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Generalized Symmetries and Infinitesimal Deformations of the Direct Sum P⊕A

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Abstract. The uniqueness of the possible infinitesimal deformation of the direct sum of the Poincaré and arbitrary internal symmetry algebra changing the structure of the direct sum is proved. The necessary restrictions on the internal symmetry algebra are obtained. The properties of this possible deformation are discussed. Its physical applications face serious difficulties.

I. Introduction

The necessity to study deformations of the direct sum $P \oplus A$, where P is the Poincaré algebra and A is any algebra of the internal symmetry of the system, arose within the general frame of relativisation of the internal symmetry after O'Raifeartaigh's "no-go" theorems [1] appeared. It was first shown by Levy-Nahas [2] that the simple De-Sitter algebras are the only nontrivial deformations of P, and the general method of connecting the contraction and deformation theories was proposed. Later it was proved that semisimplicity of internal symmetry algebras did not allow to deform $P \oplus A$ nontrivially if one wanted to make internal and space-time properties connectied [3]. The same appeared to be true also for the algebras with abelian ideal of the special form studied in the previous paper [4]. Thus the most interesting examples of internal symmetry algebras were investigated. But the possibility of existence of more complicated finite dimensional Lie algebras which meet all the necessary requirements remained. They are studied in this article.

The problem is formulated somewhat differently. What deformations of $P \oplus A$ are possible when the internal symmetry is quite general (finite dimensional Lie algebra with abelian ideal)? In Section II the theorem of uniqueness of the infinitesimal nontrivial deformation breaking the direct sum structure in $P \oplus A$ is proved. In Section III the properties of this deformation are discussed. There are great difficulties in its physical applications. Though it seems possible to solve some of them with the help of high order deformations.

II. Theorem of Uniqueness

Every finite dimensional Lie algebra A with zero characteristic may be presented in the form $A = B \dot{\oplus} G$ where G is its radical, B is semisimple and equals $B = A/G$. Let us consider the deformations of $U = P \oplus A$.

Theorem. *Among the nontrivial infinitesimal deformations of $P \oplus A$ there is only one that changes the structure of the direct sum. The corresponding deforming function is*

$$f(t, g) = h(g) \cdot t \tag{1}$$

where $t \in T$ – the translational ideal of P,
$g \in G$
$h(g)$ is a constant depending on g.
$h(g) = 0$ if $g \in A^1$ – the first commutator subalgebra of A.

Proof. The number of nontrivial infinitesimal nonequivalent deformations is not more than the dimension of the 2-cohomology group $H^2(U, U)$. Using Serré and Hochschild's formula [5] one can simplify the necessary group

$$H^2(U, U) = H^2(T \oplus G, U)^U, \tag{2}$$

where $H^2(T \oplus G, U)^U$ is the group of U-invariant elements in $H^2(T \oplus G, U)$.

All the infinitesimal nontrivial deforming functions are the elements of 2-cocycles group $Z^2(T \oplus G, U)^U$. The condition of U-invariance can be formulated for the cocycles $f \in Z^2(T \oplus G, U)$

$$\begin{aligned}(u \cdot f)(j_1, j_2) = [u, f(j_1, j_2)] &- f([u, j_1], j_2) \\ &- f(j_1, [u, j_2]) \in B^2(T \oplus G, U),\end{aligned} \tag{3}$$

where $u \in U = P \oplus A$,
$j_i \in T \oplus G$,
$B^2(T \oplus G, U)$ is the group of 2-coboundaries.

Suppose ϱ denotes the map $T \oplus G \xrightarrow{\varrho} U$, then according to the definition of the group $B^2(T \oplus G, U)$ one can reformulate the relation (3).

$$\begin{aligned}[u, f(j_1, j_2)] - f([u, j_1], j_2) - f(j_1, [u, j_2]) \\ = [j_1, \varrho(j_2)] - [j_2, \varrho(j_1)] - \varrho([j_1, j_2]).\end{aligned} \tag{4}$$

To simplify the narration we shall devide the proof into five propositions.

Proposition 1. *If $j \in T \oplus G$ the U-invariance condition for the cocycles can be presented in the following form*

$$(u \cdot f)(t_1, t_2) = [t_1, \varrho(t_2)] - [t_2, \varrho(t_1)], \tag{5}$$

$$(u \cdot f)(g_1, g_2) = [g_1, \varrho(g_2)] - [g_2, \varrho(g_1)] - \varrho([g_1, g_2]), \tag{6}$$

$$(u \cdot f)(t, g) = [t, \varrho(g)] - [g, \varrho(t)]. \tag{7}$$

These relations are the direct consequence of the linearity of all the operators used. The antisymmetry of the map $f(j_1, j_2)$ makes it possible to consider only one relation for two functions $f(j_1, j_2)$ and $f(j_2, j_1)$.

According to the domain of values let f be represented in the form $f = f_L + f_T + f_B + f_G$. This decomposition and various values of $u \in U$ can give us 16 different though not completely independent relations from each of Eqs. (5), (6), and (7). We shall make use of only some of them.

Among functions that satisfy (5) and (6) only four $f_B(t_1, t_2)$, $f_G(t_1, t_2)$, $f_L(g_1, g_2)$ and $f_T(g_1, g_2)$ can change the structure of the direct sum in U.

Proposition 2. *Functions $f_B(t_1, t_2)$ and $f_G(t_1, t_2)$ are not U-invariant. U-invariant functions $f_L(g_1, g_2)$ and $f_T(g_1, g_2)$ are coboundaries.*

If $u = b \in B$ one can obtain the following condition from the Eq. (5)

$$[b, f_B(t_1, t_2)] = 0. \tag{8}$$

From the semisimplicity of B it follows that only zero maps $f_B(t_1, t_2)$ are invariant.

Let us put $u = l \in L$ in the Eq. (5), then for $f_G(t_1, t_2)$ we have

$$f_G([l, t_1], t_2) + f_G(t_1, [l, t_2]) = 0. \tag{9}$$

The subspace T is four-dimensional. Let us chose t_1 and t_2 to be its basic elements. Then in the subspace L we can construct such a basis that it contains two elements l_{10} and l_{13} commuting with t_2 and transforming t_1 into two other basic elements of T. This gives us $f_G(t_0, t_2) = f_G(t_3, t_2) = 0$. If $f_G(t_i, t_k) = 0$ then $(u \cdot f_G)(t_i, t_k)$ is also zero. Using for u the same elements l_{10} or l_{13} of L we get

$$f_G(t_0, t_2) = f_G(t_1, t_2) = f_G(t_3, t_2) = 0. \tag{10}$$

Using other basic elements of L and the linearity condition for f_G one obtains that all the L-invariant homomorphysms $f_G(t, t')$ are zero maps.

Eq. (6) gives us for $u = l \in L$ the following relation

$$[l, f_T(g_1, g_2)] = -\varrho([g_1, g_2]). \tag{11}$$

In accordance with the standard basis one can write

$$f_T(g_1, g_2) = t_1 f_1(g_1, g_2) + t_2 f_2(g_1, g_2) + t_3 f_3(g_1, g_2) + t_4 f_4(g_1, g_2),$$

where f_i are coefficients depending on g_1 and g_2. The same can be done for $\varrho([g_1, g_2]) = \varrho_T([g_1, g_2])$. One can also choose the basic element l_{ik} for $l \in L$ and transform the Eq. (11) to

$$[l_{ik}, t_k \cdot f_k(g_1, g_2)] = -\varrho_i([g_1, g_2]) \cdot t_i \tag{12}$$

wherefrom using $C^i_{(ik)k}$ for the corresponding structure constant we get

$$f_k(g_1, g_2) = -(C^i_{(ik)k})^{-1} \cdot \varrho_i(g_1, g_2). \tag{13}$$

Taking different $l_{mn} \in L$ one can write other expressions for $f_k(g_1, g_2)$ and see that all of them are equal. (In general maps ϱ_T in the Eq. (11) are different for different elements $l \in L$ taken). So the new function

$$\begin{aligned} \tilde{\varrho}_T([g_1, g_2]) &= \Sigma t_k \cdot \tilde{\varrho}_k([g_1, g_2]) \\ \tilde{\varrho}_k([g_1, g_2]) &= (C^i_{(ik)k})^{-1} \cdot \varrho_i([g_1, g_2]) \qquad i = 1, 2, 3, 4; \qquad i \neq k \end{aligned} \tag{14}$$

is constructed so that

$$f_T(g_1, g_2) = -\tilde{\varrho}_T([g_1, g_2]) \tag{15}$$

whenever $f_T(g_1, g_2)$ is L-invariant. That means that in the case discussed coboundaries are the only invariant homomorphisms.

Similar calculations lead to the same result for $f_L(g_1, g_2)$ maps.

Now let us study Eq. (7).

Proposition 3. *Functions $f_L(t, g)$ and $f_B(t, g)$ are not U-invariant. U-invariant functions $f_G(t, g)$ are coboundaries.*

Let $u = l \in L$, then for the homomorphisms $f_L(t, g)$ one has

$$[l, f_L(t, g)] - f_L([l, t], g) = 0. \tag{16}$$

Since t here is fixed, one can always take such a basis of P that t will be its element. Then there are three corresponding basic elements of L (say l_1, l_2, l_3) that commute with t. If in (16) we check $l_{1,2,3}$-invariance of $f_L(t, g)$ we come to

$$[l_i, f_L(t, g)] = 0, \qquad i = 1, 2, 3. \tag{17}$$

But the Lorentz algebra has no such element that commutes with any of its three different basic elements. So the consequence of Eq. (17) is that all L-invariant maps $f_L(t, g)$ are zero ones.

Consider now $f_B(t, g)$ while $u = l \in L$ as before. It follows immediately from Eq. (7) that

$$f_B([l, t], g) = 0. \tag{18}$$

This means that all $f_B(t, g)$ must be zero according to the structure of the P-subalgebra.

Let us take $u = l \in L$ again and see what we have now for $f_G(t, g)$ from Eq. (7).

$$f_G([l, t], g) = [g, \varrho(t)] . \tag{19}$$

Making use of P-subalgebra structure and solving the commutator in the left hand side of the relation (19) one obtains the coboundary condition for all the L-invariant maps $f_G(t, g)$.

$$f_G(t', g) = [g, \varrho'(t')] . \tag{20}$$

So there remains only one candidate for the deforming function. This is the $f_T(t, g)$ homomorphism.

Proposition 4. *The L-invariant maps $f_T(t, g)$ can be devided into two parts. The first one called the non-diagonal $f_T^{nd}(t, g)$ is a coboundary, while the second called diagonal $f_T^d(t, g)$ is totally invariant and cocyclic.*

As far as functions $f_T(t, g)$ are concerned Eq. (7) gives us the following for $u = l \in L$

$$[l, f_T(t, g)] - f_T([l, t], g) = [t, \varrho(g)] . \tag{21}$$

Here the Poincaré algebra conventional commutation relations are necessary.

$$\begin{aligned} [l_{\mu\nu}, t_\gamma] &= -\tilde{g}_{\mu\gamma} t_\nu + \tilde{g}_{\nu\gamma} t_\mu , \\ [l_{\mu\nu}, l_{\gamma\sigma}] &= -\tilde{g}_{\mu\gamma} l_{\nu\sigma} - \tilde{g}_{\nu\sigma} l_{\mu\gamma} + \tilde{g}_{\nu\gamma} l_{\mu\sigma} - \tilde{g}_{\mu\sigma} l_{\nu\gamma} , \\ [t_\varrho, t_\sigma] &= 0 . \end{aligned} \tag{22}$$

In this basis the matrix forms of the functions f_T and ϱ are

$$\begin{aligned} f_T(t_\gamma, g) &= d_\gamma^\sigma(g) \cdot t_\sigma , \\ \varrho(g; l) &= k^{\alpha\beta}(g; l) \cdot l_{\alpha\beta}; \qquad \alpha < \beta . \end{aligned} \tag{23}$$

There may be different maps $\varrho(g)$ in Eq. (21) when various elements l are taken. This is demonstrated in formula (23). One has then the following relation for the coefficients $d_\gamma^\sigma(g)$ and $k^{\alpha\beta}(g; l)$

$$\begin{aligned} (-\tilde{g}_{\mu\sigma}\delta_{\nu\eta} + \tilde{g}_{\nu\sigma}\delta_{\mu\eta}) \cdot d_\gamma^\sigma(g) + \tilde{g}_{\mu\gamma} d_\nu^\eta(g) - \tilde{g}_{\nu\gamma} d_\mu^\eta(g) \\ = (\tilde{g}_{\alpha\gamma}\delta_{\beta\eta} - \tilde{g}_{\beta\gamma}\delta_{\alpha\eta}) k^{\alpha\beta}(g, l_{\mu\nu}) . \end{aligned} \tag{24}$$

Choosing different values of μ, ν, η and γ one obtains two groups of relations. For the diagonal part of the matrix d_ν^μ the condition is

$$d_\mu^\mu(g) = d_\nu^\nu(g) = h(g); \qquad \mu, \nu = 0, 1, 2, 3 . \tag{25}$$

The second group of relations contains only non-diagonal elements

$$d_k^i(g) = -d_i^k(g)\,, \qquad d_i^0(g) = d_0^i(g); \quad i,k = 1,2,3\,. \tag{26}$$

There is no connection between diagonal and non-diagonal parts so they are independent and may be studied separately.

Considering first the non-diagonal part of the original map $f_T^{nd}(t, g)$ it is easy to verify that the L-invariant function is the coboundary. We shall simply present the explicit form of the map $\varrho(g)$ necessary to write the coboundary condition for the function $f_T^{nd}(t_\gamma, g) = d_\gamma^\sigma(g) \cdot t_\sigma$

$$f_T^{nd}(t_\gamma, g) = [t_\gamma, \varrho(g)]\,, \tag{27}$$

$$\varrho(g) = d_\gamma^\sigma(g) \cdot (g^{\gamma\gamma})^{-1} \cdot l_{\sigma\gamma}; \qquad \sigma < \gamma = 0,1,2,3\,. \tag{28}$$

Thus only the diagonal functions $f_T^d(t, g)$ are of interest to us, for one can't find the necessary function $\varrho(g)$ to form the relation similar to (27) for them. According to Eq. (7) they are trivially T-invariant and their A-invariance leads to the following restrictions

$$f_T^d(t, [a, g]) = 0; \qquad a \in A\,. \tag{29}$$

So on all the elements $g \in A^1$ the U-invariant homomorphisms $f_T^d(t, g)$ must be zero.

It is necessary now to check whether they are the cocycles. That means that the following relation must hold for the maps $f_T^d(t, g)$

$$\sum_{P(1,2,3)} \{[j_1, f(j_2, j_3)] - f([j_1, j_2], j_3)\} = 0\,, \tag{30}$$

where the summation is over the cyclic permutation of indices. Imposing condition (29) we see that this requirement is fulfilled. So the function $f_T^d(t, g) = h(g) \cdot t$ (see condition (25)) with the properties (29) represents the element of $H^2(T \oplus G, U)$ and thus is the only possible deforming function that can change the structure of the direct sum $P \oplus A$. But only if the $H^3(U, U)$ group has zero dimension all the members of $H^2(T \oplus G, U)^U$ will give rize to deformations.

Proposition 5. *There are no obstructions to the infinitesimal deformation with the function $f_T^d(t, g) = h(g) \cdot t$, where $h(g) = 0$ when $g \in A^1$.*

According to the general rule it is necessary to show that $\dim H^3(U, U) = 0$. In our case there are non-zero elements of $H^3(U, U)$. For example the function $f_L(t_\alpha, t_\beta, g) = g^{\alpha\mu} g^{\beta\nu} l_{\mu\nu} k(g)$, where $k(g)$ is a numerical factor depending on the element $g \in G$, belongs to $H^3(U, U)$. But one must remember that particularly in the case of infinitesimal deformations not every element of $H^3(U, U)$ gives the obstruction to deformation. The only thing one must check is

$$\sum_{P(1,2,3)} f(f(j_1, j_2), j_3) \in B^3(T \oplus G, U). \tag{31}$$

If the condition (31) is fulfilled for the element of $H^2(U, U)$ this homomorphism will give rise to deformation no matter what dimension of $H^3(U, U)$ is. Using the explicit form of the function $f_T^d(t, g) = h(g) \cdot t$ it is quite easy to verify that the left hand side of the relation (31) simply equals zero. So there are no obstructions to this deformation.

This concludes the proof of the theorem.

III. Some Properties of the Deformation

First we would like to point out that the restrictions on the possible internal symmetry algebra attached to the Poincaré algebra are not severe. In the radical of such an algebra A there must be the elements that do not belong to A^1. The situation is uninteresting when the central elements of A are considered. Because if $Z(A)$ is not the subspace of A^1 then $A = B \oplus Z(A)$ and $f_T^d(t, g)$ describes the deformations of $P \oplus Z(A)$ which is the case of Abelian internal symmetry.

The second important property of this deformation is that the Poincaré subalgebra as well as the internal symmetry one remains unchanged. So the results of O'Raifeartaigh [1] can be used here to indicate that there are no self-adjoint mass operators describing descrete mass-spectrum in the deformed algebra. The situation is realized here in the following way. If there is any Kazimir operator containing the usual mass, the deformed commutation relations are such that this operator must also contain elements of G that do not form an invariant of the subalgebra A. There is no physical mass-spectrum and no possibility to classify particles using the internal symmetry subalgebra. Even the invariant operators of the internal symmetry in the initial algebra $P \oplus A$ do not help to form the mass-spectrum [6] for the internal symmetry algebra is also stable in our case.

So the applications of the generalised symmetry obtained are doubtful. It remains to be solved whether there are any nontrivial deformations of the higher order which can change the structure of P and A subalgebras and thus give physically interesting results.

To illustrate the situation we give a very simple example. Let A be the 3-dimensional solvable algebra $A = G = \{g\}$.

$$[g_1, g_2] = g_3; \quad [g_3, g_1] = 0; \quad [g_3, g_2] = 0.$$

It is possible to deform $P \oplus G$ with the deforming function $f_T(t_\alpha, g_k) = h(g_k) \cdot t_\alpha$ where $k = 1, 2$. We shall write down only those commutation relations that change

$$[t_\alpha, g_1] = t_\alpha \cdot h(g_1) \cdot c,$$
$$[t_\alpha, g_2] = t_\alpha \cdot h(g_2) \cdot c,$$

where c is the deformation parameter. All the other commutation relations are identical with those of $P \oplus G$ algebra.

It is easy to find the corresponding contraction procedure to obtain $P \oplus G$ algebra from the deformed one. The necessary limit transformation is

$$g_1 = \lim \varepsilon g_1; \quad g_2 = \lim \varepsilon g_2; \quad g_3 = \lim \varepsilon^2 g_3; \quad \varepsilon \to 0.$$

One can also see that Doebner, Melsheimer's criterium [7] of contraction into the direct sum is fulfilled.

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