On the Šilov Boundary of the Vertex Function* **

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Abstract. The Šilov boundary of the vertex function is computed without resource to analytic completion techniques.

1. Introduction

A possible systematic approach to field theory is to derive analytic expressions (integral representations) for the various Green functions. These should express the content of the linear axioms, namely, Lorentz-invariance, energy-momentum spectrum, and local-commutativity. The integral representations should then be substituted into the non-linear unitarity [1] or positive definiteness [2] relations for further investigation.

The linear axioms imply analyticity of the Green functions in certain permuted domains [3] $-D$. They may generally be continued into $E(D)$ – the envelope of holomorphy [4] of D. One then tries to set up a generalized Cauchy integral representation for the functions analytic in $E(D)$, which is the desired expression. Now, the Šilov boundary of a domain is the smallest subset of the domain on which one can hope to represent a holomorphic function by an integral representation. Hence the Šilov boundary is all that is actually needed. It is known [5] that the Šilov boundary of a domain $-S(D)$ coincides with that of its envelope of holomorphy, i.e.:

$$S(D) = S(E(D)) .$$

In view of the difficulty of finding $E(D)$ it may be interesting to calculate $S(D)$ directly.

Furthermore it is not clear whether calculating $S(E(D))$ is always easier than calculating $S(D)$. For the vertex function in configuration space this turns out to be the case. This, however, may be accidental.

In this paper we consider the vertex function. Section 2 describes the Envelope of Holomorphy – $E(D)$ – of the configuration space vertex and its Šilov boundary – $S(E(D))$. In Section 3 are described the primitive domain – D – of the configuration space vertex and its distinguished boundary – $d(D)$.

In Section 4 we compare $d(D)$ and $S(E(D)) = S(D)$, then show how to locate $S(D)$ by some simple considerations.

In Sections 5 and 6 we apply our method to specify an upper bound on the Šilov boundary of the momentum space vertex.

2. The Šilov Boundary of the Configuration Space Vertex Function

The Envelope of Holomorphy of the configuration space vertex function is an analytic polyhedron [6]. It is well known that the Šilov boundary of an analytic polyhedron is identical with its distinguished boundary (Ref. [5], § 6.4). The latter consists of the union of the intersections of any three boundary hypersurfaces. It was computed by Källén and Toll [7]. To facilitate comparison and introduce our notations we shall briefly describe these results. The Wightman function

$$F^{ABC}(x - x', x' - x'') = \langle 0| A(x) B(x') C(x'') |0\rangle$$

is the boundary value of an analytic function in the variables

$$z_1 = -(x - x' - i\eta)^2 ,$$
$$z_2 = -(x' - x'' - i\eta')^2 ,$$
$$z_3 = -(x - x'' - i(\eta + \eta'))^2$$

in the domain obtained by $\eta, \eta' \in V+$, which is called the primitive domain [3]. Due to local commutativity this analytic function may be continued into the permuted domain [3] D, which is the union of the primitive domain and the domains obtained therefrom by permutations of z_1, z_2, z_3.

The envelope of holomorphy of the permuted domain $E(D)$ is bounded by the following analytic hypersurfaces [6]:

Cuts: $\quad z_k = \varrho > 0, \quad k = 1, 2, 3 ,$

$F'_{kl}: \quad z_m = z_k + z_l - \varrho - \dfrac{z_k z_l}{\varrho} ,$

$\qquad 0 < \varrho < \infty, \quad y_m y_k < 0, \quad y_m y_l < 0 .$

$\mathfrak{F}: \quad z_1 z_2 + z_2 z_3 + z_1 z_3 - \varrho(z_1 + z_2 + z_3) + \varrho^2 = 0 ,$

$\qquad 0 < \varrho < \infty, \quad y_1 y_2 > 0, \quad y_1 y_3 > 0 .$

Here $y_k = \operatorname{Im} z_k$.

Next we describe the distinguished boundary

$$z_1 \text{cut} \wedge z_2 \text{cut} \wedge F'_{12} . \tag{A^3_1}$$

The z_1 and z_2 cuts should be approached from $y_1 y_2 > 0$. Fig. 1a illustrates the case $y_1 > 0$, $y_2 > 0$, while Fig. 1b corresponds to the completely symmetric $y_1 < 0$, $y_2 < 0$. We shall not mention these obvious symmetric situations any more.
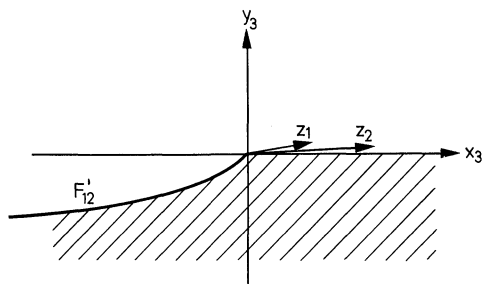

Fig. 1a. The intersection $z_1 \text{cut} \wedge z_2 \text{cut} \wedge F'_{12}$ shown in the z_3-plane for $y_1, y_2 > 0$
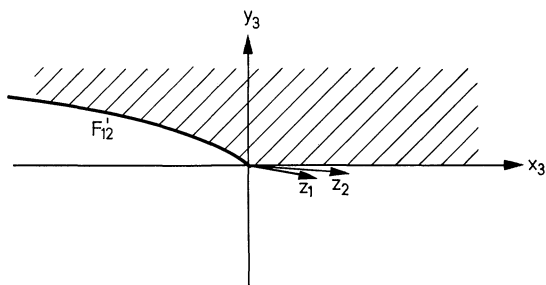

Fig. 1b. The intersection $z_1 \text{cut} \wedge z_2 \text{cut} \wedge F'_{12}$ shown in the z_3-plane for $y_1, y_2 < 0$

F'_{12} should be approached from the domain of regularity (unshaded). F'_{12} is infinitesimally deformed (Ref. [7], § 2).

$$z_1 \text{cut} \wedge z_2 \text{cut} \wedge \mathfrak{F} , \tag{A^3_2}$$

z_1 and z_2 cuts approached from $y_1 y_2 > 0$. Fig. 2 illustrates $y_1 > 0$, $y_2 > 0$. $\mathfrak{F}$ is infinitesimally deformed. Fig. 3 sums up the (A^3) contribution, i.e. $z_1 \text{cut} \wedge z_2 \text{cut} \wedge (F'_{12} \cup \mathfrak{F})$, z_1 and z_2 cuts approached from $y_1 y_2 > 0$.

$$z_1 \text{cut} \wedge z_2 \text{cut} \wedge F'_{23} . \tag{B^3_1}$$

The z_1 and z_2 cuts should be approached from $y_1 y_2 < 0$. Figs. 4a and 4b illustrate the situation $y_1 > 0$, $y_2 > 0$.

D_2 appears in Fig. 4b, where $x_2 > x_1$. In D_2 all x_k are positive and fulfill the inequality $x_2 > (\sqrt{x_1} + \sqrt{x_3})^2$:

$$z_1 \text{ cut} \wedge z_2 \text{ cut} \wedge F'_{13}, \tag{B$_2^3$}$$

illustrated in Figs. 4a and 4b.

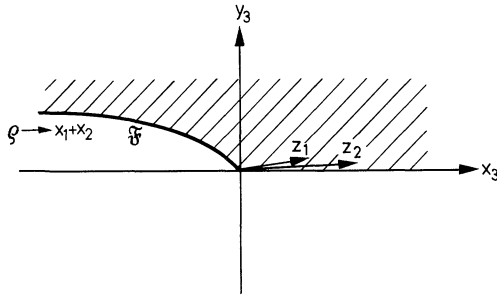$$z_1 \text{ cut} \wedge z_3 \text{ cut} \wedge (\mathfrak{F} \cup F'_{13}), \tag{A^2}$$

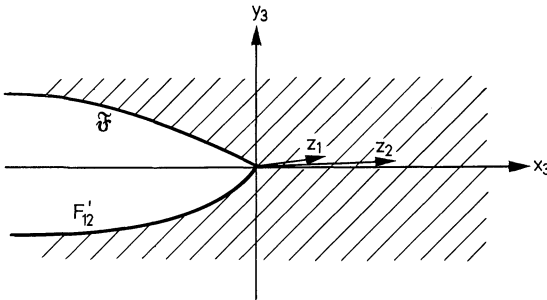Fig. 2. The intersection z_1 cut $\wedge$ z_2 cut $\wedge$ $\mathfrak{F}$ shown in the z_3-plane

Fig. 3. The intersection z_1 cut $\wedge$ z_2 cut $\wedge$ $(F'_{12} \cup \mathfrak{F})$ shown in the z_3-plane

illustrated in Figs. 5a $(x_1 > x_3)$ and 5b $(x_1 < x_3)$,

$$z_1 \text{ cut} \wedge z_3 \text{ cut} \wedge (F'_{12} \cup F'_{32}), \tag{B^2}$$

illustrated in Figs. 6a $(x_1 > x_3)$ and 6b $(x_1 < x_3)$;

$$z_2 \text{ cut} \wedge z_3 \text{ cut} \wedge (\mathfrak{F} \cup F'_{23}), \tag{A^1}$$

$$z_2 \text{ cut} \wedge z_3 \text{ cut} \wedge (F'_{13} \cup F'_{12}). \tag{B^1}$$

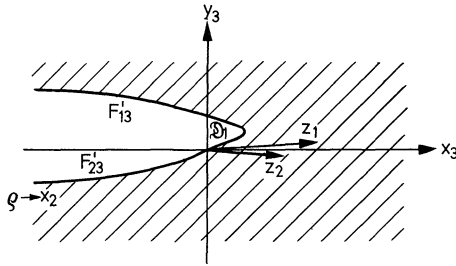These are obtained from (A^2), (B^2) by cyclic permutation of indices.

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Fig. 4a. The intersection z_1 cut $\wedge$ z_2 cut $\wedge (F'_{23} \cup F'_{13})$ shown in the z_3-plane for the case $x_2 < x_1$
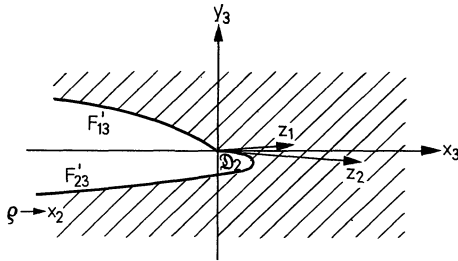

Fig. 4b. The intersection z_1 cut $\wedge$ z_2 cut $\wedge (F'_{23} \cup F'_{13})$ shown in the z_3-plane for the case $x_2 > x_1$

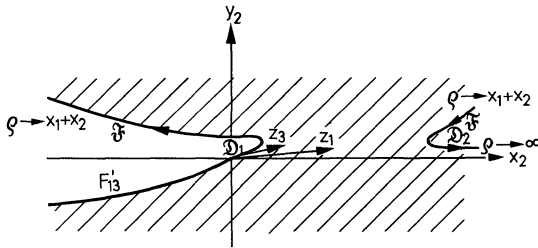

Fig. 5a. The intersection z_1 cut $\wedge$ z_3 cut $\wedge (\mathfrak{F} \cup F'_{13})$ shown in the z_2-plane for the case $x_1 > x_3$

Fig. 5b. The intersection z_1 cut $\wedge$ z_3 cut $\wedge (\mathfrak{F} \cup F'_{13})$ shown in the z_2-plane for the case $x_1 < x_3$

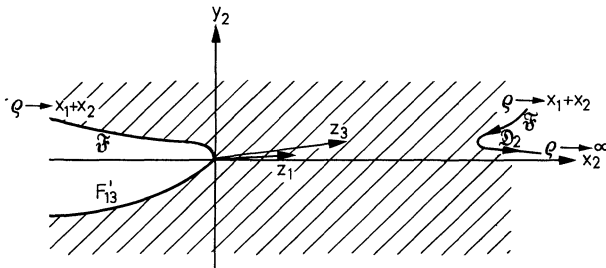Fig. 6a. The intersection z_1 cut $\wedge$ z_3 cut $\wedge$ $(F'_{12} \cup F'_{23})$ shown in the z_2-plane for the case $x_1 > x_3$

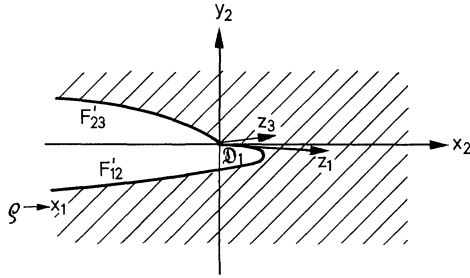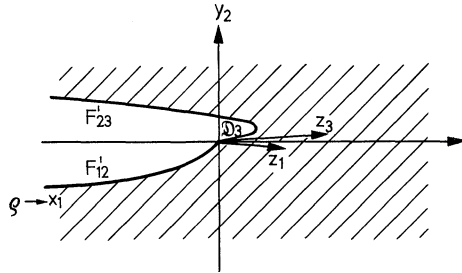
Fig. 6b. The intersection z_1 cut $\wedge$ z_3 cut $\wedge$ $(F'_{12} \cup F'_{23})$ shown in the z_2-plane for the case $x_1 < x_3$

3. The Distinguished Boundary of the Permuted Domain

Let us first describe the permuted domain D. Two characteristic situations are described in Figs. 7 and 8 (Ref. [6], p. 22). Unshaded area represents domain of regularity.

It is seen that D is again bounded by analytic hypersurfaces:

$$\text{Cuts:} \qquad z_k = \varrho > 0, \quad k = 1, 2, 3,$$

$$S_{kl}: \qquad z_m = z_k(1 - \varrho) + z_l\left(1 - \frac{1}{\varrho}\right),$$

$$0 < \varrho < \infty, \qquad y_k y_l < 0,$$

$$F_{kl}: \qquad z_m = z_k + z_l + \varrho + \frac{z_k z_l}{\varrho},$$

$$0 < \varrho < \infty, \qquad y_k y_l > 0.$$

However, D is *not* an analytic polyhedron, because an analytic polyhedron is defined by the *intersection* of domains bounded by analytic hypersurfaces (Ref. [5], § 5), whereas here one has a *union* of such domains (Ref. [6], p. 26). Nevertheless the distinguished boundary contains the

Šilov boundary also in this case. This is seen by following closely the arguments in § 5.2 of Bremermann's paper. In the present case, the distinguished boundary $d(D)$ does *not* coincide with the Šilov boundary, because the proof (Ref. [5], § 5.3) that every point on $d(D)$ is a maximum point of some function analytic in D breaks down. Let us describe the distinguished boundary of D.
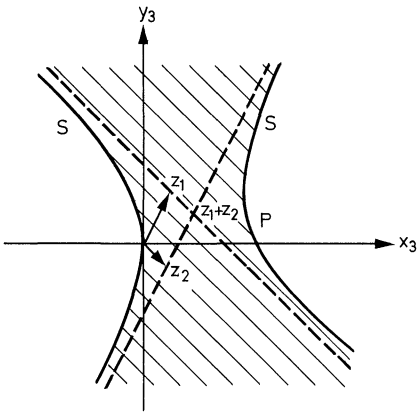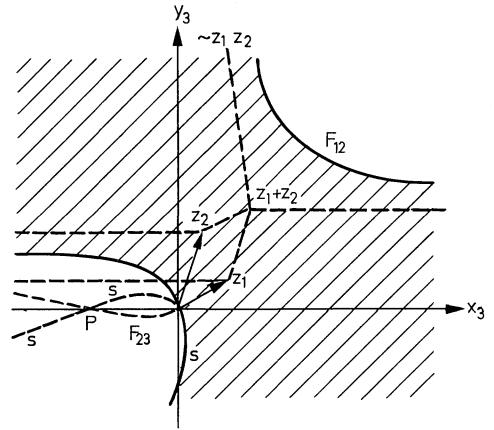
$$z_1 \text{ cut} \wedge z_2 \text{ cut} \wedge S . \qquad \qquad (\text{A}_1'^3)$$

Fig. 7

Fig. 8

Fig. 7. The domain D shown in the z_3-plane for the case $y_1 > 0$, $y_2 < 0$, $x_1 y_2 - x_2 y_1 < 0$

Fig. 8. The domain D shown in z_3-plane for the case $y_2 > y_1 > 0$; $x_1 y_2 + x_2 y_1 > 0$; $x_1 y_2 - x_2 y_1 < 0$

The cuts should be approached from $y_1 y_2 > 0$. Fig. 9 describes the case $y_1, y_2 > 0$.

$$z_1 \text{ cut} \wedge z_2 \text{ cut} \wedge F_{13} \text{ (or } F_{23}) . \qquad \qquad (\text{A}_2'^3)$$

The situation is described in Fig. 10, where the cuts are approached from $y_1, y_2 > 0$.

$$z_1 \text{ cut} \wedge z_2 \text{ cut} \wedge F_{12} . \qquad \qquad (\text{A}_3'^3)$$

The cuts should be approached from $y_1 y_2 > 0$ as in Fig. 11. We see that the intersection of our hypersurfaces passes through a fourth one; the z_3 cut. They are disentangled in the usual way [7, 9] by deforming F_{12}

infinitesimally: $F_{12} \to F_{12}^{\epsilon}$.

$$F_{12}^{\epsilon}: \qquad z_3 = z_1 + z_2 + \varrho(1 + i\varepsilon) + \frac{z_1 z_2}{\varrho}(1 + 2i\varepsilon),$$

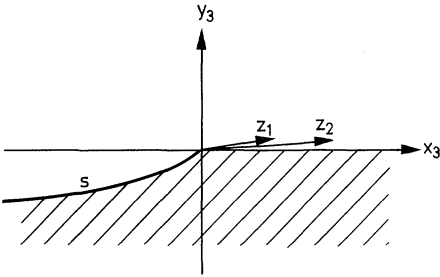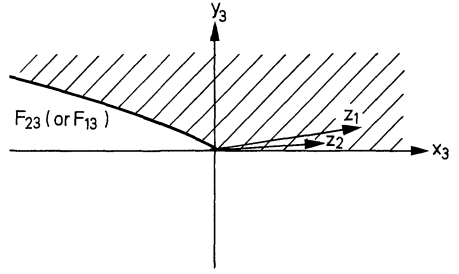$$y_1 y_2 > 0, \qquad \varepsilon y_1 > 0.$$

Fig. 9   Fig. 10

Fig. 9. The intersection z_1 cut $\wedge$ z_2 cut $\wedge$ S shown in the z_3-plane for the case $y_1, y_2 > 0$

Fig. 10. The intersection z_1 cut $\wedge$ z_2 cut $\wedge$ F_{13} (or F_{23}) shown in the z_3-plane for the case $y_1, y_2 > 0$

Thus finally the (A$'^3$) contribution z_1 cut $\wedge$ z_2 $\wedge$ $(S \cup F_{23} \cup F_{12})$, where the z_1 and z_2 cuts are approached from $y_1, y_2 > 0$, is as illustrates in Fig. 12.

$$z_1 \text{ cut} \wedge z_2 \text{ cut} \wedge S. \qquad (\text{B}'^3)$$

Looking up Fig. 7 we see that as $y_1, y_2 \to 0$ the right hand side of S approaches the z_3-cut. As P is indefinite in this limit we may discard
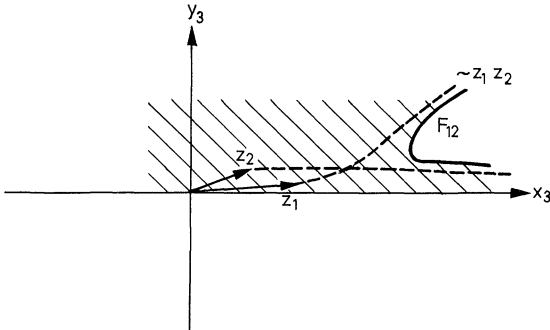

Fig. 11. The intersection z_1 cut $\wedge$ z_2 cut $\wedge$ F_{12} shown in the z_3-plane

this branch. The final situation therefore is as shown in Fig. 13. Similar contributions are obtained from (A'^2), (B'^2), (A'^1), (B'^1). Other contributions:

$$z_1 \text{cut} \wedge S \wedge S.$$

Looking up Fig. 7 we see that when $z_2 = -\alpha z_1$; $\alpha > 0$ a self intersection of S is obtained. This should be intersected with the z_1 (or z_2) cut,
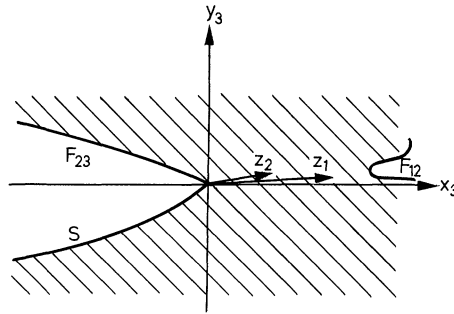

Fig. 12. The intersection z_1 cut $\wedge z_2$ cut $\wedge (S \cup F_{23} \cup F_{12})$ shown in the z_3-plane

Fig. 13. The intersection z_1 cut $\wedge z_2$ cut $\wedge S$ shown in the z_3-plane for the case $y_1 > 0$, $y_2 < 0$

giving the above mentioned and similar contributions to $d(D)$. No further contributions arise except for manifolds of lower dimensions which may be discarded (Ref. [7], p. 764). Except for the contributions above all parts of $d(D)$ belong also to $S(E(D))$. This can be easily seen by direct comparison of the corresponding figures.

4. Direct Location of the Šilov Boundary

We have compared the distinguished boundary of the permuted domain – $d(D)$ and the Šilov boundary of its envelope of holomorphy – $S(E(D))$. General considerations, indicated at the beginning of § 3, show that $d(D) \supset S(E(D))$. Our comparison shows that $d(D)$ nearly coincides with $S(E(D))$, i.e. they are identical except for the contribution due to the self intersection of S.

That this contribution should disappear in $S(E(D))$ is rather obvious when one considers the Kneser technique of analytic completion through corners (Ref. [6], § VI). We have shown, therefore, that in this particular case one is able to compute the Šilov boundary directly without resource to the difficult analytic completion procedure. It must however be admitted that as long as no method is known for writing the integral representation directly, in terms of the Šilov boundary, this analytic completion is necessary.

5. The Šilov Boundary of the Momentum Space Vertex Function

The permuted domain [3] $\hat{D}$ is now bounded by the following analytic hypersurfaces

Cuts: $\qquad z_k = \varrho \geqq a_k$,

S_{kl}: $\qquad z_m = z_k(1 - \varrho) + z_l\left(1 - \dfrac{1}{\varrho}\right)$,

$\qquad\qquad 0 < \varrho < \infty, \qquad y_k y_l < 0$.

F_{kl}: $\qquad z_m = z_k + z_l + \varrho + \dfrac{z_k z_l}{\varrho}$,

$\qquad\qquad 0 < \varrho < \infty, \qquad y_k y_l > 0$.

This is the same as the domain described in Section 3, except for the fact that the thresholds a_k may now be greater than zero.

Hence exactly the same considerations apply here, and the results should be similar to the previous ones. We shall therefore not go into detail but only state the following: The contributions to the Šilov boundary are again of the form z_i cut $\wedge$ z_j cut $\wedge$ F_{kl} or z_i cut $\wedge$ z_j cut $\wedge$ S_{kl}, and are illustrated in Figs. 12 and 13. The expression "z_i-cut" has now, however, the meaning $z_i \geqq a_i$.

Other contributions to the distinguished boundary are: $S \wedge S \wedge z_i$cut, which does not contribute to the Šilov boundary on account of the argument given in Section 4. Also $S \wedge (F_{13} \cup F_{23}) \wedge (z_1$ cut $\cup z_2$ cut) belongs to the distinguished boundary but not to the Šilov boundary, as it does not appear in the partially continued domain of the following section.

6. Partial Completion of $\tilde{D}$ and its Šilov Boundary

The completion of Källén and Wightman [6] is final when all a_k equal zero and is also a *partial* completion for positive thresholds. Let us denote this partially completed domain by $E'(\tilde{D})$. It is bounded by the following analytic hypersurfaces

Cuts: $z_k = \varrho \geqq a_k$.

F'_{kl}: $z_m = z_k + z_l - \varrho - \dfrac{z_k z_l}{\varrho}$,

        $0 < \varrho < \infty$, $y_m y_k < 0$, $y_m y_l < 0$.

$\mathfrak{F}$: $z_1 z_2 + z_2 z_3 + z_1 z_3 - \varrho(z_1 + z_2 + z_3) + \varrho^2 = 0$,

        $0 < \varrho < \infty$, $y_1 y_2 > 0$, $y_1 y_3 > 0$.

This is the same as the domain described in Section 2, except for the fact that the thresholds a_k may now be greater than zero. Hence the same considerations apply here and the results should again be similar. The contributions to the Šilov boundary are once more of the form z_i cut $\cap z_j$ cut $\cap F'_{kl}$; z_i cut $\cap z_j$ cut $\cap \mathfrak{F}$ and are illustrated in Figs. 3, 5 and 6 (z_i cut $- \{z_i : z_i = \varrho \geqq a_i\}$).

There are some contributions to the distinguished boundary which by the arguments mentioned above do not contribute to the Šilov boundary. They are $\mathfrak{F} \cap F'_{ij} \cap z_i$-cut, for all those values of z_i and z_j, where the point of intersection $\mathfrak{F} \cap l'_{ij}$ lies below the z_k threshold (it is well known that $\mathfrak{F}$ and F'_{ij} intersect on the positive z_k-axis) and contributions of the form $F'_{13} \cap F'_{23} \cap z_1$-cut, etc. (which do not appear in the domain described in Section 2). Again we see that although $\tilde{D} \neq E'(\tilde{D})$ and $d(D) \neq d(E'(\tilde{D}))$, we have $S(\tilde{D}) = S(E'(\tilde{D}))$.

The same considerations may be applied to the domain $E''(\tilde{D})$ ($\supset E'(\tilde{D})$) obtained by Brown [8], with similar results. We shall not discuss this in any detail, for Brown's domain is also only a partial completion and moreover its validity is restricted to the single threshold case.

We have now found an upper limit on the Šilov boundary of the momentum space vertex.

In view of our success in Section 4 and the coincidence of our results for $\tilde{D}$, $E'(\tilde{D})$, $E''(\tilde{D})$ we may even hope to have come down directly on $S(\tilde{D})$. However, no rigorous proof of this has been found.

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