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Symmetry Transformations from Local Currents

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Abstract. For internal symmetries it is shown that it is possible to construct automorphisms for a Haag-Araki local ring system $\{\mathscr{R}(\mathcal{O})\}$ from a local current affiliated to it. Although the "charges" Q_V for finite volume V do not converge for $V \to \infty$ we prove the convergence of the corresponding automorphisms of $\{\mathscr{R}(\mathcal{O})\}$. For external symmetries which map bounded space-time regions into unbounded ones (e.g. translations) we have to require some additional continuity condition on the isomorphisms corresponding to Q_V to get convergence.

In the usual Lagrangian formulation of Quantum Field Theory one derives in a formal way a local current $j^\mu_G(.)$ for every one-parameter transformation group G which acts nontrivially on the fields. Formally the space-integral $\int j^0_G(\boldsymbol{x}, t)\, d^3\boldsymbol{x}$ serves as infinitesimal generator for a unitary representation of G in the Hilbertspace of states. However, because of vacuum fluctuations, the local "charges" $Q_V(t) = \int\limits_V j^0(\boldsymbol{x}, t)\, d^3\boldsymbol{x}$ for finite volume V turn out not to converge in any useful way (strong or weak topology for operators) for increasing volume [1], Theorem 3.1, even if one takes care of distributional difficulties and smears the current in space and time with C_0^∞-functions. So the question arises how to construct symmetry transformations for the algebra of fields or observables from a given local current j^μ. This problem also arises in the usual formulation of the "Goldstone Theorem" [1, 2] where one assumes the existence of a group of automorphisms of the algebra of quasilocal observables generated by a local current j^μ. One may ask then if these assumptions are compatible.

Since one is not primarily interested in a global unitary transformation to implement the symmetry, which may not even exist as in the case of spontaneously broken symmetries, it would be sufficient if the local symmetry transformations $\alpha_V(\tau)\, A = e^{i\tau Q_V}\, A\, e^{-i\tau Q_V}$ for the algebra of fields or observables $\mathscr{R}(\mathcal{O})$ from some bounded space-time region $\mathcal{O}$ would converge with increasing volume V. This problem is studied in the framework of local v. Neumann algebras in the Haag-Araki [3] sense. In Section 1 we provide some mathematical tools giving the connection between the generator Q of a unitary group $\mathscr{U}(\tau)=e^{i\tau Q}$ in a Hilbert space H and the generator of the corresponding group of automorphisms $\alpha(\tau)\,A=\mathscr{U}(\tau)\,A\,\mathscr{U}^{-1}(\tau)$ of the algebra of bounded operators $\mathscr{B}(H)$ equipped with several interesting topologies.

In Section 2 we give a solution of the problem mentioned above for internal symmetries under rather natural assumptions. In Section 3 we consider the case of space-time symmetries and give a solution under the further assumption (not very natural from a field-theoretic view-point) that the local automorphisms $\alpha_V(\tau)$ are strongly continuous in τ in the uniform operator topology on the local algebras $\mathfrak{A}(\mathcal{O})$.

1. On Generators of Unitarily Implemented Automorphism Groups

Throughout this section Q is assumed to be an essentially self-adjoint (e.s.a.) operator on some domain $D(Q)$ dense in a Hilbert space H. $\mathscr{B}(H)$ denotes the algebra of bounded operators on H. Then Q^* is the uniquely determined self-adjoint extension of Q in H. It is the generator of a strongly continuous group of unitaries $\mathscr{U}(\tau)=e^{i\tau Q^*}$ which gives rise to a one-parameter group of automorphisms $\alpha(\tau)A=\mathscr{U}(\tau)A\,\mathscr{U}^{-1}(\tau)$ of $\mathscr{B}(H)$. $\alpha(.)\,A(A\in\mathscr{B}(H))$ is a continuous map of $\boldsymbol{R}^1\to\mathscr{B}(H)$ equipped with the strong or weak topology from vectors in H [1], but not in general with the norm topology of operators on $\mathscr{B}(H)$. For fixed τ, $\alpha(\tau)$ is a continuous map of $\mathscr{B}(H)\to\mathscr{B}(H)$ for all these topologies. The family $\{\alpha(\tau)\}_{\tau\in\boldsymbol{R}^1}$ is an equicontinuous set in general only for the norm topology on $\mathscr{B}(H)$.

Lemma 1. *Q e.s.a. on $D(Q)$, $A\in\mathscr{B}(H)\Rightarrow$*

$$\frac{d}{d\tau}(x,\alpha(\tau)A\,y)=i(Q^*x,\alpha(\tau)A\,y)-i(x,\alpha(\tau)A\,Q^*y)\ \ \forall x,y\in D(Q^*).$$

Proof. Lemma 1 is an immediate consequence of *Stone's* theorem [4].

Lemma 2. *Q e.s.a. on $D(Q)$, $A\in\mathscr{B}(H)$, $A\,D(Q^*)\subset D(Q^*)\Rightarrow$*

$$\frac{d\alpha(\tau)}{d\tau}Ax=s\cdot\lim_{h\to 0}\frac{\alpha(\tau+h)-\alpha(\tau)}{h}Ax=i[Q^*,\alpha(\tau)A]\,x,\ \ \forall x\in D(Q^*).$$

Proof.

$$\left\|\left(\frac{\alpha(\tau+h)-\alpha(\tau)}{h}A-i[Q^*,\alpha(\tau)A]\right)x\right\|$$
$$\leqq\left\|\left(\frac{\mathscr{U}(\tau+h)-\mathscr{U}(\tau)}{h}A\,\mathscr{U}^*(\tau)-iQ^*\mathscr{U}(\tau)A\,\mathscr{U}^*(\tau)\right)x\right\|$$

[1] In the following "weak" is always to be interpreted in this sense.

$$+\left\|\left(\mathscr{U}(\tau+h)A\frac{\mathscr{U}^*(\tau+h)-\mathscr{U}^*(\tau)}{h}+i\mathscr{U}(\tau+h)A\mathscr{U}^*(\tau)Q^*\right)x\right\|$$
$$+\|(\mathscr{U}(\tau)-\mathscr{U}(\tau+h))A\mathscr{U}^*(\tau)Q^*x\|$$
$$=\left\|\left(\frac{\mathscr{U}(h)-1}{h}\mathscr{U}(\tau)A\mathscr{U}^*(\tau)-iQ^*\mathscr{U}(\tau)A\mathscr{U}^*(\tau)\right)x\right\|$$
$$+\left\|\left(\mathscr{U}(\tau)A\mathscr{U}^*(\tau)\frac{\mathscr{U}^*(h)-1}{h}+i\mathscr{U}(\tau)A\mathscr{U}^*(\tau)Q^*\right)x\right\|$$
$$+\|(1-\mathscr{U}(h))A\mathscr{U}^*(\tau)Q^*x\|$$

which tends to zero for $h\to 0$ for all $x\in D(Q^*)$.

Lemma 3. Q e.s.a. *on* $D(Q)$, $A\in\mathscr{B}(H)$, $AD(Q)\subset D(Q^*)$,

$$\|[Q^*,A]x\|\leqq c\|x\|, \quad \forall x\in D(Q)\Rightarrow AD(Q^*)\subset CD(Q^*).$$

Proof. Let $x\in D(Q^*)$ arbitrary, then there exists a sequence $x_n\in D(Q)$ with $x_n\xrightarrow[n\to\infty]{}x$ and $Qx_n\xrightarrow[n\to\infty]{}Q^*x$ since Q^* is the closure of Q. We derive $Q^*Ax_n=AQx_n+[Q^*,A]x_n\xrightarrow[n\to\infty]{}AQ^*x+[Q^*,A]^-x$ [2]. *From* Q^* being closed we get $Ax\in D(Q^*)$.

Lemma 4. *For a map* $u(.)$ *from* $\mathbf{R}^1$ *into a Banachspace* X *the following statements are equivalent:*

i) $u(.)$ *is analytic at* $t=0$.

ii) $u(.)$ *is infinitely differentiable in some neighbourhood* $|t|<\delta$ *of* $t=0$ *and there exist* $M>0$, $a>0$ *with* $\|u^{(n)}(t)\|\leqq Mn!a^n$ *for all* $|t|<\delta$ *and* $n\in\mathbf{N}$.

Proof. i)$\Rightarrow$ii): $u(.)$ may be continued to a holomorphic function $\tilde{u}(.)$ in some disk $|z|<R$ and we get

$$\tilde{u}^{(n)}(z)=\frac{n!}{2\pi i}\int\limits_{|\zeta|=R/2}\frac{\tilde{u}(\zeta)\,d\zeta}{(\zeta-z)^{n+1}},\quad \forall|z|<R/2.$$

If we set $M=\sup\limits_{|z|=R/2}\|\tilde{u}(z)\|$ and $a=2/R$ we get the desired estimate for $u^{(n)}(t)$ in $|t|<R/2$.

$$\text{ii)}\Rightarrow\text{i): }\quad u(t)=\sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}t^k+\int\limits_0^1\frac{t^n(1-\tau)^{n-1}}{(n-1)!}u^{(u)}(t\tau)\,d\tau,\qquad \forall n\in\mathbf{N}$$

$$\Rightarrow\left\|u(t)-\sum_{k=0}^{n-1}\frac{u^{(u)}(0)}{t!}t^k\right\|\leqq M(at)^n\xrightarrow[n\to\infty]{}0\quad\text{for}\quad |t|<1/a.$$

[2] B^- denotes the closure of B.

Now we are able to prove some propositions which give us the announced connection between the infinitesimal properties of $\alpha(.)$ and $\mathscr{U}(.)$:

Proposition 1. Qe.s.a. *on* $D(Q)$, $A \in \mathscr{B}(H)$, $\alpha(\tau)A = e^{i\tau Q^*} A e^{-i\tau Q^*}$ *then the following statements are equivalent:*

i) $A D(Q) \subset D(Q^*)$, $\|\mathrm{ad}\, Q^* A x\| \leqq c\|x\|$, $\forall x \in D(Q)$,[3]

ii) $\alpha(.)A : \mathbf{R}^1 \to \mathscr{B}(H)$ *is weakly differentiable*[4],

iii) $\alpha(.)A : \mathbf{R}^1 \to \mathscr{B}(H)$ *is strongly differentiable,*

and under the extra assumption that $\alpha(.)\, \mathrm{ad}\, Q^* A$ *is continuous in the uniform (norm) topology on* $\mathscr{B}(H)$,

iv) $\alpha(.)A : \mathbf{R}^1 \to \mathscr{B}(H)$ *is differentiable in norm;*

from i)–iv) *it follows*

$$\frac{d\alpha(\tau)}{d\tau} A = i(\mathrm{ad}\, Q^* \alpha(\tau) A)^{-} = i\alpha(\tau)(\mathrm{ad}\, Q^* A)^{-}.$$

Proof. iv) $\Rightarrow$ iii) $\Rightarrow$ ii) are trivial.

ii) $\Rightarrow$ i): From Lemma 1 we know

$$\frac{d}{d\tau}(x, \alpha(\tau)A y) = i(Qx, \alpha(\tau)A y) - i(x, \alpha(\tau) A Q y), \quad \forall x, y \in D(Q)$$

$$\begin{aligned} &\Rightarrow |(Qx, \alpha(\tau)Ay)| \leqq |(x, \alpha(\tau) A Q y)| + \left|\left(x, \frac{d\alpha(\tau)}{d\tau} A y\right)\right| \\ &\leqq \|x\| \left(\|\alpha(\tau) A Q y\| + \left\|\frac{d\alpha(\tau)}{d\tau} A y\right\| \right), \quad \forall x, y \in D(Q) \Rightarrow \\ &\alpha(\tau) A y \in D(Q^*) \end{aligned} \tag{1.1}$$

for all $y \in D(Q)$ according to the Riesz representation theorem. So we have $\alpha(\tau) A D(Q) \subset D(Q^*)$ and from Eq. (1.1) we deduce $\frac{d\alpha(\tau)}{d\tau} A = i(\mathrm{ad}\, Q^* \alpha(\tau) A)^{-} = i\alpha(\tau)(\mathrm{ad}\, Q^* A)^{-}$ because $[\mathscr{U}(\tau), Q^*] \subseteqq 0$.

So we get $\|\mathrm{ad}\, Q^* A x\| \leqq \|\delta A\|\, \|x\|$, $\forall x \in D(Q^*)$ with the definition

$$\delta A := \frac{d\alpha(\tau)}{d\tau} A|_{\tau=0}\,.^4$$

i) $\Rightarrow$ iii): From Lemma 3 we have $A D(Q^*) \subset D(Q^*)$ so we can apply Lemma 2 to get

$$\frac{d\alpha(\tau)}{d\tau} A x = i\, \mathrm{ad}\, Q^* \alpha(\tau) A x\,, \qquad \forall x \in D(Q^*)\,.$$

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[3] $\mathrm{ad}\, Q^* A$ denotes $[Q^*, A]$, inductively $(\mathrm{ad}\, Q^*)^n A = [Q^*, (\mathrm{ad}\, Q^*)^{n-1} A]$.

[4] Differentiability means existence of the limit $\lim\limits_{h\to 0} \frac{\alpha(\tau+h) - \alpha(\tau)}{h} A$ in $\mathscr{B}(H)$.

Using the identity

$$\frac{\alpha(h)-1}{h}Ax=\int_0^1 \frac{d\alpha(th)}{dt}Ax\,dt=i\int_0^1 \operatorname{ad}Q^*\alpha(th)Ax\,dt\,,\qquad \forall x\in D(Q^*)$$

we arrive at

$$\left\|\frac{\alpha(h)-1}{h}Ax\right\|\leqq\int_0^1 \|\alpha(th)\operatorname{ad}Q^*Ax\|\,dt\leqq\|\operatorname{ad}Q^*A\|\,\|x\|\,,\qquad \forall x\in D(Q^*)$$

$$\Rightarrow\left\|\frac{\alpha(h)-1}{h}A\right\|\leqq\|\operatorname{ad}Q^*A\|\leqq C\,.$$

Consequently the family $\{1/h(\alpha(\tau+h)-\alpha(\tau))\,A\}_{h\in\boldsymbol{R}^1}$ of bounded operators on H is equi-bounded since $\|\alpha(\tau)A\|\leqq\|A\|$, converging strongly on the dense set $D(Q^*)$ for $h\to 0$. Thus it converges strongly on all of H and

$$\frac{d\alpha(\tau)}{d\tau}Ax=i(\operatorname{ad}Q^*\alpha(\tau)A)^{-}x\,,\qquad \forall x\in H\,.$$

iii) $\Rightarrow$ iv):

$$\left\|\left(\frac{\alpha(\tau+h)-\alpha(\tau)}{h}A-i\alpha(\tau)\operatorname{ad}Q^*A\right)x\right\|$$

$$=\left\|\int_0^1(\alpha(th+\tau)-\alpha(\tau))\operatorname{ad}Q^*Ax\,dt\right\|$$

$$\leqq\sup_{|t|\leqq 1}\|(\alpha(th+\tau)-\alpha(\tau))\operatorname{ad}Q^*A\|\,\|x\|$$

$$\Rightarrow\left\|\frac{\alpha(\tau+h)-\alpha(\tau)}{h}A-i\alpha(\tau)\operatorname{ad}Q^*A\right\|$$

$$\leqq\sup_{|t|\leqq 1}\|(\alpha(\tau+th)-\alpha(\tau))\operatorname{ad}Q^*A\|\xrightarrow[h\to 0]{}0\,;$$

$\Rightarrow$ iv) using the assumed continuity of $\alpha(.)\operatorname{ad}Q^*A$ in norm.

Corollary. *For $k\in\boldsymbol{N}$, $1\leqq k\leqq\infty$ the following statements are equivalent:*

i) $(\operatorname{ad}Q^*)^{n-1}A\,D(Q)\subset D(Q^*)$, $\|(\operatorname{ad}Q^*)^nAx\|\leqq c_n\|x\|$, $\forall x\in D(Q)$, $1\leqq n\leqq k$,

ii) $\alpha(.)A:\boldsymbol{R}^1\to\mathscr{B}(H)$ *is k-times weakly differentiable,*

iii) $\alpha(.)A:\boldsymbol{R}^1\to\mathscr{B}(H)$ *is k-times strongly differentiable,*

and under the extra assumption that $\alpha(.)(\operatorname{ad}Q^*)^nA$ *is continuous in the norm topology of* $\mathscr{B}(H)$ *for* $1\leqq n\leqq k$,

iv) $\alpha(.)A:\boldsymbol{R}^1\to\mathscr{B}(H)$ *is k-times differentiable in norm.*

i)–iv) $\Rightarrow$ $\alpha^{(n)}(\tau)A=\alpha(\tau)\,((i\operatorname{ad}Q^*)^nA)^{-}$, $1\leqq n\leqq k$.

Proof. By induction using Proposition 1.

Proposition 2. *If* Q *is* e.s.a. *and* $Q \in \mathscr{B}(H)$ *the following statements are equivalent:*

i) $(\operatorname{ad} Q^*)^{n-1} A D(Q) \subset D(Q^*)$ *and there exist* $M>0$, $a>0$ *with* $\|(\operatorname{ad} Q^*)^n A x\| \leqq M n!\, a^n \|x\|$, $\forall x \in D(Q)$, $n \in \boldsymbol{N}$,

ii) $\alpha(.)A : \boldsymbol{R}^1 \to \mathscr{B}(H)$ *is weakly analytic,*

iii) $\alpha(.)A : \boldsymbol{R}^1 \to \mathscr{B}(H)$ *is strongly analytic,*

iv) $\alpha(.)A : \boldsymbol{R}^1 \to \mathscr{B}(H)$ *is analytic in norm.*

Proof. iv) $\Rightarrow$ iii) $\Rightarrow$ ii) are trivial.

iv) $\Rightarrow$ i): From $\alpha(.)A$ being infinitely differentiable we get

$$(\operatorname{ad} Q^*)^n A D(Q) \subset D(Q^*)\,,\quad \alpha^{(n)}(\tau) A x = ((i \operatorname{ad} Q^*)^n \alpha(\tau) A)^{-} x, \forall x \in H\,,\ n \in \boldsymbol{N}\,.$$

Lemma 4 gives the existence of $M>0$ and $a>0$ with

$$\|\alpha^{(n)}(0) A\| = \|(\operatorname{ad} Q^*)^n A\| \leqq M n!\, a^n, \qquad \forall n \in \boldsymbol{N}\,.$$

i) $\Rightarrow$ ii): Firstly we notice $\alpha(.)A$ being infinitely often weakly differentiable and $|(x, \alpha^{(n)}(\tau) A y)| = |(x, \alpha(\tau)(\operatorname{ad} Q^*)^n A y)| \leqq \|x\|\, \|y\|\, M n!\, a^n$ for all $\tau \in \boldsymbol{R}^1$, $n \in \boldsymbol{N}$. It follows that $(x, \alpha(\tau) A y)$ is analytic for all $x, y \in H$ using again Lemma 4 for $X = \boldsymbol{C}$.

ii) $\Rightarrow$ iii) (compare [7], p. 52, Lemma 3): $\sum_{n=0}^{\infty} (\operatorname{ad} Q^*)^n Ay|\tau|^n/n!$ converges weakly for $|\tau| < 1/a$, so $\{\|(\operatorname{ad} Q^*)^n A y\|\, |\tau|^n/n!\}_{n \in \boldsymbol{N}}$ is bounded for all $y \in H$. We choose $\varepsilon > 0$ with

$$(1+\varepsilon)|\tau| < 1/a \Rightarrow \sum_{n=0}^{\infty} \|(\operatorname{ad} Q^*)^n A y\|\, |\tau|/n!$$
$$= \sum_{n=0}^{\infty} \|(\operatorname{ad} Q^*)^n A y\| ((1+\varepsilon)|\tau|)^n (n!)^{-1} (1+\varepsilon)^{-n} \leqq C \sum_{n=0}^{\infty} (1+\varepsilon)^{-n}\,.$$

iii) $\Rightarrow$ iv): For $y \in H$ there exists $M(y)$ with $\|\alpha^{(n)}(0) A y\| \leqq n!\, a^n M(y)$ by Lemma 4. So we get

$$\|\alpha(\tau) A y\| \leqq \sum_{n=0}^{\infty} \|(\operatorname{ad} Q^*)^n A y\|\, |\tau|^n/n! \leqq M(y)(1 - \alpha|\tau|)^{-1}\,.$$

By the uniform-boundedness principle we obtain $\|(1 - a|\tau|\, \alpha(\tau) A\| \leqq C'$. Now we can apply a known theorem [5], p. 365 giving the desired result.

Remark. As the reader may have already noticed there is no extra condition for concluding iv) from i)–iii) in this case.

2. Internal Symmetries

For the following we assume that we are given a local ring system in the Haag-Araki sense [3], the $\mathscr{R}(\mathcal{O})$ are assumed to be v. Neumann algebras of operators on some Hilbertspace H. We consider a local current j^μ affiliated to $\{\mathscr{R}(\mathcal{O})\}$. From j^μ we construct local "charge" operators $Q_{r,\alpha}$ [1] by $Q_{r,\alpha}=j^0(f_r\otimes\alpha)$ with $f_r\in C_0^\infty(\boldsymbol{R}^3)$, $\alpha\in C_0^\infty(\boldsymbol{R}^1)$

$$f_r(\boldsymbol{x})=\left\{\begin{matrix}1 & \text{for} & |\boldsymbol{x}|\leqq r\\ 0 & \text{for} & |\boldsymbol{x}|\geqq r+1\end{matrix}\right\}, \qquad \int \alpha(t)\,dt=1\,.$$

The charges Q_r (we keep α fixed and suppress the index α from now on) are assumed essentially self-adjoint on some common domain $D\subset H$ giving rise to automorphisms $\alpha_r(\tau)A=e^{i\tau Q_r^*}A\,e^{-i\tau Q_r^*}$ of $\mathscr{B}(H)$. From the relative locality of j^μ with respect to $\mathscr{R}(\mathcal{O})$ we deduce for bounded $\mathcal{O}$ the existence of r_0 such that

$$\begin{aligned}&(Q_r x, A y)-(x, A Q_r y)=(Q_{r'} x, A y)-(x, A Q_{r'} y) \quad \text{for} \quad r, r'\geqq r_0\,,\\ &\forall A\in\mathscr{R}(\mathcal{O})\,, x, y\in D\,.\end{aligned} \tag{2.1}$$

For the definition of an internal symmetry we follow Ref. [6]:

Definition. A symmetry is called "internal" if $\alpha_r(\tau)\,\mathscr{R}(\mathcal{O})\subset\mathscr{R}(\mathcal{O})$, $\forall\tau\in\boldsymbol{R}^1$ and r sufficiently big.

A symmetry which is not internal we call "external".

Our statement now is that for internal symmetries and bounded $\mathcal{O}$ the restrictions $\alpha_r(.)\,|\,\mathscr{R}(\mathcal{O})$ of $\alpha_r(.)$ to $\mathscr{R}(\mathcal{O})$ all coincide for sufficiently big r, thus $\lim\limits_{r\to\infty}\alpha_r(.)\,|\,\mathscr{R}(\mathcal{O})$ exists trivially.

Theorem 1. *Let Q_r be essentially self-adjoint on a common domain $D\subset H$*

$$\alpha_r(\tau)A=e^{-i\tau Q_r^*}A\,e^{-i\tau Q_r^*}\in\mathscr{R}(\mathcal{O}) \quad \textit{for} \quad A\in\mathscr{R}(\mathcal{O})\,,\ r\geqq r_0$$

and Eq. (2.1) for, $r\geqq r_0, x, y\in D, A\in\mathscr{R}(\mathcal{O})$, then $\alpha_r(\tau)A=\alpha_{r'}(\tau)A$, $\forall A\in\mathscr{R}(\mathcal{O})$, $\tau\in\boldsymbol{R}^1, r, r'\geqq r_0$.

Proof. We consider $\mathscr{R}(\mathcal{O})$ equipped with the weak topology from vectors of H as a quasicomplete locally convex topological vector space [5]; then $\alpha_r(.)A$ is a continuous map from $\boldsymbol{R}^1$ into $\mathscr{R}(\mathcal{O})$ for all $A\in\mathscr{R}(\mathcal{O})$. All elements of $\mathscr{R}(\mathcal{O})$ are weakly exponential [6] vectors for $\alpha_r(.)$ since $|(x, \alpha(\tau)A\,y)|\in\|A\|\;\|x\|\;\|y\|$. So we can apply a generalization of a theorem of Gårding to quasicomplete locally convex topological vector spaces [7]

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[5] The topology of $\mathscr{R}(\mathcal{O})$ is defined by the family of seminorms $p(A)=\sum\limits_{k=1}^{n}|(x_k, A\,y_k)|$ with $x_k, y_k\in H$ arbitrary.

[6] A vector A is called weakly exponential for $\alpha(.)$ if for any continuous linear functional φ on $\mathscr{R}(\mathcal{O})$ there exist constants $a>0$ and $b>0$ with $\varphi(\alpha(\tau)A)\leqq a\,e^{b|\tau|}$. See Ref. [7].

which asserts the existence of a dense supply of analytic vectors for each $\alpha_r(.)$ which we denote by C_r^ω. We want to show that $C_r^\omega = C_{r'}^\omega$ for $r, r' \geqq r_0$ Assume therefore $A \in C_r^\omega$ then $(\mathrm{ad}\, Q_r^*)^n A D \subset D(Q_r^*)$ and there exist $M_r > 0$, $\alpha_r > 0$ with $\|(\mathrm{ad}\, Q_r^*)^n A\| \leqq M_r n! a_r^n$, $\forall n \in N$ according to Proposition 2. From Eq. (2.1) we get

$$|(Q_{r'} x, A y)| \leqq |(x, A Q_{r'} y)| + |(x, \mathrm{ad}\, Q_r^* A y)| \leqq \|x\| (\|A Q_{r'} y\| + \|\mathrm{ad}\, Q_r^* A y\|)$$

for $\forall x, y \in D$ i.e. $A D \subset D(Q_{r'}^*)$. Again from (2.1) we deduce $(\mathrm{ad}\, Q_r^* A)^- = (\mathrm{ad}\, Q_{r'}^* A)^-$ for $r, r' \geqq r_0$. Repeating this argument we find $(\mathrm{ad}\, Q_{r'}^*)^n A D \subset D(Q_{r'}^*)$ and $((\mathrm{ad}\, Q_r^*)^n A)^- = ((\mathrm{ad}\, Q_{r'}^*)^n A)^-$ for $\forall n \in N$. Therefore $\|(\mathrm{ad}\, Q_{r'}^*)^n A\| \leqq M_r n! a_r^n$ i.e. $A \in C_{r'}^\omega$. Thus we have $C_r^\omega \subset C_{r'}^\omega$; starting with $C_{r'}^\omega$ we arrive at $C_{r'}^\omega \subset C_r^\omega$, so we have proved $C_r^\omega = C_{r'}^\omega$, for $r, r' \geqq r_0$. Furtheron we have shown

$$\alpha_r^{(n)}(0) A = (i\, \mathrm{ad}\, Q_r^*)^n A^- = \alpha_{r'}^{(n)}(0) A \quad \text{for} \quad n \in N,\ A \in C_r^\omega .$$

Thus

$$\alpha_r(\tau) A = \sum_{n=0}^{\infty} \frac{\alpha_r^{(n)}(0)}{n!} A \tau^n = \alpha_{r'}(\tau) A \quad \text{for} \quad A \in C_r^\omega,\ r, r' \geqq r_0 .$$

Since the C_r^ω lie dense in $\mathscr{R}(\mathcal{O})$ and the $\alpha_r(\tau)$ are continuous maps of $\mathscr{R}(\mathcal{O}) \to \mathscr{R}(\mathcal{O})$ we may extend this equality to all of $\mathscr{R}(\mathcal{O})$.

Remark 1. We notice that all we need to prove Theorem 1 is a weakly closed subspace of $\mathscr{B}(H)$ which fulfills condition (2.1) for sufficiently big r and r'. So, if there exists a bounded $\mathcal{O}_1$ such that $\left(\bigcup_{\tau \in \mathbf{R}^1} \alpha_r(\tau) \mathscr{R}(\mathcal{O}) \right)'' \subset \mathscr{R}(\mathcal{O}_1)$ for big r and r', the assumptions of Theorem 1 hold.

Remark 2. It would be desirable to have some sufficient condition on the Q_r that reveals the fact that they give rise to an internal symmetry. The condition $\mathrm{ad}\, Q_r^* A \in \mathscr{R}(\mathcal{O})$ for a dense set of $A \in \mathscr{R}(\mathcal{O})$ is clearly not sufficient.

3. External Symmetries

From Remark 1 to Theorem 1 we conclude that there is no problem with space rotations but only with translations and pure Lorentz-rotations. The construction of the global automorphism $\alpha(\tau) = \lim_{r \to \infty} \alpha_r(\tau)$ from local "charges" relies on the equality of the corresponding infinitesimal generators $\mathrm{ad}\, Q_r^*$ for big r. At first sight one may have the impression that it should work equally well for external symmetries since only infinitesimal neighbourhoods of a given bounded region $\mathcal{O}$ seem to be involved. Unfortunately we have used analytic vectors which are generally constructed by smoothing $\alpha_r(.)$ with analytic functions: $A_f = \int f(\tau)\, \alpha_r(\tau) A\, d\tau$ (f analytic). These A_f do generally not belong to any $\mathscr{R}(\mathcal{O}_1)$ with bounded $\mathcal{O}_1$ if the symmetry changes the region $\mathcal{O}$. So we do not have a dense supply of analytic vectors in the local algebras $\mathscr{R}(\mathcal{O})$ to integrate up the equality of the generators of the α_r. What we thus need is another method of reconstructing the α_r from their generators. There can be found several such methods in the literature [4, 8] but they all seem to require equicontinuity of $\{\alpha_r(\tau)\}_{\tau\in \mathbf{R}^1}$ in τ which we only know in the norm topology of $\mathscr{R}(\mathcal{O})$. Therefore we now require that $\alpha_r(.)A:\mathbf{R}^1 \to \mathscr{B}(H)$ is continuous in the norm topology. It would be interesting to know if there exists any method not requiring equicontinuity and which reproduces α_r from its infinitesimal generator.

We proceed now to prove the existence of $\lim\limits_{r\to\infty} \alpha_r$ for norm continuous $\alpha_r(.)$[7]. It is natural to work with local concrete C^*-Algebras $\mathfrak{A}(\mathcal{O})$ in that case. Clearly the $\alpha_r(\tau)$ can be extended to the quasi-local algebra $\mathfrak{A} = \bigvee\limits_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$[8].

Theorem 2. *Let* Q_r *be* e.s.a. *on* $D\subset H$, $\alpha_r(\tau)A = e^{i\tau Q_r^*} A e^{-i\tau Q_r^*} \in \mathfrak{A}$, $\forall A \in \bigcup\limits_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$;

assume the existence of numbers r_T *such that for all* $A \in \overline{\bigcup\limits_{|\tau|\leqq T} \alpha_r(\tau)\mathfrak{A}(\mathcal{O})}$ *we have*

$$(Q_r x, A y) - (x, A Q_r y) = (Q_{r'} x, A y) - (x, A Q_{r'} y), \qquad \forall x, y \in D,\ r, r' \geqq r_T \tag{3.1}$$

and further the continuity of $\alpha_r(.)A : \mathbf{R}^1 \to \mathfrak{A}$ *(in norm) then* $\lim\limits_{r\to\infty} \alpha_r(\tau)$ *exists on* $\mathfrak{A}$, $\forall \tau \in \mathbf{R}^1$.

Remark. Condition (3.1) expresses the fact that the symmetry belonging to Q_r maps a bounded region $\mathcal{O}$ into some bounded region $\mathcal{O}_T$ if $|\tau| \leqq T$. Intuitively one would even expect that $\alpha_r(\tau)\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}_\tau)$ (τ fixed, r sufficiently big) where $\mathcal{O}_\tau$ is the transformed region.

Before proving Theorem 2 we give a simple lemma on the resolvent of the generator of $\alpha_r(.)$.

Lemma 5. *Let* $\alpha(.)$ *be a continuous one-parameter group of contractions on a Banach space* X *(i.e.* $\|\alpha(\tau)x\| \leqq \|x\|$, $\forall x \in X$, $\tau \in \mathbf{R}^1$. *If* δ *denotes* $\left.\frac{d\alpha(\tau)}{d\tau}\right|_{\tau=0}$ *the generator of* $\alpha(.)$ *and* $R(z) = (z-\delta)^{-1}$ *its resolvent then*

$$R(z) = \int_0^T e^{-z\tau} \alpha(\tau)\, d\tau + e^{-zT} \alpha(T) R(z), \qquad \operatorname{Re} z > 0.$$

[7] For a discussion of this norm-continuity see Ref. [9].

[8] $\bigvee\limits_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ denotes the algebra generated by $\bigcup\limits_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$.

Proof. We define $R_T(z) = \int_0^T e^{-z\tau}\alpha(\tau)\,d\tau$ then for $x \in X$

$$\alpha(t)\,R_T(z)x = \int_t^{T+t} e^{-z(\tau-1)}\alpha(\tau)\,d\tau\,x$$

$$\Rightarrow \frac{d\alpha(t)}{dt} R_T(z)\,x\Big|_{t=0} = \delta R_T(z)x = z R_T(z)x + e^{-zT}\alpha(T)x - x$$

$$\Rightarrow R_T(z)x = R(z)x - e^{-zT}\alpha(T)\,R(z)\,.$$

Proof of Theorem 2. Let δ_r denote the generator of $\alpha_r(.)$, $R_r(.)$ its resolvent. We want to show the existence of $\lim_{r\to\infty} R_r(z)$ on $\mathfrak{A}$ for $\operatorname{Re} z > 0$. Assume $A \in \mathfrak{A}(\mathcal{O})$, then we may write

$$(R_r(z) - R_{r'}(z))A = \delta_{r'}R_{r'}(z)\,R_r(z)A - R_{r'}(z)\,\delta_r R_r(z)A \quad \text{for} \quad \operatorname{Re} z > 0$$

since $R_{r(')}(z)A$ lies in the domain of $\delta_{r(')}$. For $R_r(z)$ we use Lemma 5 to get $R_r(z) = \int_0^T e^{-z\tau}\alpha_r(\tau)\,d\tau + e^{-zT}\alpha_r(T)\,R_r(z)$.

Choosing r and $r' \geqq r_T$ and setting $A_T(z) = \int_0^T e^{-z\tau}\alpha_r(\tau)\,A\,d\tau$ we deduce from Eq. (3.1):

$$|(Q_{r'}x, A_T(z)y)| \leqq \|x\|\,(\|A_T(z)Q_r y\| + \|\operatorname{ad} Q_r^* A_T(z)y\|)\,, \qquad \forall x, y \in D\,.$$

That means $A_T(z)D \subset D(Q_{r'}^*)$ and (again using (3.1) and Proposition 1)

$$\delta_{r'}A_T(z) = (\operatorname{ad} Q_{r'}^* A_T(z))^- = (\operatorname{ad} Q_r^* A_T(z))^- = \delta_r A_T(z)\,.$$

We arrive at

$$(R_r(z) - R_{r'}(z))A = \delta_{r'}R_{r'}(z)e^{-zT}\alpha_r(T)\,R_r(z)A - R_{r'}(z)\delta_r e^{-zT}R_r(z)\alpha_r(T)A\,.$$

Using $\|\delta_{r(')}R_{r(')}(z)\| \leqq 1$, $\|R_{r(')}(z)\| \leqq \frac{1}{\operatorname{Re} z}$ we get $\|(R_r(z) - R_{r'}(z))A\| \leqq e^{-T\operatorname{Re} z}\frac{2\|A\|}{\operatorname{Re} z}$ for $r, r' \geqq r_T$. For $T \to \infty$ we get the existence of $\lim_{r\to\infty} R_r(z)A = R(z)A$ for $A \in \mathfrak{A}(\mathcal{O})$ from which the existence of the limit for all $A \in \mathfrak{A}$ follows by the equiboundedness of $R_r(z)$.

Next we want to show that the range of $R(z)$ is dense in $\mathfrak{A}$. For that we assume $A \in \mathfrak{A}(\mathcal{O})$ for bounded $\mathcal{O}$ then we get for $n \geqq 1$

$$\|nR(n)A - A\| \leqq \|(nR(n) - nR_r(n))A\| + \|nR_r(n)A - A\|$$
$$\leqq 2e^{-nT}\|A\| + \|nR_r(n)A - A\|$$

for r sufficiently big, which can be made arbitrarily small since $\lim_{n\to\infty} nR_r(n)A = A$ (see Ref. [4], p. 241). So we conclude $\lim_{n\to\infty} nR(n)A = A$ for all $A \in \mathfrak{A}(\mathcal{O})$. Since $\|nR(n)\| \leqq 1$ we get $\lim_{n\to\infty} nR(n)A = A$ for all $A \in \mathfrak{A}$.

$R(.)$ satisfying the resolvent equation $R(z) - R(z') = (z' - z) R(z) R(z')$ because the $R_r(.)$ do, we can apply Lemma 1′ of Ref. [4], p. 217 which asserts that $\overline{\text{range } R(z)} = \{A \in \mathfrak{A} : \lim_{n \to \infty} n R(n) A = A\} = \mathfrak{A}$.

Now we are prepared to apply the Trotter-Kato-Theorem Ref. [4], p. 269 on the convergence of semigroups proving the convergence of $\alpha_r(\tau)$ on $\mathfrak{A}$ for $\tau \geqq 0$. The proof for $\tau < 0$ runs along the same lines. The limit $\alpha(\tau) = \lim_{n \to \infty} \alpha_r(\tau)$ is clearly a C^*-automorphism of $\mathfrak{A}$.

Finally we want to remark that the statements made above apply equally well to Quantum Statistical Mechanics, except time translations where condition (3.1) does not hold.

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