the origin of the basis vectors of $\bar{M}$ for $p>2$.

The factor group G_0/G'_0 is the dilation group of M.

According to the theorem it follows that G is the semi-direct product of the translation group with the direct product of the dilation group with the subgroup G'_0 of $SL(N, R)$.

Finally G'_0 is a topological subgroup of $SL(N, R)$. It follows from Lemma 1 and from its consequence that $f \in G'_0$ if and only if $f(Q+a) = Q+f(a)$, $a \in M$, and $f(0)=0$.

Let $\{f_n\}_{n=1,2,\ldots} \subset G'_0$ be a sequence with $f=\lim_{n\to\infty} f_n$. Since Q is closed, it follows that $f(x)=\lim_{n\to\infty} f_n(x) \in Q+f(a)$ for any $x \in Q+a$. One obtains $f \in G'_0$ and G'_0 is a closed subgroup of $SL(N, R)$. Hence G'_0 is a Lie group [8]. $\lrcorner$

Conclusion. In particular for the four-dimensional space-time M when the positive cone C is the Minkowski cone, the theorem was proved by Zeeman [1]. We proved that the causal group is a subgroup of the affine group of M not requiring a concrete form of the positive cone. It is enough to suppose the compatibility of the causal relation with the linearity of the space on which is defined the causal relation, the antiisotony for time inversion and the closing of the positive cone.

The authors would like to thank Prof. C. Teleman for a helpful discussion and Dr. I. Zamfirescu for reading an earlier draft of the present paper.

References

1. Zeeman, E. C.: J. Math. Phys. **5**, 490 (1964).
2. Fuchs, L.: Partially ordered algebraic systems. New York: Pergamon Press 1963.
3. Bourbaki, N.: Espaces vectoriels topologiques; Chap. II. Paris: Hermann 1953.
4. Barucchi, G., and G. Teppati: Nuovo Cimento **52** A, 50 (1967).
5. Alonso, J. L., and F. Y. Yndurain: Commun. Math. Phys. **4**, 349 (1967).
6. Teppati, G.: Nuovo Cimento **54** A, 800 (1968).
7. Robertson, A., and N. Robertson: Topological vector spaces. Cambridge: University Press 1964.
8. Helgason, L.: Differential geometry and symmetric spaces. New York: Academic Press 1962.

C. Gheorghe
E. Mihul
Institute of Atomic Physics
P.O. Box 35, Bukarest, Rumänien