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On the Equivalence of Additive and Analytic Renormalization

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Abstract. The equivalence of additive and analytic renormalization is proved for any choice of finite renormalizations and any fixed generalized evaluator.

In recent years two rigorous renormalization schemes for the perturbation series of the Green's functions in Lagrangian quantum field theory have been shown to be closely related, the additive renormalization (R-operation) of Bogoliubov and Parasiuk [1, 2] and the analytic renormalization of Speer [3]. Speer has shown that an analytic renormalization starting from any generalized evaluator leads to an additive renormalization for certain finite renormalizations. For pedagogical reasons we remark that the converse is true in the following sense:

In the notation of [2] and [3] let $\mathscr{F}$ be a generalized evaluator and $G(V_1, \ldots V_n; \mathscr{L})$ any graph in Lagrangian quantum field theory. Let $U_1, \ldots U_m$ be any partition of $\{V_1, \ldots V_n\}$, let $\hat{\mathscr{X}}_\varepsilon(U_1), \ldots \hat{\mathscr{X}}_\varepsilon(U_m)$ be finite renormalizations and let $\mathscr{T}_{\underline{\lambda}, \varepsilon, r}(U_1, \ldots U_m)$ be the analytically regularized Feynman amplitude of $G(U_1, \ldots U_m, \mathscr{L})$,

$$\mathscr{T}_{\underline{\lambda}, \varepsilon, r}(U_1, \ldots U_m) = \prod_{i=1}^{m} \hat{\mathscr{X}}_\varepsilon(U_i) \prod_{\text{conn}} \Delta^l_{\lambda(l), \varepsilon, r} \,. \tag{1}$$

According to [3] $\mathscr{F}\mathscr{T}_{\underline{\lambda}, 0, 0}(U_1, \ldots U_m) \in \mathscr{S}'(\mathbb{R}^{4n})$.

Let $\{\tilde{\mathscr{X}}_\varepsilon(U)\}$ be any choice of finite renormalizations for all

$$U \subset \{V_1, \ldots V_n\}$$

and

$$\tilde{\mathscr{R}}\mathscr{T}_{1, \varepsilon, r}(V_1, \ldots V_n)$$

be the R-operation on

$$\mathscr{T}_{1, \varepsilon, r}(V_1, \ldots V_n)$$

including the $\{\tilde{\mathscr{X}}_\varepsilon(U)\}$. According to [1–3]

$$\lim_{\varepsilon \downarrow 0} \lim_{r \downarrow 0} \tilde{\mathscr{R}}\mathscr{T}_{1, \varepsilon, r}(V_1, \ldots V_n) \in \mathscr{S}'(\mathbb{R}^{4n}) \tag{2}$$

is an additive renormalization.

Theorem. *Given any generalized evaluator $\mathcal{F}$, any graph $G(V_1, \ldots V_n; \mathcal{L})$ and any choice of finite renormalizations $\{\tilde{\mathcal{X}}(U)\}$. Then there exist a unique set of finite renormalizations $\{\hat{\mathcal{X}}_\varepsilon(U)\}$, such that*

$$\lim_{\varepsilon\downarrow 0}\lim_{r\downarrow 0}\tilde{\mathcal{R}}\mathcal{T}_{1,\varepsilon,r}(V_1, \ldots V_n) = \mathcal{F}\sum_P \mathcal{T}_{\lambda,0,0}(U_1^P, \ldots U_{m(P)}^P), \tag{3}$$

where $\sum_P$ extends over all partitions P of $\{V_1, \ldots V_n\}$, and the $\mathcal{T}_{\lambda,\varepsilon,r}(U_1^P, \ldots U_{m(P)}^P)$ are defined by (1) *starting from $\hat{\mathcal{X}}_\varepsilon(U_1^P), \ldots \hat{\mathcal{X}}_\varepsilon(U_{m(P)}^P)$.*

Proof. Let $\{\hat{\mathcal{X}}_\varepsilon(U)\}$ be any choice of finite renormalizations. By [3], Theorem 3, one has

$$\mathcal{F}\mathcal{T}_{\lambda,0,0}(U_1, \ldots U_m) = \lim_{\varepsilon\downarrow 0}\lim_{r\downarrow 0}\mathcal{R}^{\mathcal{F}}\mathcal{T}_{1,\varepsilon,r}(U_1, \ldots U_m) \tag{4}$$

where $\mathcal{R}^{\mathcal{F}}$ is the R-Operation on $\mathcal{T}_{1,\varepsilon,r}(U_1, \ldots U_m)$ including the finite renormalizations

$$\mathcal{X}_\varepsilon^{\mathcal{F}}(U_1', \ldots U_s') = \begin{cases} \hat{\mathcal{X}}_\varepsilon(U_1') & \text{for } s=1 \\ 0 \quad \text{for} & G(U_1', \ldots U_s'; \mathcal{L})\, IPR \\ \mathcal{F}\mathcal{M}\mathcal{T}_{\underline{\lambda},\varepsilon,0}(U_1', \ldots U_s') & \text{otherwise} \end{cases}$$

for any $\{U_1', \ldots U_s'\} \subset \{U_1, \ldots U_m\}$.

Furthermore

$$\mathcal{R}^{\mathcal{F}}\mathcal{T}_{1,\varepsilon,r}(U_1, \ldots U_m) = \sum_Q \mathcal{R}\mathcal{T}_{1,\varepsilon,r}(W_1^Q, \ldots W_{k(Q)}^Q) \tag{6}$$

where $\sum_Q$ is over all partitions $W_1^Q, \ldots W_{k(Q)}^Q$ of $\{U_1, \ldots U_m\}$ and $\mathcal{R}$ is the R-operation without finite renormalizations starting from the vertex parts $\mathcal{X}_\varepsilon^{\mathcal{F}}(W_i^Q) = \mathcal{X}_\varepsilon^{\mathcal{F}}(U_{i1}, \ldots U_{ij})$, if $W_i^Q = \{U_{i1}, \ldots U_{ij}\}$.

For every $U = \{V_1', \ldots V_m'\} \subset \{V_1, \ldots V_n\}$ we define the finite renormalization $\tilde{\mathcal{X}}_\varepsilon(U)$ by

$$\tilde{\mathcal{X}}_\varepsilon(U) = \hat{\mathcal{X}}_\varepsilon(U) + \begin{cases} \sum_Q{}' \mathcal{X}_\varepsilon^{\mathcal{F}}(W_1^Q, \ldots W_{k(Q)}^Q), \\ \text{if} \quad G(W_1^Q, \ldots W_{k(Q)}^Q; \mathcal{L}) \quad \text{is } IPI \\ 0, \quad \text{otherwise} \end{cases} \tag{7}$$

where $\sum_Q{}'$ extends over all partitions $W_1^Q, \ldots W_{k(Q)}^Q$ of $\{V_1', \ldots V_m'\}$ into $k(Q) > 1$ sets. We claim that (3) holds. For, use (4) and (6) for the right-hand side of (3). Expand the left-hand side of (3),

$$\tilde{\mathcal{R}}\mathcal{T}_{1,\varepsilon,r}(V_1, \ldots V_n) = \sum_P \mathcal{R}\mathcal{T}_{1,\varepsilon,r}(U_1^P, \ldots U_{m(P)}^P); \tag{8}$$

where the $\mathcal{T}_{1,\varepsilon,r}(U_1^P, \ldots U_{m(P)}^P)$ are defined from the $\tilde{\mathcal{X}}_\varepsilon(U_i^P)$. The $\mathcal{R}\mathcal{T}_{1,\varepsilon,r}(U_1^P, \ldots U_{m(P)}^P)$ are multilinear in their vertex parts and can be expanded, if (7) is inserted. The resulting sums over R-operations on the left- and right-hand side of (3) are identical for $\varepsilon \downarrow 0, r \downarrow 0$.

One sees from (7) that one can recursively obtain any choice of finite renormalizations $\{\tilde{\mathscr{X}}_\varepsilon(U)\}$ by a unique choice of $\{\hat{\mathscr{X}}_\varepsilon(U)\}$, since the second term only depends on contributions from proper partitions of $\{V_1', \dots V_m'\}$. QED.

Speer has introduced one particular generalized evaluator, which is given by a multiple contour integral in the region of holomorphy of $\mathscr{T}_{\underline{\lambda}, \varepsilon, 0}(V_1, \dots V_n)$. It is useful to know for studying the analytic structure of perturbation theory that the R-operation for any choice of finite renormalizations can be effected by this concrete integral representation.

References

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