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Macroscopic Causality and Physical Region Analyticity in S-Matrix Theory

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Abstract. An equivalence is proved between a certain macroscopic causality condition and the normal analytic structure of the physical-region S-matrix. The normal analytic structure is this: each scattering function has physical-region singularities only on positive-α Landau surfaces and near these surfaces it is the limit from certain well-defined directions of a unique analytic function. The macroscopic causality condition is formulated in terms of S-matrix concepts. It expresses the requirement that in an appropriate classical macroscopic limit all transition amplitudes fall off in the way indicated by classical estimates. This result gives, on the one hand, a physical basis for the basic physical-region analyticity properties of the S matrix. On the other hand, it gives, alternatively, a basis for a space-time description of phenomena starting from momentum space properties having no *a priori* space-time content.

I. Introduction

Space-time causality properties are intimately related to momentum space analyticity. This connection is the basis of dispersion theory, and of certain forms of field theory. In this paper, we study one aspect of this general connection, namely the connection between *physical region*, analyticity properties and *macroscopic* causality properties. We prove, in particular, the mathematical equivalence of a certain set of physical-region analyticity properties of the S matrix, called the normal analytic structure, to a certain macroscopic causality property, called macrocausality. This connection has special significance because it deals with the quantities most closely related to actual physical measurements: it involves neither conceptions of causality that attribute a fundamental

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physical significance to the idealization of the space-time point, nor extensions of the S matrix either off the mass shell or away from the physical-region.

The precise description of the analyticity properties that define the normal analytic structure is given in Chapter II. These properties, suggested first by perturbation theory, are, in brief, that the S-matrix decomposes into a sum of connected parts, each having only one energy-momentum conservation law delta function (cluster decomposition property); that each connected part divided by the delta function – called a scattering function – is furthermore analytic except on positive-α Landau surfaces; and that near these surfaces they are "plus $i\varepsilon$" limits of certain analytic functions (plus $i\varepsilon$ rule).

The description of the macrocausality condition is given in Chapter III. First, a weak concept of macroscopic space-time localization is introduced. It makes precise the notion that for appropriate forms of their momentum space-wave functions, particles are *asymptotically* localized in the direction of the energy momentum.

Space-time displacements of the initial and final particles of a scattering process are then considered, and macrocausality is formulated by means of a macroscopic limit involving a space-time dilation. This formulation refers only to the above asymptotic notion of localization, and is such that in a dilated coordinate system a *classical* description becomes relevant.

The macroscopic causality condition expresses the requirement that a certain classical idea of causality should become valid in the macroscopic limit. This classical idea is that dynamical effects are carried over very large distances only by (stable) physical particles [1]. In particular, all transfers of energy-momentum not ascribable to physical particles are required to give effects having finite range (i.e. effects that fall off exponentially under space-time dilation, in the asymptotic limit).

This condition is imposed by demanding that if, in the macroscopic limit, there is no classical multiple-scattering process connecting the initial and final particles, then the transition amplitude must fall off in that limit. Moreover, the rate of fall-off should be of the type calculated by classical arguments, and, in particular, exponential under appropriate conditions on the form of the (initial and final) wave functions[1].

The derivation of the normal analytic structure from macrocausality is given in Chapter IV 2. The converse is given in Chapter IV 3.

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[1] The formulation of Chapter III refers directly to the connected part of the transition amplitude, and requires fall-off unless some *connected* multiple-scattering process is possible. This form of the condition can be derived from a more general expression of the same physical ideas in terms of the complete transition probability itself [2].

Applications of these results in S-matrix theory and field theory are discussed in the Conclusion (Chapter V). The connection between micro-causality and macrocausality is also discussed there.

Several mathematical results concerning relationships between bounded functionals in L^2 norm (with respect to each variable separately), tempered distributions, and boundary values of analytic functions used in this work are derived in Appendix I.

The present work is a direct extension of an earlier work along similar lines [1]. The macrocausality condition stated here is a stronger mathematical expression of essentially the same physical ideas used before. From this stronger version, we obtain analyticity in place of the infinite differentiability obtained previously.

Extensions of the ideas and methods of this paper are made in Ref. [2]. In particular, an equivalence is shown between, on the one hand, the normal analytic structure together with the discontinuity formulas and, on the other hand, macroscopic causality together with certain factorization properties in the cases where classical multiple scattering is possible. A general discussion of space-time properties in S-matrix theory is also given there.

II. Analyticity Properties

1. Scattering Functions [2]

We consider for simplicity a theory with no superselection rules and with only spinless particles. The arguments extend immediately to the more general cases.

The basic observable in a scattering experiment can be considered to be the functional $S_{nm}(\psi_1, \ldots \psi_n; \varphi_1, \ldots \varphi_m)$ the square of which is the transition probability [3] from an initial system of m particles represented by unit norm wave functions $\varphi_1 \ldots \varphi_m$ to a final system of n particles represented by unit norm wave functions $\psi_1 \ldots \psi_n$.

The norm of a wave function φ_i (or ψ_i) is the L^2 norm in the on-mass-shell momentum space of particle i:

$$|\varphi_i| = [(2\pi)^{-3} \int |\varphi_i(p_i)|^2 \, \delta(p_i^2 - \mu_i^2) \, \theta(p_{i0}) \, d^4 p_i]^{1/2}, \qquad (1)$$

where μ_i is the mass of particle i.

[2] The approach to S-matrix theory used here is that of Ref. [1, 3]. A comparison to the approach that starts with the operator S that maps a Hilbert space $\mathscr{H}$ of free particle states onto itself is given in Ref. [2] (see also Appendix I).

[3] This *strict* probability interpretation is not crucial in this paper and could be replaced by a weaker condition.

The momentum space wave functions can be assumed to span either the space L^2 of square integrable functions or a dense subspace, for instance the Schwartz space $\mathscr{D}$ of infinitely differentiable functions with compact support.

The functional S_{nm} is assumed to be linear in the wave functions of the initial particles and antilinear in the wave functions of the final particles.

This linearity, together with the boundedness of the S-matrix (probability interpretation):

$$|S_{nm}(\psi_1 \dots \psi_n; \varphi_1 \dots \varphi_m)| \leqq \prod_i |\varphi_i| \prod_j |\psi_j| \tag{2}$$

implies the existence of the momentum space kernels $S_{nm}(q_1, \dots, q_n, p_1, \dots, p_m)$ as tempered distributions in the space of all on-mass-shell momenta (see Appendix I).

Conservation of energy-momentum requires this distribution to be concentrated on the set $\mathscr{M}$, defined as the intersection of the hyperplane $\sum p_i - \sum q_j = 0$ with the above space of all on-mass-shell momenta. This distribution – when restricted to the set of points at which at least two of the initial and final 4-vectors p_i, q_j are not collinear – is furthermore the product of a δ function $\delta(\sum p_i - \sum q_j)$ with a corresponding kernel[4].

The *connected* kernels S^c_{nm} are uniquely defined in terms of the kernels S_{nm} by the iterative solution to the system of equations:

$$S_{nm}(Q_n, P_m) = \sum_{\{k\}} \beta_{\{k\}} \prod_{k \in \{k\}} S^c_{n_k m_k}(Q_{n_k}, P_{m_k}), \qquad S^c_{11}(Q, P) = S_{11}(Q, P), \tag{3}$$

where the $\{k\}$ are all subdivisions of the set (n, m) into subgroups (n_k, m_k), P_m and Q_n denote the sets of the initial and final momenta, and the coefficients $\beta_{\{k\}}$ are phase factors that are equal to plus one or minus one[5].

The kernel $S^c_{nm}(Q_n, P_m)$ is a linear combination of products of various S_{rt} and is likewise a tempered distribution that is the product of a δ function of energy momentum conservation with a kernel $T_{nm}(Q_n, P_m)$ defined on $\mathscr{M}$ [4].

The kernel $T_{nm}(Q_n, P_m)$ is called a *scattering function.*

The analyticity properties described in this Chapter refer to these scattering functions.

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[4] The derivatives of $\delta(\sum p_i - \sum q_j)$ are excluded [4] because of the boundedness (2) of the S-matrix.

[5] They are all equal to plus one if no fermions are involved. If fermions are involved, then these phases are plus or minus one, depending on the relative order of the subdivisions $\{n_k, m_k\}$ and $\{n, m\}$. (Even though we only consider spinless particles, we need not admit here the spin statistics connection.)

2. Landau Surfaces

Consider a multiple scattering[6] *connected* graph G associated with the initial and final particles of a process, the set I of its internal lines and a set of corresponding 4-vectors k_i. Each line has an orientation (sense) that specifies the direction of flow of the energy-momentum k_i.

Let f denote a closed loop that can be constructed on the internal lines of G. Define n_{if} as follows: $n_{if}=0$ if loop f does not contain the line i, $n_{if}=+1$ (resp. -1) if f contains i with the correct (resp. opposite) orientation.

The Landau equations of the graph G consist of one loop equation

$$\sum_{i\in I} \alpha_i k_i n_{if} = 0 \tag{4}$$

for each closed loop f, plus the equations of energy-momentum conservation at each vertex and the (real, positive-energy) mass-shell constraint for each internal and external line. (The α_i are independent of f and are not all zero.)

To each internal line of G, let us now associate (in addition to the "sense" defined above) a "sign" σ_i, and let the α_i in the Landau equations associated with G be restricted by the further condition $\sigma_i \alpha_i > 0$.

The Landau surface $L(G)$ is obtained by eliminating the coefficients α_i and the internal k_i. [$L(G)$ is confined to the space $\mathscr{M}$ of external momenta.][7]

A graph with all $\sigma_i > 0$ is denoted G^+ and the set $L(G^+)$ is called a positive α Landau surface. An important fact emphasized by Coleman and Norton [5] is this: Any solution of the Landau equations of G^+ corresponds to a nontrivial[8] classical multiple-scattering process having the structure of G, and conversely. The vector $\alpha_i k_i$ represents the space-time interval between the creation and annihilation of internal particle i.

Certain definitions will prove useful: $\mathscr{M}_0$ is the subset of $\mathscr{M}$ where two (or more) initial or two (or more) final energy-momentum 4-vectors are collinear.

[6] The phrase "multiple scattering" signifies here only the *topological* aspect that the lines of G are classified as initial, final and intermediate; that the initial and final lines have no vertices at their trailing or leading end points respectively; and that the intermediate lines have vertices on both ends. It is not required of G that there actually exist a corresponding physical multiple-scattering process. In particular, the vertices of G are not required to be partially ordered by any condition that particles move "forward in time."

[7] Graphs G that arise from insertions of vertices on lines of a set of lines that all begin at one vertex and all end at one vertex of some other diagram $\hat{G}$ will be excluded, since these G do not give new Landau surfaces. Graphs with lines that begin and end at the same vertex will be excluded also, for the same reason.

[8] A nontrivial connected multiple-scattering process is one with at least one intermediate particle.

$L_0(G)$ is the subset of $L(G) - \mathcal{M}_0$ that lies on no $L(G')$ where G' is a nontrivial contraction[9] of G.

The *physical side* of $L_0(G^+)$ is the (well-defined) side to which it would be shifted by a formal scale transformation that increases by a common factor the masses associated with the internal lines of G.

The *physical region* of G is the subset of $\mathcal{M}$ defined by the Landau equations for G without the loop equations (i.e. by the mass-shell and conservation-law constraints alone).

Some properties of Landau surfaces are these: (i) The union L_c^+ of all positive-α Landau surfaces is the union of all $L_0(G^+)$, plus $\mathcal{M}_0$. (This is trivial.)

(ii) Each surface $L_0(G^+)$ is a real analytic submanifold of $\mathcal{M} - \mathcal{M}_0$ of codimension 1 [1].

(iii) The number of graphs G^+ that give positive-α Landau surfaces $L_0(G^+)$ that enter bounded portions of the physical region is finite [6].

(iv) $L_0(G^+)$ lies on the boundary of the physical region of G^+. Furthermore, in some neighborhood of any point of $L_0(G^+)$, the physical region of G^+ either coincides with $L_0(G^+)$ or lies on its physical side [1, 7].

(v) If two $L_0(G^+)$ coincide in a neighborhood of a point, then their physical sides coincide at this point [1].

By virtue of (v), the physical side of $L_0(G^+)$ is an intrinsic characteristic of the surface that does not depend on the particular G^+ used to define it.

Property (ii) asserts that for any point P of $L_0(G^+)$ there is a real neighborhood $\mathcal{U}$ and a real analytic function l (of real analytic local coordinates[10] of $\mathcal{M} - \mathcal{M}_0$ near P) such that grad $l \neq 0$ in $\mathcal{U}$ and $L_0(G^+)$ is defined by $l = 0$ in $\mathcal{U}$. *The sign of l is conventionally chosen to be positive* on the physical side of $L_0(G^+)$.

The power series for l at P defines a function $\boldsymbol{l}$ analytic in a certain complex neighborhood of P.

The function $\boldsymbol{l}$ should always be understood as an abbreviation of $\boldsymbol{l}[z; P, L_0(G^+)]$, where the z are analytic local coordinates of $\mathcal{M} - \mathcal{M}_0$ near P of $L_0(G^+)$. Different functions $\boldsymbol{l}$ could be used to define the same surface $L_0(G^+)$ at P, but the properties described below do not depend on the particular choice of $\boldsymbol{l}$.

3. Analyticity Properties

The normal analytic structure is defined by the following properties (1) through (3):

[9] A *contraction* of G is a graph obtained by equating the two end points of certain lines of G and then removing all lines that begin at their own end points. G is considered a trivial contraction of itself.

[10] Real analytic local coordinates are local coordinates such that all energy momentum 4-vectors are real analytic functions of them and conversely.

Property 1. *(Positive-α rule). Each scattering function is analytic*[11] *at P unless P lies on the union L_c^+ of positive-α Landau surfaces of the process.*

[By virtue of properties (i) through (iii) of Chapter II 2, the set L_c^+ has dimension one less than that of $\mathscr{M}$.]

Property 2. *(Plus iε rule for simple and quasisimple points.) If P lies on the Landau surface $L_0(G^+)$ of a given G^+, and if any other $L_0(\hat{G}^+)$ containing P coincides with $L_0(G^+)$ near P, then P has a real neighborhood $\mathscr{U}$ and a complex neighborhood $\boldsymbol{\mathscr{U}}$ that contains $\mathscr{U}$, such that the scattering function in $\mathscr{U}$ is the boundary value of a function $f(q)$ analytic in a domain that lies in $\operatorname{Im} \boldsymbol{l} > 0$. This domain is the intersection of $\boldsymbol{\mathscr{U}}$ with an open cone C_P in $\operatorname{Im} \boldsymbol{q}$ that can be made arbitrarily close to the open half space $\operatorname{Im} \boldsymbol{l} > 0$ in $\boldsymbol{\mathscr{U}}$ by taking $\boldsymbol{\mathscr{U}}$ sufficiently small. The function $f(q)$ also satisfies a bound of the form $|f(q)| < C |\operatorname{Im} \boldsymbol{l}|^{-m}$, where C and m are positive constants that are independent of $\boldsymbol{\mathscr{U}}$, for $\boldsymbol{\mathscr{U}}$ sufficiently small.*

By virtue of the edge of the wedge theorem [8], properties (1) and (2) ensure the existence of a "plus $i\varepsilon$" analytic continuation of the scattering function past the surface $L_0(G^+)$.

The points which satisfy the conditions of property (2) are called *quasisimple*. A *simple* point P is one such that P lies in the closure of $L_0(G^+)$ for precisely one G^+. The class of quasisimple points thus includes, in addition to simple points, the points P that lie on some $L_0(\hat{G}^+)$ that coincides with $L_0(G^+)$, or that are boundary points of some $L_0(G'^+)$. In the latter case, G [or one of the $\hat{G}$ such that $L_0(\hat{G}^+)$ coincides with $L_0(G^+)$] is a contraction of G' and the consistency of the property (2) at P with property (2) at points that lie on $L_0(G'^+)$ in the neighborhood of P is a consequence of the following property:

(vi) The gradients to $L_0(G^+)$ are continuous on L_c^+, in the sense that the directions of the gradients at points of L_c^+ in some sufficiently small neighborhood of P lie arbitrarily close to the set of directions defined by positive linear combinations of the gradients at P.

Property (2) gives the plus $i\varepsilon$ rule at quasisimple points. By virtue of properties (i) through (v) of Section 2, almost every point lying on the set of positive-α Landau surfaces is quasisimple. The only remaining points are those of $\mathscr{M}_0$ – which are confined to a set of dimension 2 less than the dimension of the complete set L_c^+ – and the points that lie on two distinct surfaces $L_0(G^+)$ – which are confined to a set of dimension 1 less than the dimension of L_c^+.

Most of the latter points are covered by a third property, which is the generalization of property (2) to points P of $L_c^+ - \mathscr{M}_0$ such that for some fixed G^+, any $L_0(\hat{G}^+)$ that contains P coincides near P with

[11] A function is analytic at P if it is an analytic function of analytic local coordinates.

the $L_0(G_c^+)$ of some contraction G_c^+ of G^+. These points are called *semisimple*. It is shown in Ref. [1] that the domains $\operatorname{Im} \boldsymbol{l}_i > 0$ – where the index i labels the (necessarily finite) number of distinct surfaces $L_0(G_c^+)$ that contain P – have a nonempty intersection that has a real neighborhood $\mathscr{U}$ of P on its boundary.

Property 3. *(Plus iε rule for semisimple points.) The property analogous to* (2) *holds at semisimple points, with the intersection of the* $\operatorname{Im} \boldsymbol{l}_i > 0$ *replacing the domain* $\operatorname{Im} \boldsymbol{l} > 0$.

Quasisimple points are special cases of semisimple points and property (2) is a special case of property (3).

III. Macroscopic Causality

1. Space-Time Localization of Particles

To formulate a causality condition, some notion of space-time position is needed. There is no difficulty with space-time *displacements:* we accept that they are generated by $\exp(ip\,\Delta x)$. That is, the four-vector a in the equation

$$\varphi^a(\boldsymbol{p}) = \varphi(\boldsymbol{p})\, e^{ipa} \tag{5}$$

is interpreted as a space-time displacement.

On the other hand, the *position* of a relativistic particle cannot be determined precisely. The sharpest definition allowable either in practice or mathematically is of the order of the particle Compton wavelength.

For the purposes of this work, it is sufficient to introduce a weak notion of a macroscopic space-time localization that expresses roughly the idea that the *asymptotic* space-time localization of a particle represented by $\varphi(\boldsymbol{p})$ lies in its velocity cone V, which is defined as the closure of the set of all space-time points that can be reached by traveling from the origin along any direction that is parallel to any on-mass-shell four-vector p such that $\varphi(\boldsymbol{p})$ is nonvanishing.

To make this idea precise, consider a given arbitrarily small open region R, the closure of which *does not* intersect the velocity cone V, and a dilation that takes each point u' to the point $u'\tau$. This dilation takes the region R to the dilated region R_τ.

The region R_τ becomes infinitely large as $\tau \to \infty$. Our basic notion is that it makes sense in the asymptotic domain $\tau \to \infty$ to speak of the probability $P(R_\tau)$ that the particle represented by $\varphi(\boldsymbol{p})$ can be considered to be in R_τ. Moreover, this probability should become small when the space-time wave function corresponding to $\varphi(\boldsymbol{p})$ becomes uniformly small in a region R'_τ, where R' is an open set that contains the closure of R. (Notice that R'_τ contain all points that lie at distance $\alpha\tau$ from points R_τ, where α is positive and τ goes to infinity.)

Various space-time functions can be defined. Common forms are

$$f(x) = \int \varphi(\boldsymbol{p})\, e^{-ipx} \frac{d\boldsymbol{p}}{2p_0}, \tag{6}$$

and

$$\tilde{\varphi}(x) = \int \varphi(\boldsymbol{p})\, e^{-ipx} \frac{d\boldsymbol{p}}{(2p_0)^{1/2}}, \tag{7}$$

where $p_0 = (\boldsymbol{p}^2 + m^2)^{1/2}$.

It is not possible in general to attach a strict physical meaning to the values of either $f(x)$ or $\tilde{\varphi}(x)$ at a *given point* x. However, both $\max_{x \in R_\tau} |f(x)|$ and $\max_{x \in R_\tau} |\tilde{\varphi}(x)|$ have a rapid (i.e. faster than any inverse power of τ) fall-off, under certain conditions on $\varphi_\tau(\boldsymbol{p})$. Under certain other conditions, the fall-off is exponential. Thus, although the actual values of these different space-time functions are different, the *nature* of the rate of fall-off is the same. And this nature is the same also for the various other quantities that have been proposed as measures of the space-time densities (cf. Ref. [2]). Thus it seems reasonable to take the *nature* of the rate of fall-off given by these functions to be a reliable indication of the nature of the rate of fall-off of $P(R_\tau)$.

For example, suppose the wave function $\varphi(\boldsymbol{p})$ belongs to the Schwartz space $\mathscr{D}$. If the closure of R does not intersect V, then Ruelle's lemma [9] implies that both $\max_{x \in R_\tau} |f(x)|$ and $\max_{x \in R_\tau} |\tilde{\varphi}(x)|$ have a rapid fall-off with τ.

If we consider a set of relatively displaced regions R_u centered at different space-time points u, then this fall-off is also uniform on compact subset of u's, as long as the closure of the set of R_u does not intersect V.

Consider next gaussian-type wave functions that shrink with the displacement parameter τ [10]. In particular, consider the class of wave functions of the form

$$\varphi_\tau(\boldsymbol{p}) = \chi(\boldsymbol{p})\, e^{-(\boldsymbol{p}-\boldsymbol{P})^2 \gamma\tau}, \tag{8}$$

where $\boldsymbol{P}$ is a given 3-vector, γ is a positive constant and the $\chi(\boldsymbol{p})$ are restricted to the class of functions of $\mathscr{D}_K$ that have compact supports confined to some fixed compact set K; that are analytic in some fixed complex neighborhood N of $\boldsymbol{P}$; and that satisfy $|\chi(\boldsymbol{p})| \leqq 1$ if $\boldsymbol{p}$ is real or in N.

Theorem. *Both* $\max_{x \in R_\tau} |\tilde{\varphi}(x)|$ *and* $\max_{x \in R_\tau} |f(x)|$ *have an exponential fall-off if the intersection of the closure of* R *with the line parallel to the* 4*-vector* $P(P_0 = (\boldsymbol{P}^2 + m^2)^{1/2}, \boldsymbol{P})$ *drawn from the origin is void. This fall-off is, moreover, uniform on any compact subset of* u*'s over which this intersection remains void. Furthermore, the constant of exponential fall-off decreases no faster than linearly in* γ *in the* $\gamma \to 0$ *limit.*

In particular, for all γ smaller than some fixed γ_0, the following inequalities hold:

$$\max_{x\in R_\tau} |f_{\chi,\gamma}(x)| < C e^{-\alpha\gamma\tau}, \qquad \max_{x\in R_\tau} |\tilde{\varphi}_{\chi,\gamma}(x)| < D e^{-\beta\gamma\tau} \tag{9}$$

where C and α (resp. D and β) do not depend on χ, γ, or τ.

The proof is given in Appendix III.

The fall-off indicated by the symbol $\Rightarrow 0$ will mean a fall-off of the type specified by (9) when the $\varphi_\tau(\boldsymbol{p})$ under consideration are of the form (8). The $\varphi(\boldsymbol{p})$ in $\mathscr{D}$ of interest will be the $\gamma=0$ limit of the $\varphi_\tau(\boldsymbol{p})$ of (8). The meaning of $\Rightarrow$ switches to rapid fall-off in this case.

Our general assumption about space-time localization is that for any $\varphi_\tau(\boldsymbol{p})$ of the form (8) (including $\gamma=0$), $P(R_{u\tau})\Rightarrow 0$ uniformly in u on compact sets such that the intersection of the closure of the union of the R_u with V is void. The set V is the velocity cone of $\varphi_\tau(\boldsymbol{p})=\chi(\boldsymbol{p})$ for $\gamma=0$, but for $\gamma \neq 0$, it is the line through the origin with direction P. Thus V is defined by the "effective support" of $\varphi_\tau(\boldsymbol{p})$, which for the case $\gamma \neq 0$ is simply the point $\boldsymbol{p}=\boldsymbol{P}$.

2. *Macroscopic Causality*

Let $\varphi_{i\tau}$ be a set of initial and final wave functions of the form (8) and let $\varphi_{i\tau}^{u_i\tau}$ be a corresponding set of displaced wave functions:

$$\varphi_{i\tau}^{u_i\tau}(p_i) = \varphi_{i\tau}(p_i)\, e^{i(p_i u_i)\tau}. \tag{10}$$

It is convenient to visualize the limit $\tau \to \infty$ in a space-time coordinate system scaled to τ. Then the displacements $u_i\tau$ become fixed displacements u_i and the regions R_τ of the preceding section become fixed regions R.

In this system, the particle i becomes localized, in the limit $\tau\to\infty$, in its displaced velocity cone $V_i^{u_i}$ with tip at u_i. (That is, the probability $P(R)\Rightarrow 0$ as $\tau\to\infty$, if the closure of R does not intersect the corresponding $V_i^{u_i}$.)

The requirement that the transfer of energy-momentum from the initial particles to the final particles can be ascribed to (stable) physical particles is the requirement that one can draw a network of lines representing a *classical* multiple scattering process that is compatible with both the momentum-energy requirements on the various initial and final particles and also the space-time requirements entailed by the positions of the various initial and final velocity cones $V_i^{u_i}$. Specifically, the energy-momentum of each external line of the classical process must satisfy the appropriate mass shell constraint, and its momentum must lie in the effective support of $\varphi_{i\tau}(p)$. Moreover, the particular vertex of the multiple scattering diagram that represents the interaction involving the external particle i must lie in the corresponding $V_i^{u_i}$.

For a fixed set of $\varphi_{i\tau}$, there may be certain sets $u \equiv \{u_i\}$ of displacements such that it is possible to find a classical multiple scattering process that satisfies the conditions just described. These u are called *causal* with respect to $\varphi_\tau \equiv \{\varphi_{i\tau}\}$. The set consisting of the u's that are noncausal is called $\mathscr{A}(\{\varphi_{i\tau}\})$. The set consisting of the u's such that one can find no *connected* classical process is called $\mathscr{A}_c(\{\varphi_{i\tau}\})$. In the case $\gamma > 0$, the $\mathscr{A}(\{\varphi_{i\tau}\})$ and $\mathscr{A}_c(\{\varphi_{i\tau}\})$ depend only on $P \equiv \{P_i\}$ and they will often be denoted simply by $\mathscr{A}(P)$ and $\mathscr{A}_c(P)$ respectively.

Macrocausality is the requirement that

$$|S^c(\{\varphi_{i\tau}^{u_i\tau}\})| \Rightarrow 0 \tag{11}$$

uniformly on compact subsets of $\mathscr{A}_c(\{\varphi_{i\tau}\})$ [12].

For $\gamma = 0$, the symbol $\Rightarrow 0$ denotes a rapid fall-off and for $\gamma > 0$, it denotes a bound of the type (9):

$$|S^c(\{\varphi_{i\tau}^{u_i\tau}\})| < C e^{-\alpha\gamma\tau}. \tag{11'}$$

This bound (11′) is to hold for γ smaller than some fixed γ_0, and C and α are independent of γ, τ and the χ_i's, provided the χ_i's satisfy the conditions listed under (8) [13].

The form of the bound in (11) is justified on the basis of the classical model. All transfers of energy momentum *not* ascribable to stable physical particles in accordance with the classical ideas are assumed to give only effects that fall off at large distances with some arbitrary but fixed exponential rate. For sufficiently small γ, these terms are all masked by effects $\sim e^{-\alpha\gamma\tau}$ that are classically ascribable to stable physical particles: these latter effects come from classical multiple scattering processes in which some external particles are not localized in their effective velocity cones (or the momenta are not in the effective supports). Similar classical effects occur in the $\gamma = 0$ limit, where they give contributions having rapid fall-off. Thus, the dominant effect for small enough γ has the same type of fall-off as $P(R_\tau)$ itself. This is the assertion of (11).

[12] The physical justification of macrocausality described below requires that the velocity cones used in the construction of $\mathscr{A}$ or $\mathscr{A}_c$ be "infinitesimally larger" than the ones defined by the effective supports of the $\varphi_{i\tau}$. This modification of the definitions of $\mathscr{A}$ and $\mathscr{A}_c$ does not significantly alter any arguments, as long as P is kept away from $\mathscr{M}_0$ [1]. In this work, we consider only a restricted macrocausality condition that excludes in particular all points of $\mathscr{M}_0$. Hence this slight complication will be ignored.

[13] The rapid and exponential fall-off rates can depend on the compact sets in $\mathscr{A}_c$. The rapid fall-off rate can depend also on χ.

A final remark: Suppose $Z(\boldsymbol{p})$ is some function of all the initial plus final variables $\boldsymbol{p}_i$ that is analytic with an appropriate choice of units in each (complex) component $p_{i\mu}$ of each vector $\boldsymbol{p}_i$ for $|p_{i\mu} - P_{i\mu}| \leqq 1$. (The corresponding domain of analycity in $\boldsymbol{p} \equiv \{\boldsymbol{p}_i\}$ is N.) Then consider the function $\hat{\chi}(\boldsymbol{p}) = Z(\boldsymbol{p})\chi(\boldsymbol{p})$, where χ is a product $\prod_i \chi_i$ of functions of the form (8) that has its compact support in N.

Macrocausality says *a priori* nothing unless Z has itself a product form[14]. However, it is easily seen – by using a multidimensional Cauchy expansion for Z around the point $P \equiv \{P_i\}$ – that (11′) still holds for this nonproduct $\hat{\chi}(\boldsymbol{p})$ (with some different C)[15].

3. Causal Displacements [1]

In this section, we describe some basic properties, which will be used later, of the set $\mathscr{A}_c$ of noncausal displacements. [By causal displacements, we mean here only those that are associated with *connected* diagrams (i.e. that do not belong to $\mathscr{A}_c$)].

(i) If the (effective) support of $\varphi_\tau \equiv \prod_i \varphi_{i\tau}$ contains no positive-α Landau point, then all nontrivial displacements with respect to φ_τ belong to $\mathscr{A}_c(\varphi_\tau)$.

The trivial displacements with respect to φ_τ are displacements that consist of any common displacement of all particles, plus any set of displacements of the various particles along lines that lie in their own velocity cones.

Property (i) is a consequence of the Coleman-Norton [5] remark that the positive-α Landau equations are precisely equations of classical kinematics: if the Landau equations cannot be satisfied, then there can be no causal displacements except trivial ones.

Let P be a point of $\mathscr{M} - \mathscr{M}_0$. Let u represent a $4(n+m)$-dimensional displacement vector. Let $\{\hat{u}_k\}$ be a set of $l = 3(n+m) - 4$ of these vectors such that the corresponding $q_k = \hat{u}_k \cdot (p - P)$ are real local analytic coordinates of $\mathscr{M} - \mathscr{M}_0$ at P. (A specific set of $\hat{u}_k$ and q_k is exhibited at the beginning of Chapter IV 1.) Let Γ be the space spanned by the set $\{\hat{u}_k\}$ and let any u in Γ be written as $u = \sum u_k \hat{u}_k$. Then $u \cdot (p - P) = \sum u_k q_k \equiv u \cdot q$.

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[14] We have specifically restricted the statement of macrocausality to the case of product wave functions $\varphi_{i\tau}^{u_i\tau}$, because the classical arguments are more compelling in this case. In any case, the physical arguments are difficult to extend to "wave functions" that mix initial and final variables.

[15] The assumption that C and α are independent of the χ_i [in the class defined below (8)] is used precisely here; all functions of the form $\prod_\mu (p_{i\mu} - P_{i\mu})^{n^\mu} \chi_i(\boldsymbol{p}_i)$ belong to this class.

The following properties are proved in Ref. [1]:

(ii) If the support of $\chi = \prod_i \chi_i$ is sufficiently small, then the intersection of Γ with the set of displacements that are trivial with respect to χ is the null vector.

Properties (i) and (ii) ensure that if P does not lie on L_c^+, then all nonzero displacements in the set Γ belong to $\mathscr{A}_c(P)$, and also to $\mathscr{A}_c(\chi)$ if the support of χ is sufficiently small.

(iii) If P is a simple or quasisimple point of L_c^+, then there is only one causal direction u in the set Γ at P [i.e. which does not belong to $\mathscr{A}_c(P)$] and this direction is precisely the direction of the gradient at P to the Landau surface L_0 that contains P. That is, $u_k = (\partial l/\partial q_k)\, h$, where h is positive.

Moreover, given any open cone C_p^+ that contains the causal direction at P, there is a neighborhood N of P sufficiently small so that all nonzero u's in Γ lying in the complement C_P' of C_P^+ lie also in $\mathscr{A}_c(\chi)$, for all χ with support in N.

(iv) At a semisimple point P, the analogous property holds, with the direction of the gradient to L_0 at P being replaced by the closed convex cone C_P^+ that is formed of positive linear combinations of vectors $u_c\,(P)$ that are gradients at P to the various L_0 that contain P.

The closed convex cone C_P^+ of causal displacements at a semisimple point P is precisely the "polar cone" of the cone C_P of analyticity defined in property (3) of Chapter II. That is, either cone is defined in terms of the other by the condition $u \cdot \operatorname{Im} q > 0$ for all u in C_P^+ and $\operatorname{Im} q$ in C_P.

IV. Macrocausality and the Normal Analytic Structure

1. Introductory Remarks

In Section 2 below, the normal analytic structure of the S-matrix, that is, properties (1) through (3) of Chapter II 3, is derived from the macrocausality condition (11). The linearity in γ, for small γ, of the constant of exponential fall-off, and the rapid fall-off at $\gamma = 0$, are crucial in this proof; we have not succeeded in deriving the analyticity properties simply from a condition of exponential fall-off at one fixed value of γ. (This is discussed in Appendix II.) On the other hand, (11) can be slightly weakened: condition (11′) can be replaced by:

$$S^c(\{\varphi_{i\tau}^{u_i\tau}\}) < \mathscr{P}(\tau)\, e^{-\alpha\gamma\tau}, \tag{12}$$

where $\mathscr{P}(\tau)$ is a fixed polynomial in τ. The derivation of the normal analytic structure is only slightly affected by this change.

In Section 3, various converse properties are derived. In Subsection (a) the rapid fall at $\gamma=0$ demanded by (11) is derived from the normal analytic structure. The support of χ is assumed to contain only points of the type referred to by the normal analytic structure: i.e., they must be either non-Landau (not on any positive-α Landau surface) or semisimple. In Subsection b), the fall-off property (12) is derived for $\gamma>0$, provided P is non-Landau or semisimple. This proves the equivalence of the normal analytic structure to a weak form of the macrocausality condition (11). This form is restricted to the points referred to by the normal analytic structure, and has (12) in place of (11).

The physical space-time arguments for macrocausality give (11′) rather than (12). In Subsection c), the stronger condition (11′) is derived from the normal analytic structure, together with the boundedness condition on the S-matrix, provided P is non-Landau or quasisimple. We see no reason why (11′) could not be proved also for semisimple P, but the additional effort seems unwarranted.

From the full macrocausality condition (11), one can derive analyticity properties also at points not covered by properties (1) though (3). At all points except the very rare type-II points [1], the scattering function decomposes into a sum of terms corresponding to different parent diagrams. The proofs and results are direct generalizations of those given in Ref. [1] and are not further discussed here.

2. *Derivation of the Normal Analytic Structure from Macrocausality*

Consider the $3(n+m)$ components of the $(n+m)$ vectors $\boldsymbol{p}_i-\boldsymbol{P}_i$. For simplicity, we may choose from among them a set of $l=3(n+m)-4$ analytic local coordinates of $\mathcal{M}-\mathcal{M}_0$ at P. [This set can be chosen in the following way: Since P is not in $\mathcal{M}_0$, there must be two noncollinear 4-vectors P_{i_1}, P_{i_2} and two components $P_{i_1\mu}, P_{i_2\mu}$ of their 3-momenta such that $P_{i_1\mu}(P_{i_1 0})^{-1} \neq P_{i_2\mu}(P_{i_2 0})^{-1}$. We exclude from the set of $3(n+m)$ variables $\{\boldsymbol{p}_i-\boldsymbol{P}_i\}$ the two corresponding components, plus two other components with different values of μ. The remaining set of l components is a set $\{q_k\}$ of real analytic coordinates at P. That is, any p_i^μ can be expressed, locally, as a real analytic function of the q_k.] These q_k are written as $q_k=\hat{u}_k\cdot(p-P)$. Then the set Γ of Chapter III 3 is the set of u of the form $u=\sum u_k\hat{u}_k$, and $u\cdot q\equiv\sum u_k q_k$, where sums on k run from 1 to l. (This particular choice of q_k and u_k is made only for definiteness; it is not important in the proofs.)

The initial and final wave functions will be restricted to a class of functions $\{\varphi_{i\tau}^{u_i\tau}\}$ of the form (8), where the support of $\chi=\prod_i \chi_i$ lies in the neighborhood N defined at the end of Chapter III 2. The neighborhood N is taken to be small enough so that the intersection of N with the complex mass shell $\mathcal{M}$ lies in the region covered by the local analytic coordinate system at P.

The connected amplitude $S_c(\{\varphi_{i\tau}^{u_i\tau}\})$ of the process can now be written, taking $|u|=1$,

$$e^{i(Pv)} S_c(\{\varphi_{i\tau}^{u_i\tau}\}) = \int T(q)\,\chi(q)\,J(q)\,e^{-iqv}\,e^{-\gamma|v|\mu(q)}\,dq\,, \tag{13}$$

where $v=\tau u$, $T(q)$ denotes the scattering function, $J(q)$ is the Jacobian that comes from the elimination of the δ functions of energy-momentum conservation and of mass-shell constraint, and $\mu=\sum_i(\boldsymbol{p}_i-\boldsymbol{P}_i)^2$, where i ranges over all the initial and final particles.

The function $\mu(q)$ is positive for real q and vanishes for $q=0$. The functions $J(q)$ and $\mu(q)$ are analytic in complex neighborhoods of the origin and N is taken small enough to lie in both of these. Finally, $J(0)$ is nonzero.

We introduce

$$\hat{T}(v,t)=\int F(q)\,e^{-t\mu(q)}\,e^{-iqv}\,dq\,, \tag{14}$$

with

$$F(q)=T(q)\,\chi(q)\,J(q)\,. \tag{15}$$

Then (13) gives

$$\hat{T}(v,\gamma|v|)=S_c(\{\varphi_{i\tau}^{u_i\tau}\})\,e^{i(P\cdot v)}\,. \tag{16}$$

In Subsection a) below, the analyticity of $F(q)$ at $q=0$ is derived from macrocausality, in case P of $\mathcal{M}$ lies on no positive-α Landau surface. In Subsection b), the plus $i\varepsilon$ rule for $F(q)$ at $q=0$ is obtained for semi-simple points P. The corresponding properties of T itself, considered as a *distribution*, follow[16] from these properties of $F(q)$, together with the facts that $J(q)$ is analytic and nonzero at $q=0$, and that $\chi(q)$ is analytic and can be taken nonzero at $q=0$.

a) Analyticity at Non-Landau Points. Consider a point $P\equiv\{P_i\}$ of $\mathcal{M}$ that does not lie on L_c^+. Then choose the support of χ sufficiently small so that it contains no point of L_c^+, and so that the set of all unit normed u's of the form $(u_1 \dots u_l)$ is a compact subset of $\mathcal{A}_c(\{\varphi_{i\tau}\})$ for $\gamma\geqq 0$ (see Chapter III 3).

The special case $\gamma=0$ gives the rapid fall-off condition:

$$\hat{T}(v,0)\Rightarrow 0 \tag{17}$$

for $|v|\to\infty$, uniformly in all directions of v.

Then (14) shows that $F(q)$ is the Fourier transform of $\hat{T}(v,0)$:

$$F(q)=(2\pi)^{-l}\int \hat{T}(v,0)\,e^{iqv}\,dv\,. \tag{18}$$

[16] This is proved in Ref. [1], Appendix D. The analytic representation applies also for product test functions in L^2. See Appendix I 3 below.

The uniform rapid fall-off of $\hat{T}(v,0)$ entails that $F(q)$ be infinitely differentiable.

To show that $F(q)$ is also analytic at $q=0$, we write (18) in the form:

$$(2\pi)^l F(q) = \int e^{iqv} dv \left\{ \hat{T}(v, \gamma_0|v|)\, e^{\gamma_0|v|\mu(q)} - \int_0^{\gamma_0|v|} dt \frac{\partial}{\partial t} [\hat{T}(v,t)\, e^{t\mu(q)}] \right\}. \tag{19}$$

Eqs. (11′) and (16) immediately entail that the term $F_1(q)$ coming from the first term under the integral is convergent and, moreover, analytic in the domain $\mathscr{E}$ defined as the intersection of N with

$$\mathrm{Re}\mu(q) + \frac{|\mathrm{Im}\, q|}{\gamma_0} < \alpha, \tag{20}$$

where α is the fall-off parameter in (11′).

The domain in $\mathscr{E}$ includes the origin since $\mu(0)=0$.

The integral $F_2(q)$ associated with the second term under the integral of Eq. (19) is a well-defined function for q real in $\mathscr{E}$ since F and F_1 are both well defined. We shall now show that it is also analytic in $\mathscr{E}$.

Introducing (14) into the term of the integrand of (19) that corresponds to F_2, one obtains

$$e^{iqv} \frac{\partial}{\partial t} [\hat{T}(v,t)\, e^{t\mu(q)}] = \int F(q')\, e^{i(q-q')v}\, e^{t(\mu(q)-\mu(q'))}\, [\mu(q)-\mu(q')]\, dq'. \tag{21}$$

(The transference of $\partial/\partial t$ to the integrand is allowed because $F(q)$ belongs to $\mathscr{D}$.)

Hefer's Theorem [11] allows one to write

$$\mu(q) - \mu(q') = \varrho(q,q') \cdot (q-q'), \tag{22}$$

where the components ϱ_j ($j=1,2,\ldots,l$) of ϱ are analytic in the product of the domains N in q and q' space. Eq. (21) then becomes

$$e^{iqv} \frac{\partial}{\partial t} [\hat{T}(v,t)\, e^{t\mu(q)}] = \nabla_v [e^{iqv}\, e^{t\mu(q)}\, \boldsymbol{H}(q,v,t)], \tag{23}$$

where

$$\boldsymbol{H}(q,v,t) = -\int i F(q')\, e^{-iq'v}\, e^{-t\mu(q')}\, \varrho(q,q')\, dq'. \tag{24}$$

We may thus write

$$(2\pi)^l F_2(q) = -\lim_{R\to\infty} \left[\int_{|v|<R} dv \int_0^{\gamma_0|v|} dt\, \nabla_v [e^{iqv}\, e^{t\mu(q)}\, \boldsymbol{H}(q,v,t)] \right]. \tag{25}$$

For fixed R, we may change the order of integration and perform a first integration over v in the domain $t/\gamma_0 < |v| < R$. Then Gauss' theorem gives this integral as a difference of integrals over the boundary surfaces at $|v| = R$ and $|v| = t/\gamma_0$.

The contribution $F_2^R(q)$ to $F_2(q)$ associated with the surface $|v| = R$ vanishes in the $R \to \infty$ limit (this is proved in Appendix IV).

The function $F_2(q)$ is thus equal to the contribution from the surface at t/γ_0:

$$(2\pi)^l F_2(q) = \gamma_0 \int e^{iqv} e^{\gamma_0 |v| \mu(q)} \hat{v} \cdot \boldsymbol{H}(q, v, \gamma_0 |v|) \, dv \,, \tag{26}$$

where $\hat{v} = |v|^{-1} v$.

Since each component H_j is an analytic function of q in N, $F_2(q)$ is analytic in $\mathscr{E}$ by virtue of (11′) (and the remark at the end of Chapter III 2). The argument is essentially the same as for $F_1(q)$.

This proves the analyticity of $F(q)$ [and thus of $T(q)$] at $q = 0$, and completes the proof of property (1) of Chapter II.

b) The Plus iε Rule. Consider the cone C_P^+ of causal u at a semisimple point P and a slightly larger cone C^+ that contains C_P^+. We choose the support of χ sufficiently small so that all $v \neq 0$ in the complement C' of C^+ lie in the acausal set $\mathscr{A}_c(\{\varphi_{i\tau}\})$ for $\gamma \geqq 0$.

The cone C^+ has a nonempty open polar cone C, which is the set of all vectors $y = (y_1 \dots y_l)$ such that $y \cdot v > 0$ for all $v \neq 0$ in the closure of C^+.

The proof of the plus $i\varepsilon$ rule proceeds as follows. The $\hat{T}(v, t)$ of (14) is written as:

$$\hat{T}(v, t) = \theta(C^+) \, \hat{T}(v, t) + \theta(C') \, \hat{T}(v, t) \,,$$

where $\theta(C)$ is one for v in C and zero otherwise.

The distribution $F(q)$ divides accordingly. The first term,

$$F_C(q) = \int \theta(C^+) \, \hat{T}(v, 0) \, e^{iqv} \, dv (2\pi)^{-l} \,, \tag{27}$$

is a boundary value of a function analytic for $\operatorname{Im} q$ in C due to the exponential fall-off of e^{iqv} as $|v| \to \infty$. [$\hat{T}(v, 0)$ is bounded by a constant (see (2)).]

The second term is

$$F_{C'}(q) = \int \theta(C') \, \hat{T}(v, 0) \, e^{iqv} \, dv (2\pi)^{-l} \,.$$

The intersection of $|u| = 1$ with C' is a compact subset of $\mathscr{A}_c(\chi)$. Thus the arguments of Subsection a) can be used again.

The only difference is that there is an extra surface term at the boundary of C'. This term is confined to the boundary of C^+ and thus, similarly to $F_C(q)$, it is a boundary value of a function analytic in the intersection of $\operatorname{Im} q$ in C with a neighborhood of the origin. Details are given in Appendix IV.

The sum $F(q)$ is, therefore, analytic in the intersection of $\operatorname{Im} q$ in C with some neighborhood of $q = 0$.

This proves property (3) of Chapter II, and property (2) is a special case. The bound $|T(\boldsymbol{q})| < C|\mathrm{Im}\,\boldsymbol{l}(\boldsymbol{q})|^{-l}$ for $\boldsymbol{q}$ in $C_P^+ \cap \mathcal{U}$ follows from (27) and the remarks at the end of Appendix IV.

3. The Converse

In this section, we prove the fall-off properties for a given u of $\mathcal{A}_c$. The proof of uniformity on compact subsets of $\mathcal{A}_c$ follows from continuity arguments.

a) Proof of Rapid Fall-Off (11) *for* $\gamma = 0$ [17]. It is a simple matter to construct a set of local analytic coordinates $q_k = \hat{u}_k \cdot (p - P)$ such that the given u of $\mathcal{A}_c(\chi)$ belongs to the corresponding set Γ (see Chapter III 3). Thus we may still write

$$e^{iPv}\, S^c(\{\chi_i^{u_i\tau}\}) = \int T(q)\,\chi(q)\,J(q)\,e^{-iqv}\,dq = \hat{T}(v)\,, \tag{28}$$

where $v = \tau u$ and J is a Jacobian that is analytic (and nonzero) at all points that lie in the (sufficiently small) support of χ. (If the support of χ is not sufficiently small, we use a suitable partition of unity [13] and prove the rapid fall-off (11) for each corresponding term.)

By virtue of property (1) of Chapter II, Eq. (28) entails immediately the rapid fall-off (11) if the support of χ contains no positive-α Landau point.

The set $\mathcal{A}_c(\chi)$ lies in the intersection of the $\mathcal{A}_c(P)$ over all P in the support of χ. This is evident from their definitions. Thus the given u in $\mathcal{A}_c(\chi)$ must lie in all the $\mathcal{A}_c(P)$ for P in supp. χ. If the support of χ is sufficiently small, and contains *only* non-Landau and semisimple points, then by virtue of properties (1) and (3) of Chapter II, and property (iv) of Chapter III, there is a fixed direction of $\mathrm{Im}\,q$ that satisfies $\mathrm{Im}\,q \cdot u < 0$ such that for all $\mathrm{Re}\,q$ in the support of χ the point q lies in the domain of analyticity of $T(q)$ for sufficiently small but positive $\mathrm{Im}\,q$ having the fixed direction.

Consider a small box $\mathcal{N}$ in q-space that contains the support of χ, and such that the above-described analyticity property holds also for $\mathrm{Re}\,q$ in $\mathcal{N}$. The codimension-one faces of $\mathcal{N}$ are defined by equations $\hat{q}_i = \pm a_i$. Now define

$$\xi(q) = \prod_{i=1}^{l} \exp\left(-\frac{1}{\hat{q}_i^2 - a_i^2}\right).$$

This function is analytic inside $\mathcal{N}$; we choose $\mathcal{N}$ sufficiently small so that u also belongs to $\mathcal{A}_c(\xi)$. (If this were not possible for the function χ considered, again use a suitable partition of unity.)

[17] A special case of this result has been proved in Ref. [12].

The function $\hat{T}(v)$ in (28) can be written as the convolution of the Fourier transform $\hat{\chi}_\xi(v)$ of the function $\chi(q)\,J(q)\,(\xi(q))^{-1}$ with the Fourier transform $\hat{T}_\xi(v)$ of the distribution $T(q)\,\xi(q)$ [see Appendix I 3 (c)]:

$$\hat{T}(v) = \hat{\chi}_\xi(v) * \hat{T}_\xi(v)\,. \tag{29}$$

The function $\hat{\chi}_\xi(v)$ has a uniform rapid fall-off in all directions of v. On the other hand, the function $\hat{T}_\xi(v)$ has a uniform rapid fall-off in all directions of v that belong to compact subsets of $\mathscr{A}_c(\xi)$. To see this, it is sufficient to consider a distortion of the interior of $\mathscr{N}$ into the part of the complex q-space that lies in the domains of analyticity of $T(q)$ and $\xi(q)$ and such that $\operatorname{Im} q \cdot u < 0$. The result then follows from the fact that ξ approaches zero on boundary of $\mathscr{N}$ faster than any power of $\operatorname{Im} q$ (for appropriate distorted contour $\mathscr{N}'$), whereas $T(q)$ can grow no faster than an inverse power of $\operatorname{Im} q$ [14].

Finally, $\hat{T}_\xi(v)$ has a slow growth (no faster than a polynomial) in the causal directions of v, since $T(q)$ is a tempered distribution.

Eq. (29) can be written:

$$\hat{T}(v) = \int \hat{\chi}_\xi(v - v')\,\hat{T}_\xi(v')\,dv'\,.$$

To prove the rapid fall-off of $\hat{T}(v)$ in the direction v of $\mathscr{A}_c(\xi)$, divide the v'-space as follows:

(i) The points that lie outside a sphere S_α of radius $\alpha|v|$ centered at the origin, where α is large compared to one ($|v - v'| \approx |v'|$, $|v'| \geqq \alpha|v|$).

(ii) The points that lie inside S_α but outside a sphere S_β of radius $\beta|v|$ centered at the point v, where β is chosen such that S_β does not intersect the cone C_ξ of causal directions with respect to ξ. ($|v - v'| \geqq \beta|v|$, $|v' \leqq \alpha|v|$).

(iii) The points inside S_β.

Each term is easily seen to have the rapid fall-off of (11): use the rapid fall-off of $\hat{\chi}_\xi$ and the slow growth of $\hat{T}_\xi$ for the first two domains, and the rapid fall-off of $\hat{T}_\xi$ in the directions of $\mathscr{A}_c(\xi)$ for the third.

b) Proof of (12). In the case of $\gamma > 0$, the proofs are adapted from Appendix III. Consider a local coordinate system q at P of $\mathscr{M} - \mathscr{M}_0$ and suppose first that the support of χ is sufficiently small so that all points in that support are covered by the above coordinate system. (If not, use a suitable partition of unity, as is described later.)

Since $\mu(0) = 0$ and $\mu(q)$ is analytic and positive, it is equal to a positive quadratic form in the components of q, plus a third-order term. Thus the region in real q-space defined by $\mu(q) \leqq \alpha$ grows monotonically with α for small α.

If the support of χ contains no positive-α Landau point, then the bound (11′) follows from the same arguments that give (9). (The condition that u be noncausal, hence nontrivial, replaces the condition in (9) that x be not along P.)

Suppose the support of χ contains Landau points and that P is non Landau or semisimple. If the support of χ is sufficiently small (if not, use also here a partition of unity), then the scattering function $T(q)$ is equal (see Appendix I 2b)) to some derivative of a continuous function $C(q)$ *which has the same analyticity properties* as T:

$$T(q) = C(q)\frac{d^n}{dq^n}. \tag{30}$$

As is discussed in Appendix B of Ref. [1], the given noncausal displacement u is equal, up to a trivial displacement, to some noncausal $\bar{u}$ in Γ, where Γ is the space associated with the coordinate system q. Then, by virtue of properties (1) and (3) of Chapter II, and property (iv) of Chapter III 3, there is a fixed direction of $\operatorname{Im} q$ satisfying $\operatorname{Im} q \cdot \bar{u} < 0$ such that for all $\operatorname{Re} q$ in a sufficiently small neighborhood $\mathcal{N}$ of $q=0$, the point q is in the domain of analyticity of $T(q)$ provided $\operatorname{Im} q$ is sufficiently small but positive and has the fixed direction.

The region $\mathcal{N}$ will be distorted into $\mathcal{N}'$ by shifting each point along this fixed direction, and the bound (12) follows from the same arguments that give (9). The polynomial $\mathcal{P}(\tau)$ arises because of the derivatives in q on the function $e^{i(p-P)u\tau}$. The derivatives on the function χ itself obliges us *a priori* to admit a dependence of the coefficients of $\mathcal{P}$ on some derivatives of χ, but we now show how this can be eliminated. The following method also allows one to treat the cases when the support of χ is not sufficiently small (or is not even finite).

Consider a partition of unity in the space of the initial (resp. final) 3-momenta into a part that has a sufficiently small support around the point $P_{in} \equiv \{P_i\}_{i \in in}$ (resp. P_f) and a part that vanishes in the neighborhood of P_{in} (resp. P_f). This defines a corresponding partition of unity in the space of all momenta into a part that has its support in a neighborhood $\mathcal{N}$ of $q=0$ and a part that vanishes in a neighborhood of that point. The contribution of the latter is easily seen, by using *the boundedness of the S-matrix* [18], to have a bound of the type (11′) itself. The above partitions are chosen so that $\mathcal{N}$ is (much) smaller than the domain $\mathcal{N}$ of analyticity of χ. Then all the occurring derivatives of χ in $\mathcal{N}$ will be bounded by virtue of the bound on χ in $\mathcal{N}$. (Use a multidimensional Cauchy formula to see this.)

[18] Instead of (2), we use in the remainder of this chapter the general condition

$$|S_{nm}(\psi_n, \varphi_m)| \leqq |\varphi_m| \, |\psi_n|,$$

where φ_m and ψ_n involve only the initial and final variables respectively, and are square integrable (see also Appendix T).

c) Proof of (11′). The methods of Subsection b) are sufficient to prove (11′) in case P is not a positive-α Landau point. In this subsection, we prove (11′) in case P is a quasisimple point.

Consider a local coordinate system q at P of the type described at the beginning of Chapter IV 1, and a (sharp [19]) separation of the real q-space into a neighborhood $\mathcal{N}$ of $q=0$ and an external part. The neighborhood $\mathcal{N}$ is chosen to be a product of two neighborhoods. These two neighborhoods depend on the coordinates q_{in} and q_f respectively, where the q_{in} and q_f are components of the initial and final $\boldsymbol{p}_i - \boldsymbol{P}_i$ respectively. They are made small enough so that: (i) all points of χ are covered by the above coordinate system; (ii) $\chi(q)$ is analytic in $\mathcal{N}$; (iii) $\mu(q)$ is analytic in a region $\mu(q) \leqq \hat{\alpha}$ which contains $\mathcal{N}$; (iv) the region $\mu(q) \leqq \alpha_1$ grows monotonically for $\alpha_1 \leqq \hat{\alpha}$ [see subsection (b)]; and (v) any part of L_c^+ in $\mathcal{N}$ is "almost flat in $\mathcal{N}$; i.e. the normal to L_c^+ does not change appreciably in $\mathcal{N}$.

The contribution of the external part is again easily seen by using the boundedness of the S-matrix, to have a bound of the form (11′).

As in Subsection b), there is again a fixed direction of $\operatorname{Im} q$ satisfying $\operatorname{Im} q \cdot \bar{u} < 0$ along which $\mathcal{N}$ can be distorted. But, instead of using (30) we now show that, by using an appropriate choice of coordinates and an appropriate $\mathcal{N}$, one can obtain an $\mathcal{N}'$ that completely avoids the singularity surface L_c^+. Then the scattering function $T(q)$ is analytic and bounded on this $\mathcal{N}'$ and the bound (11′) is obtained. (The justification of the use of a "sharp" division is given in Appendix I 3.)

We denote by $\hat{N}$ the direction of the gradient at P to the surface L_c^+. We then chose $\mathcal{N}$ to be a product of boxes in the spaces of variables q_{in} and q_f such that the direction $\hat{d}$ of some one of the one-dimensional edges of $\mathcal{N}$ satisfies the conditions:

$$\begin{aligned} \hat{d} \cdot \bar{u} &< 0 , \\ \hat{d} \cdot \hat{N} &> 0 , \end{aligned}$$

and such that the Landau surface intersects the boundary of $\mathcal{N}$ only along the (closed) codimension 1 faces that contain this direction. (This is possible unless $\bar{u}_{in}$ and $\bar{u}_f$ are both parallel to $\hat{N}_{in}$ and $\hat{N}_f$ respectively. In this latter case, it is easy to show that one may find a coordinate system q' for which these conditions are not satisfied.)

One may then use an imaginary distortion along the direction $\hat{d}$ to shift $\mathcal{N}$ away from the Landau surface. (Recall that by virtue of the Fundamental Theorem of Cauchy-Poincaré [15], one can shift $\mathcal{N}$ continuously into $\mathcal{N}'$ provided only each boundary point of $\mathcal{N}$ is shifted in a way that maintains the various boundary equations satisfied at

[19] In contrast to Subsection b), we do not use here a partition of unity.

that point. For the codimension one faces of $\mathcal{N}$, there is only one equation, in the space of l dimensions. For the one-dimensional edges of $\mathcal{N}$, there are $l-1$ equations, and hence only one (complex) degree of freedom. At the zero-dimensional corners, there are l equations. Hence these corners cannot be moved at all. See also Appendix C of Ref. [16].)

Dimensional arguments indicate that this argument should be able to be extended to any semisimple point P. However, when the number of Landau surfaces through P becomes large, it is not so easy to construct the required distortion, which can no longer have a single fixed direction. This problem is left for possible future work.

V. Conclusions

1. Applications to S-Matrix Theory

In S-matrix theory, the analyticity properties are derived from the principle of maximal analyticity. This principle says that the analyticity structure of S is the simplest one consistent with its unitarity and cluster decomposition properties.

For the physical region, this principle is interpreted as follows: The insertion of the cluster decomposition of S and S^{-1} into $SS^{-1}=I$ gives an integral equation for the scattering function that has terms having explicit singularities. Inductive or iterative procedures give other expressions for the scattering function having terms with other explicit singularities.

It has been shown that all explicit singularities generated in this way by the combination of the unitarity and cluster properties are confined to Landau surfaces. The first specific content of maximal analyticity is, accordingly, the assertion that all physical-region singularities of the scattering functions are confined to Landau surfaces.

One cannot conclude in the same way that the singularities should be confined to *positive-*α Landau surfaces, for the individual terms in the expansions do not have this property. And even if the singularities were confined to the positive-α surfaces, it is not apparent that the plus $i\varepsilon$ rules should hold, since these rules do not hold for individual terms. In fact, these terms generally do not continue via any path into the functions defined on other sides of these singularity surfaces: the functions in different sectors are different analytic functions. Unitarity places very strong conditions on the analytic structure of scattering functions, and it may ultimately be possible to prove that the normal analytic structure is the only one consistent with unitarity, given the fact that the singularities are confined to Landau surfaces. So far, however, it has always been necessary to assume at least certain features of the normal analytic structure.

The justification for these assumptions is provided by macrocausality. This justification is completely in the framework of S-matrix ideas: no appeal to ideas based on microcausality or locality properties is needed.

If the normal analytic structure is someday proved from maximal analyticity, then our converse results yield the interesting conclusion that maximal analyticity entails macroscopic space-time causality properties: space-time properties emerge automatically from a framework that contains no *a priori* space-time notions.

The idea that the wave functions defined by Fourier transformation are related to space-time properties was introduced into the above discussion as an *a priori* notion, in connection with the direct proof. The converse result provides a basis for this notion.

In the above discussion, the principle of maximal analyticity was accepted as the basic principle – macrocausality was used only to restrict the singularities to the *positive-*α Landau surfaces, and to give $i\varepsilon$ rules Alternatively, macrocausality can be accepted as the prior principle: macrocausality would then be used to justify the entire normal structure, and maximal analyticity would then merely *extend* this primitive domain of analyticity. There are evidently a variety of ways dividing the roles played by maximal analyticity and macrocausality.

A broad general conclusion to be drawn from our work is that the validity of the normal analytic structure and of strong macroscopic space-time causality properties does not provide a basis for believing in microscopic locality properties: from macroscopic space-time properties, it is enough to have physical-region analyticity properties, and *vice versa*.

2. Applications to Field Theory

It has so far not been possible to derive any analyticity in the physical region directly from general field theoretic axioms. Thus macrocausality would seem to be independent of microcausality. On the other hand, some analyticity in the elastic region of the $2 \to 2$ process has been obtained if a certain extra smoothness condition is imposed [17]. Macrocausality is thus presumably related to this smoothness condition.

If one wants to use physical assumptions instead of purely technical ones, then macrocausality could be added to the field theory axioms. This would yield, by virtue of the above work, the normal analytic structure. But the proof would not depend on the field theoretic substructure.

Alternatively, one might try to derive from field theory (plus smoothness) the macrocausality condition. The feasibility of this approach is suggested by the work of Williams [18]. If macrocausality can be derived from field theory, then the normal analytic structure would follow.

Appendix I

1. Scattering Functions as Tempered Distributions

Suppose first that the functional $S_{nm}(\psi_1, \ldots, \psi_n; \varphi_1, \ldots, \varphi_m)$ is defined for all wave packets φ_i (or ψ_j) in the space $L^2(R^3)$ of square integrable functions. Then the continuity (2) of the functional S_{nm} in L^2-norm *with respect to each separate variable* implies the continuity in the topology of the Schwartz space $\mathscr{S}$, which in turn implies that S_{nm} is a tempered distribution (nuclear theorem): that is, S_{nm} can be extended to all functions φ_{n+m} of $\mathscr{S}(R^{3n+3m})$ and this extension is linear and continuous in the topology of $\mathscr{S}$. On the other hand, the nuclear theorem is not valid for L^2 (which is not a nuclear space) and S_{nm} cannot be extended to all functions φ_{n+m} of $L^2(R^{3n+3m})$. That is, there are square integrable φ_{n+m} of nonproduct form for which $S(\varphi_{n+m})$ is not defined (i.e. is infinite).

The continuity (2) in L^2-*norm* allows one to show that the *order* [20] of the distribution S_{nm} is not greater than $2(n+m)$.

Proof [19]. First write

$$S_{nm} = S_{nm} * \prod_i \delta_i(\boldsymbol{p}_i) \prod_j \delta_j(\boldsymbol{q}_j), \tag{31}$$

where $*$ denotes convolution.

Then use the distribution identity

$$\delta_i(\boldsymbol{p}_i) = (\Delta_i - \alpha^2) \cdot \frac{e^{-\alpha|\boldsymbol{p}_i|}}{|\boldsymbol{p}_i|}.$$

$\left(\text{Here } \alpha \text{ is positive and } \Delta_i = \frac{\partial^2}{\partial^2 p_{i1}} + \frac{\partial^2}{\partial^2 p_{i2}} + \frac{\partial^2}{\partial^2 p_{i3}}.\right)$

This gives

$$S_{nm} = \prod_i (\Delta_i - \alpha^2) \prod_j (\Delta_j - \alpha^2)\, C_{nm}, \tag{32}$$

where

$$C_{nm} = S_{nm} * \prod_i \frac{e^{-\alpha|\boldsymbol{p}_i|}}{|\boldsymbol{p}_i|} \prod_j \frac{e^{-\alpha|\boldsymbol{q}_j|}}{|\boldsymbol{q}_j|}. \tag{33}$$

Since the function $e^{-\alpha|\boldsymbol{p}-\boldsymbol{p}'|}/|\boldsymbol{p}-\boldsymbol{p}'|$ of $\boldsymbol{p}'$ belongs to $L^2(R^3)$, the bound (2) implies the existence of C_{nm} for any value of the $\boldsymbol{p}_i$ and $\boldsymbol{q}_j$,

[20] A distribution is of order l if it can be expressed as a finite sum of lth derivatives of continuous functions.

and it is easy to show that it also implies that C_{nm} is a continuous function of these variables. This completes the proof.

Suppose next that the functional $S_{nm}(\psi_1, \dots, \psi_n; \varphi_1, \dots, \varphi_m)$ is originally assumed to be defined, and to satisfy (2), only for wave functions φ_i and ψ_j in the dense subspace $\mathscr{D}$ of L^2. We show now that S_{nm} has a unique linear extension to functions in L^2 for which (2) still holds, and hence that all the above results still remain true.

A functional $F(f)$ that is defined for f in $\mathscr{D}$ and that is linear and continuous in L^2-norm:

$$|F(f)| \leqq |f|$$

has a unique linear continuous extension to functions $\hat{f}$ in L^2. This well-known result is obtained by considered a sequence $\{f_p\}$ of functions in $\mathscr{D}$ which approach $\hat{f}$ in L^2-norm. They then satisfy the Cauchy condition:

$$|f_p - f_{p'}| < \varepsilon, \quad \text{if} \quad p, p' > N(\varepsilon).$$

By virtue of the boundedness condition, differences $F(f_p) - F(f_{p'})$ also satisfy the Cauchy condition:

$$\begin{aligned} |F(f_p) - F(f_{p'})| &= |F(f_p - f_{p'})| \quad &\text{(linearity)} \\ &\leqq |f_p - f_{p'}| \quad &\text{(continuity)}. \end{aligned}$$

The sequence of numbers $F_p \equiv F(f_p)$ thus has a limit and it is easy to show that the extension of F thus defined is still linear and continuous.

This result cannot be used directly because (2) holds only for product wave functions. [In any case, (2) cannot be extended to functions that mix the initial and final variables.]

Consider a set $\{\hat{f}_i\}_{i=1,\dots,m+n}$ of functions in L^2. Each $\hat{f}_i$ may be approached by a sequence $\{f_{i,j}\}$ of functions in $\mathscr{D}$. Then write

$$\begin{aligned} &S(f_{1,p_1}, \dots, f_{i,p_i}, \dots) - S(f_{1,p'_1}, \dots, f_{i,p'_i}, \dots) \\ &\quad = S(f_{1,p_1}, \dots, f_{i,p_i}, \dots) - S(f_{1,p'_1}, f_{2,p_2}, \dots, f_{i,p_i}, \dots) \\ &\quad\quad + S(f_{1,p'_1}, f_{2,p_2}, \dots) - S(f_{1,p'_1}, f_{2,p'_2}, f_{3,p'_3}, \dots) \\ &\quad\quad + \cdots \\ &\quad\quad + S(f_{1,p'_1}, \dots, f_{m+n-1,p'_{m+n-1}}, f_{m+n,p_{m+n}}) - S(f_{1,p'_1}, \dots, f_{m+n,p'_{m+n}}). \end{aligned} \tag{34}$$

Then we may use the linearity and continuity with respect to each variable to show that the sequence in the left-hand side satisfies the Cauchy condition. The existence of the extension of S_{mn} to functions $\hat{f}_i$ in L^2 follows, and it is easy also here to show that this extension is linear and continuous (in L^2-norm).

2. Distributions and Boundary Values of Analytic Functions

In this Section and in Section 3, we consider only distributions with compact support: in the applications to Chapter IV, the scattering function T can always be replaced by its product with a function of $\mathscr{D}$ that is equal to one over a sufficiently large domain.

(a) Consider any distribution $T(x)$ (where x is an l-dimensional variable) and define:

$$\hat{T}(z) = T\left[\prod_{i=1}^{l} \frac{(-2\pi i)^{-1}}{x_i - z_i}\right]. \tag{35}$$

This function is a well-defined analytic function of z when all $\operatorname{Im} z_i \neq 0$. Then define:

$$\bar{T}_\varepsilon(x) = \sum \left(\prod_i \eta_i\right) \hat{T}(\{x_i + i\varepsilon_i \eta_i\}) = T(x) * \prod_i \left(\frac{-\varepsilon_i/\pi}{x_i^2 + \varepsilon_i^2}\right), \tag{36}$$

where $\eta_i = \pm 1$ and the sum runs over the 2^l combinations of the values -1 and $+1$.

It is well known [20] that

$$T(z) = \lim \bar{T}_\varepsilon(x) \tag{37}$$

in the sense of distributions.

It is easy to see [21] that if $T(x)$ is written as the derivative $(d^n/dx^n)\, C(x)$ of a continuous function $C(x)$, then

$$\hat{T}(z) = \frac{d^n}{dz^n} \hat{C}(z),$$

and that consequently each term on the right-hand side of (36) approaches the real x space no faster than inverse powers of the ε_i:

$$|\hat{T}(x + i\varepsilon\eta)| < \frac{C}{\Pi(\varepsilon_i)^{n_i+1}}, \tag{38}$$

where C is independent of x and ε.

Finally, if a function is analytic for all $\operatorname{Im} z_i \neq 0$ and has a bound of the type (31), then its boundary value in the limit $\operatorname{Im} z_i \to 0$ is a distribution [21].

(b) Suppose now that the distribution $T(x)$, when restricted to test functions with support in a given compact set K, is also the boundary value of a *single* function $T'(z)$ from a *single* well-defined open convex cone $C[T'(z)$ is analytic for $\operatorname{Re} z$ in K and $\operatorname{Im} z$ in the intersection of C (having apex at $\operatorname{Im} z = 0$) with a neighborhood of $\operatorname{Im} z = 0]$.

If x is one-dimensional, then the edge of the wedge theorem implies immediately that $T'(z)$ has also a bound of the type (38): According to

(37), one has in the distribution sense

$$T(x) = \lim_{\varepsilon \to 0} \hat{T}(x + i\varepsilon) - \hat{T}(x - i\varepsilon).$$

If $T(x)$ is also equal to $\lim_{\varepsilon \to 0} T'(x + i\varepsilon)$, then the restrictions to $\mathrm{Im}\, z > 0$ of T' and $\hat{T}$ are equal up to a function analytic in a neighborhood that contains K. Thus T' must have the bound (38), since $\hat{T}$ has it.

We do not know if this result can be extended to the many-dimensional case. On the other hand, if we suppose that T' has a bound of type (38), then the distribution $T(x)$ is (in K) the multiple derivative of a continuous function $C(x)$ which is also the boundary value of a single function $C'(z)$ analytic in the same domain as T'. To see this, one can integrate $T'(z)$ from some fixed point in the domain of analyticity. The power bound $\Pi(y_i)^{-m_i}$ then entails that after some finite number of integrations, one will get in the limit a continuous function.

Applications. From macrocausality, one derives a power bound of the form (38) (see Chapter IV 1). This bound is used in the converse proof of Chapter IV 2a) ($\gamma = 0$ case). The result of the above paragraph is used in Chapter IV 2b) [$\gamma > 0$ case with polynomial $\mathscr{P}(\tau)$]. The uniqueness of the analytic function that represents the scattering function is proved in the following subsection.

(c) The representation of a distribution as a limit of 2^l analytic functions in highly nonunique [21]. On the other hand, the representation as the limit of a single analytic function is unique, if it exists.

Lemma. *Let K' be an open neighborhood in $\mathbf{R}^l$ with closure K. Let $\mathscr{D}_K$ be the space of infinitely differentiable functions of compact support K. Let z be in $\mathbf{C}^l$ and let $f(z)$ be a function analytic at all points of the strip*

$$S = \{\mathrm{Re}\, z \text{ in } K,\ \mathrm{Im}\, z = \varepsilon e,\ e = (e_1 \dots e_l),\ e_j > 0,\ 0 < \varepsilon < \varepsilon_0\}.$$

For any φ in $\mathscr{D}_K$, define

$$T_\varepsilon[\varphi] \equiv \int T(x + i\varepsilon e)\, \varphi(x)\, dx.$$

Suppose for any φ in $\mathscr{D}_K$ the functional

$$T[\varphi] \equiv \lim_{\varepsilon \to 0} T_\varepsilon[\varphi] \tag{39}$$

exists (is finite). Then $T(z)$ is unique in S.

Proof. Suppose there were two $T(x + i\varepsilon e)$ that satisfied (39). Then their difference $D(x + i\varepsilon e)$ would give, for all φ in $\mathscr{D}_K$,

$$\lim_{\varepsilon \to 0} D_\varepsilon[\varphi] = 0.$$

For φ in $\mathscr{D}_{K''}$, where $K'' \Subset K$, the function $D_\varepsilon[\varphi]$ is an analytic function of ε for $\varepsilon_0 > \operatorname{Re}\varepsilon > 0$, $|\operatorname{Im} z| < a$, for some sufficiently small a. And it approaches zero as $\operatorname{Re}\varepsilon \to 0$, for all $|\operatorname{Im} z| < a$. This analytic function of one variable is therefore identically zero. But, if $D_\varepsilon[\varphi] = 0$ for $\varepsilon > 0$ for all φ in $\mathscr{D}_{K''}$, then $D(x + i\varepsilon e)$ must be identically zero.

(d) The following lemma is used in Section 3.

Lemma. *Let $T[\varphi]$ and $T(x + i\varepsilon e)$ be as in the preceding lemma. Suppose, moreover, that $T[\varphi]$ has a representation*

$$T[\varphi] = \int C(x)\, \varphi(x)\, dx\,,$$

where $C(x)$ is infinitely differentiable. And suppose finally that $T(z) < B(\operatorname{Im} z)^{-m}$ in S, where B and m are fixed positive constants. Then, for any x in K',

$$\lim_{\varepsilon \to 0} T(x + i\varepsilon e) = C(x)\,.$$

Proof. Let $\bar{x}$ be an arbitrary fixed point in K'. Let N be a real neighborhood of $\bar{x}$ that lies in K and is defined by

$$N = [(x_j - \bar{x}_j)^2 < a_j^2 \quad \text{all} \quad j]\,.$$

Consider the test function that is given in N by

$$\varphi(x, \bar{z}) = \prod_j \left[\frac{\exp - [a_j^2 - (x_j - \bar{x}_j)^2]^{-1}}{(x_j - \bar{z}_j)\, 2\pi i} \right],$$

where $\bar{z} = \bar{x} + i\bar{\varepsilon} e$ is in S. Let C be a contour in S, defined by $\varepsilon = \varepsilon(x)$, where $\varepsilon(x)$ is continuous for x in $\bar{N}$, positive for x in N, and zero for x on $\partial N = \bar{N} - N$. Let $C(\varepsilon')$ be the particular contour C defined by

$$\varepsilon(x) = \min \left[\prod_j [a_j^2 - (x_j - \bar{x}_j)^2], \varepsilon' \right].$$

Let $T^{\varepsilon'}[\varphi(z, \bar{z})]$ be

$$\int_{C(\varepsilon')} T(z)\, \varphi(z, \bar{z})\, dz\,.$$

Then, for any $\varepsilon' < \bar{\varepsilon}$,

$$T[\varphi(x, \bar{z})] = T^{\varepsilon'}[\varphi(z, \bar{z})]\,.$$

In particular, the contour originally defined by $\lim \varepsilon \to 0$ can be shifted to the fixed contour $C(\varepsilon')$ by means of the generalized theorem of Cauchy-Poincaré [15]. The contribution to the integrals from near $\partial N = \bar{N} - N$ are exponentially damped by the exponential factor in $\varphi(z, \bar{z})$ (cf. Ref. [14]).

Let ε' be gradually increased. For sufficiently small $\bar{\varepsilon}$, the contour $C(\varepsilon')$ passes through the multiple-pole singularity of $\varphi(z, \bar{z})$ when $\varepsilon' = \bar{\varepsilon}$.

Then for $\varepsilon' > \bar{\varepsilon}$, one has

$$T[\varphi(x, \bar{z})] = T^{\varepsilon'}[\varphi(z, \bar{z})] + T(\bar{z}) \prod_j \exp -[a_j^2 - (\bar{z}_j - \bar{x}_j)^2]^{-1} . \quad (40)$$

One obtains this result by shifting the contour successively through each pole singularity of $\varphi(x, \bar{z})$; the difference gives a delta function in place of the pole.

Now, fix $\varepsilon' > \bar{\varepsilon}$ and let $\bar{\varepsilon} \to 0$ $(\bar{z} \to \bar{x})$. The contribution from $T^{\varepsilon'}[\varphi(z, \bar{z})]$ gives an analytic function of $\bar{z}$ for $\bar{z} \approx \bar{x}$. The contribution from $T[\varphi(x, \bar{z})]$ is equal to the convolution product of the infinitely differentiable function $C(x)$ with a distribution with compact support (a pole gives a well-defined distribution in the $\varepsilon \to 0$ limit); it is thus an infinitely differentiable function [22].

From (40), the limit $\varepsilon \to 0$ of $T(x + i\varepsilon)$ is thus also *infinitely differentiable* and it is then evidently equal to $T(x)$ as a function since it is equal to it in the distribution sense.

3. Bounded Functionals and Boundary Values of Analytic Functions

(a) Consider any functional T that satisfies the boundedness condition:

$$|T(\varphi', \varphi'')| \leqq |\varphi'| \, |\varphi''| \quad (41)$$

for any φ' and φ'' in $L^2(R^p)$ and $L^2(R^q)$ respectively. Then we know (Section 1) that T is a distribution, which can be written as a sum of boundary values of $2^{(p+q)}$ functions [see (36)].

We show below that this representation holds equally well for functions φ' and φ'' in L^2.

Lemma. *For any φ in L^2 (with compact support), one has*

$$\lim_{\varepsilon \to 0} |\varphi_\varepsilon - \varphi| = 0 ,$$

where φ_ε is defined from φ as in (36).

One sees by direct calculation that the Fourier transform $\tilde{\varphi}_\varepsilon(\tilde{x})$ of φ_ε is equal to

$$\tilde{\varphi}_\varepsilon(\tilde{x}) = \tilde{\varphi}(\tilde{x}) \prod_{i=1}^{p} \exp(-\varepsilon_i |\tilde{x}_i|) ,$$

where $\tilde{\varphi}$ is the Fourier transform of φ.

Since $\tilde{\varphi}$ is in L^2, the result follows.

Using this lemma and the bound (41), one obtains

$$T(\varphi', \varphi'') = \lim_{\varepsilon', \varepsilon'' \to 0} T(\varphi'_{\varepsilon'}, \varphi''_{\varepsilon''}) . \quad (42)$$

Since $\varphi'_{\varepsilon'}$ and $\varphi''_{\varepsilon''}$ are smooth (and in fact analytic) functions and the same is true of $\overline{T}_{\varepsilon'\varepsilon''}$, one can apply the results of [20] to show that

$$\begin{aligned} T(\varphi'_{\varepsilon'}, \varphi''_{\varepsilon''}) &= \lim_{\eta', \eta'' \to 0} T(\varphi'_{\varepsilon'+\eta'}, \varphi''_{\varepsilon''+\eta''}) \\ &= \lim_{\eta', \eta'' \to 0} \overline{T}_{\varepsilon'\varepsilon''}(\varphi'_{\eta'}, \varphi''_{\eta''}) \\ &= \overline{T}_{\varepsilon'\varepsilon''}(\varphi', \varphi'') . \end{aligned}$$

Thus one obtains from (42) that

$$T(\varphi', \varphi'') = \lim_{\varepsilon', \varepsilon'' \to 0} \overline{T}_{\varepsilon'\varepsilon''}(\varphi', \varphi'') . \tag{43}$$

Thus the representation of T by 2^{p+q} functions holds also for functions φ' and φ'' in L^2.

(b) Suppose now that T is, as a *distribution* on $\mathscr{D}_K$, the boundary value from the open convex cone C of a single analytic function $T'(z)$ which has a power bound of type (38) [see Section 2b)].

We show below that this respresentation holds equally well for functions φ' and φ'' in L^2, at least from certain directions in the cone C.

We denote by x' and x'' the subsets of variables associated with φ' and φ'' respectively. Let φ' and φ'' have their supports in sets K' and K'', with $K' \times K''$ smaller than K, and let $\chi(x) = \chi'(x')\,\chi''(x'')$ be a function of $\mathscr{D}^{p+q}$ that has its support in K and is equal to one over $K' \times K''$. We denote by T^χ the functional:

$$T^\chi(\varphi', \varphi'') = T(\chi'\varphi', \chi''\varphi'') .$$

Using the same methods as in Subsection a) and then the convolution theorem [23], one gets [see end of Subsection d)]

$$\begin{aligned} T^\chi(\varphi', \varphi'') &= \lim_{\varepsilon', \varepsilon'' \to 0} \overline{T}^\chi_{\varepsilon'_1\varepsilon''_1}(\varphi'_{\varepsilon'_2}, \varphi''_{\varepsilon''_2}) \\ &= \lim_{\varepsilon \to 0} \int \tilde{T}^\chi(v)\, \tilde{\varphi}'(-v')\, \tilde{\varphi}''(-v'')\, exp(-\textstyle\sum \varepsilon_i |v_i|)\, dv , \end{aligned} \tag{44}$$

where $\varepsilon' = \varepsilon'_1 + \varepsilon'_2$ (resp. $\varepsilon'' = \varepsilon''_1 + \varepsilon''_2$), the ε_i are the components of the vector $\varepsilon(\varepsilon', \varepsilon'')$ and $\tilde{T}^\chi(v)$ is the Fourier transform of the distribution T^χ. It is thus an infinitely differentiable function and it is bounded by virtue of (41).

On the other hand, using the methods of Chapter IV 3a), one shows that $\tilde{T}^\chi(v)$ has a uniform rapid fall-off in all directions of a cone C^+ that is arbitrarily close to the polar cone of C. We write correspondingly

$$\tilde{T}^\chi(v) = \theta(C^+)\,\tilde{T}^\chi(v) + (1 - \theta(C^+))\,\tilde{T}^\chi(v) ,$$

where $\theta(C^+)$ is one inside C^+ and zero otherwise.

The contribution $T^\chi_2(\varphi', \varphi'')$ associated with the second term is easily evaluated: by virtue of the fall-off of $(1 - \theta(C^+))\,\tilde{T}^\chi(v)$, one can take the limit $\varepsilon \to 0$ inside the integral and obtain, using again the convolution theorem,

$$T_2^\chi(\varphi', \varphi'') = \int T_2^\chi(x)\, \varphi'(x)\, \varphi''(x'')\, dx\,, \tag{45}$$

where $T_2^\chi(x)$ is the Fourier transform of $(1-\theta(C^+))\,\tilde{T}^\chi(v)$ and is thus infinitely differentiable.

The contribution $T_1^\chi(\varphi', \varphi'')$ associated with the term $\theta(C^+)\,\tilde{T}^\chi(v)$ is then well-defined in the $\varepsilon \to 0$ limit. We show below that it is equal to

$$T_1^\chi(\varphi', \varphi'') = \lim_{\varepsilon \to 0} \int T_1^\chi(x + i\bar{\varepsilon})\, \varphi'(x')\, \varphi''(x'')\, dx\,, \tag{46}$$

where (i) $T_1^\chi(x+i\varepsilon)$ is the Fourier transform of $\theta(C^+)\,\tilde{T}^\chi(v)\, e^{-\varepsilon \cdot v}$ and is thus analytic in the polar cone of C^+ – which is arbitrarily close to the original cone C; (ii) besides the condition $\varepsilon v > 0$, the direction $\bar{\varepsilon}$ along which the limit is taken is required to satisfy

$$\begin{aligned} \bar{\varepsilon}' v' &\geqq 0\,, \\ \bar{\varepsilon}'' v'' &\geqq 0\,, \end{aligned} \tag{47}$$

for all v in C^+.

To prove (46), consider a given direction $\bar{\varepsilon}$ which satisfies (47) and a coordinate system in which the first axis in v' space (resp. v'' space) is along the direction $\bar{\varepsilon}'$ (resp. $\bar{\varepsilon}''$). Then $v_1' = |v_1'|$ and $v_1'' = |v_1''|$. Since the function $\theta(C^+)\,\tilde{T}^\chi(v) \exp - [\varepsilon_1' v_1' + \varepsilon_1' v_1'']$ still belongs to L^2 by virtue of the term $\theta(C^+)$, one can move to inside the integral in (44) the limits when all other $\varepsilon_i \to 0$. Fourier transformation then gives the result (46).

Inserting (46) into the original representation, one obtains the distribution equation

$$T_2^\chi(x) = \lim_{\bar{\varepsilon} \to 0} [T'(x + i\bar{\varepsilon}) - T_1^\chi(x + i\bar{\varepsilon})]\,,$$

where $T_2^\chi(x)$ is infinitely differentiable. Then the lemma of Section 2d) insures that this result holds also in the pointwise sense. The result stated at the beginning of this subsection then follows.

Applications. We admit here, as in Chapter IV 3 c), the following boundedness condition for the S-matrix:

$$|S_{nm}(\psi_n, \varphi_m)| \leqq |\psi_n|\, |\varphi_m| \tag{48}$$

for any ψ_n and φ_m in $L^2(R^{3n})$ and $L^2(R^{3m})$ respectively.

It is then easy to construct a similar bound for the scattering function $T(q)$ itself with respect to L^2 functions in the variables q_{in} and q_f respectively: one simply takes special wave functions that depend only on the variables q_{in} and q_f except at large distances from the points that satisfy the mass-shell constraints.

The same proof holds with only a slight modification for the product of T with the function $e^{-\gamma\tau\mu(q)} J(q)$ and the result described [in Subsection b) above] then justifies the use of the representation $T = \lim_{\varepsilon \to 0} T_{\bar{\varepsilon}}$ for all square integrable wave functions [of compact support] and in particular for the "sharply cut" wave functions of Chapter IV 3 c).

A final remark: the proof of Subsection b) just above is similar to the proof from macrocausality [1] that the scattering function is, as a *distribution*, the boundary value of an analytic function; then φ is in $\mathscr{D}$ and $\tilde{\varphi}(v)$ thus has a rapid fall-off. In that case, the limit $\varepsilon \to 0$ in (44) can then be taken inside the integral and a factor $e^{-\varepsilon v}$ can be reintroduced for the term T_1^χ for any ε such that $\varepsilon v > 0$ for all v in C^+.

(c) *Proof of*(29).

$$\begin{aligned}\int T(q)\, J(q)\, \chi(q)\, e^{-iqv}\, dq &\equiv \langle T, \chi J e^{-iqv}\rangle \\ &\equiv \langle T\chi J, e^{-iqv}\rangle = \mathscr{I}[T\chi J] \\ &= \mathscr{I}[(T\xi)(\chi J/\xi)] = \mathscr{I}[T\xi] * \mathscr{I}[\chi J/\xi]\,.\end{aligned}$$

The second line follows from Ref. [24], since $T\chi J$ has compact support and the last line follows from [23], since $T\xi$ is in $\mathscr{S}'$ and $\chi J/\xi$ is in the dual of $\mathscr{O}_c'$, the space $\mathscr{O}_M$ of infinitely differentiable functions with slow growth.

For (44), one uses the converse theorem that for T in $\mathscr{O}_c'$ and U in $\mathscr{S}'$

$$\mathscr{I}[T * U] = (\mathscr{I}\, T)(\mathscr{I}\, U)\,.$$

Appendix II. Exponential Fall-Off and Analyticity

If a function $\hat{F}(v)$ has a uniform exponential fall-off,

$$|\hat{F}(v)| < C e^{-\alpha|v|}\,, \tag{49}$$

then its Fourier transform $F(q)$ is analytic in the domain $|\mathrm{Im}\, q| < \alpha$ for all values of $\mathrm{Re}\, q$.

The result of Chapter IV 2 constitutes in some sense a generalization of this result to the case when $F(q)$ is *not* analytic for all real values of $\mathrm{Re}\, q$. The exponential fall-off then refers to the function

$$\hat{F}_\gamma(v) = \int F(q)\, e^{-iqv}\, e^{-\gamma|v|\mu(q)}\, dq\,, \tag{50}$$

where γ is positive, $\mu(0) = 0$, $\mu(q)$ is positive for q real and is analytic at $q = 0$.

The analyticity of the distribution $F(q)$ at $q = 0$ follows from the condition (11). Our proof can be carried under slightly weaker conditions, but it depends crucially on the fact that the constant of exponential fall-off of $\hat{F}_\gamma(v)$ is proportional to γ for small γ, and on some fall-off condition for $\gamma = 0$ (see Appendix IV).

In the special case when q and v are one-dimensional variables, the analyticity of $F(q)$ at $q = 0$ follows simply from a condition of exponential fall -off of $\hat{F}(v)$ at a given value of γ (for instance, $\gamma = 1$). The question thus arises whether this stronger result can be obtained also in the many-dimensional case.

A proof in the one-dimensional case is given below. It makes use of the fact that any distribution $F(q)$ is a sum of boundary values of two functions analytic in the upper half plane and lower half plane respectively. The analogous result when q belongs to R^l is that $F(q)$ is the sum of 2^l boundary values of analytic functions in corresponding domains. We have tried to use this to generalize the one-dimensional proof but have not succeeded. Neither have we found any counter example. Thus we do not know whether our stronger conditions are actually necessary.

One-Dimensional Case. Consider the function

$$\hat{F}(v) = \int F(q)\, e^{-iqv}\, e^{-|v|\mu(q)}\, dq\,, \tag{51}$$

where $\mu(q)$ has the properties stated below (50). We suppose for simplicity that the distribution $F(q)$ has a compact support, but this condition could be weakened. The function $\hat{F}(v)$ is supposed to fall off exponentially in the $v \to +\infty$ limit.

Define the functions

$$G_+(q) = \int_0^\infty \hat{F}(v)\, e^{iqv}\, e^{v\mu(q)}\, dv; \tag{52}$$

$$G_-(q) = \int_{-\infty}^0 \hat{F}(v)\, e^{iqv}\, e^{-v\mu(q)}\, dv\,. \tag{53}$$

It is clear that G_+ and G_- are analytic in the intersection of a neighborhood of the origin with the domains:

$$\operatorname{Re}\mu(q) - \operatorname{Im} q < \alpha\,, \tag{52'}$$

$$\operatorname{Re}\mu(q) + \operatorname{Im} q < \alpha\,, \tag{53'}$$

where α is the constant of exponential fall-off.

Both domains contain the origin.

Consider, on the other hand, the function $G_+(q)$ itself, and suppose first (see below for contrary case) that F is a continuous function. Then one has, from (51) and (52),

$$G_+(q) = \int_0^\infty dv \int F(q')\, e^{i(q-q')v}\, e^{(\mu(q)-\mu(q'))v}\, dq'\,.$$

The integrand is absolutely convergent in the domain

$$\mathrm{Re}\,\mu(q) - \mathrm{Im}\,q < 0 .$$

By changing the order of integration in that domain, we obtain

$$G_+(q) = \int F(q')\,[(q-q')\,(i+\varrho(q,q'))]^{-1} dq', \tag{54}$$

where we have used the equality

$$\mu(q) - \mu(q') = \varrho(q,q')\,(q-q') . \tag{55}$$

The function $\bar{G}_+(q)$ defined by the above integral is analytic in a neighborhood of the real axis with the exception of a cut along the real axis itself.

Consider the values of $\bar{G}_+(q)$ at $q+i\varepsilon$ and $q-i\varepsilon$ for q real and $\mu(q)<\alpha$. It is easy to show that

$$\lim_{\varepsilon\to\infty}[\bar{G}_+(q+i\varepsilon) - \bar{G}_+(q-i\varepsilon)] = \frac{2\pi}{1-i\varrho(q,q)} F(q) . \tag{56}$$

Since $\bar{G}_+(q)$ is equal to G_+ above the cut, it can be analytically continued into the domain defined by (52'). $F(q)$ thus appears as a boundary value of a single function $H_-(q)$ analytic in the intersection of the lower half plane with a neighborhood of the origin.

Using $G_-(q)$, one can similarly show that F is the boundary value of a function $H_+(q)$ analytic in the intersection of the upper half plane with a neighborhood of the origin.

The edge of the wedge theorem [8] then guarantees that F is itself analytic in the neighborhood of the origin.

If F is not known *a priori* to be a continuous function, but is a distribution, then it is equal to some derivative of a continuous function C:

$$F(q) = C(q)\frac{\partial^l}{\partial q^l} . \tag{57}$$

That is,

$$G_+(q) = \int_0^\infty dv \int C(q')\frac{d^l}{d^l q'}\,[e^{i(q-q')v}\,e^{(\mu(q)-\mu(q'))v}]\,dq' . \tag{58}$$

It is still easy to show the absolute convergence in the domain $\mathrm{Re}\,\mu(q) - \mathrm{Im}\,q < 0$, and $G_+(q)$ is then equal to

$$\bar{G}_+(q) = \int C(q')\,dq'\,\frac{d^l}{d^l q'}\,[(q-q')\,(i+\varrho(q,q'))]^{-1} , \tag{59}$$

which is also analytic in a neighborhood of the real axis with a cut along the real axis.

It is still possible to show that (56) holds in the sense of distributions:

$$\int C(q)\frac{d^l}{d^l q}(\varphi(q))\,dq = \lim \int [\bar{G}_+(q+i\varepsilon) - \bar{G}_+(q-i\varepsilon)](1 - i\varrho(q,q)\varphi(q)\,dq) \tag{60}$$

for any test function φ in $\mathcal{D}$, and the end of the proof is unchanged, since the edge of the wedge theorem applies equally well to distributions.

Appendix III. Fall-Off Properties of Space-Time Wave Functions

Consider the wave functions (8): $\varphi_{(\gamma\tau)}(\boldsymbol{p}) = \chi(\boldsymbol{p})\,e^{-(\boldsymbol{p}-\boldsymbol{P})^2\gamma\tau}$ where χ is analytic in a neighborhood of $\boldsymbol{P}$ and γ is positive. We study in this appendix the fall-off properties of the space-time wave functions defined in (6).

We shall adopt the notation $\hat{f}(x)$ for either one of the functions f or $\tilde{\varphi}$ and the notation $\hat{\chi}(\boldsymbol{p})$ for the function $\chi(\boldsymbol{p})/2p_0$ or $\chi(\boldsymbol{p})/(2p_0)^{1/2}$ respectively.

We prove the exponential fall-off of $\hat{f}(u\tau)$ in the $\tau\to\infty$ limit for a given 4-vector u that is not parallel to P $((P^2+m^2)^{1/2}, \boldsymbol{P})$, and the precise bound of (9)

If we use the variable $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{P}$ and denote from now on by $\chi(\boldsymbol{q})$ the function $\hat{\chi}(\boldsymbol{P}+\boldsymbol{q})$ [$\chi(\boldsymbol{q})$ is analytic at $\boldsymbol{q}=0$ in either case], we can write:

$$\hat{f}(u\tau) = e^{i(Pu)\tau}\int \chi(\boldsymbol{q})\exp\left(-\tau[\boldsymbol{q}^2\gamma - i(\boldsymbol{q}\boldsymbol{u} - u_o[((\boldsymbol{P}+\boldsymbol{q})^2+m^2)^{1/2} - P_0]\right)d\boldsymbol{q}\,. \tag{61}$$

Consider, for any positive α, the set of surfaces Σ_α in the space of complex $\boldsymbol{q}$:

$$\mathrm{Re}[\boldsymbol{q}^2\gamma - i(\boldsymbol{q}\boldsymbol{u} - u_0[((\boldsymbol{P}+\boldsymbol{q})^2+m^2)^{1/2} - P_0])] = \alpha\gamma\,. \tag{62}$$

The intersection of Σ_α with the space of real $\boldsymbol{q}$ is the sphere $S_\alpha : \boldsymbol{q}^2 = \alpha$.

If it is possible to find on Σ_α a 3-dimensional surface L_α bounded by S_α, continuous in α, closed and bounded and lying in the domain of analyticity of χ [and of $((\boldsymbol{P}+\boldsymbol{q})^2+m^2)^{1/2}$], then $\hat{f}(u\tau)$ is easily seen to have an exponential fall-off $e^{-\alpha\gamma\tau}$ in the $\tau\to\infty$ limit: to see this, divide the region of integration over real $\boldsymbol{q}$ into the two parts $|\boldsymbol{q}|^2 > \alpha$ and $|\boldsymbol{q}|^2 < \alpha$. The contribution of the part $|\boldsymbol{q}|^2 > \alpha$ has the fall-off $e^{-\alpha\gamma\tau}$ because of the presence of the gaussian in the integrand. The contribution of the part $|\boldsymbol{q}|^2 < \alpha$ is evaluated by distorting the contour to the position L_α. The fall-off $e^{-\alpha\gamma\tau}$ is thus obtained.

We will not determine here the maximum value of α for which the properties required on Σ_α are satisfied[21], but will prove the existence of a positive α independent of γ such that the surface Σ_α satisfies these properties for all γ smaller than or equal to γ_0.

We denote by $\boldsymbol{x}$ and $\boldsymbol{y}$ the real and imaginary parts of $\boldsymbol{q}$, and choose a coordinate system in which the first axis is along the direction of the vector $\boldsymbol{d} = \boldsymbol{u} - (\boldsymbol{P}/P_0) u_0$.

Consider the surface L_α obtained by setting $y_2 = y_3 = 0$:

$$L_\alpha \begin{cases} \gamma(\boldsymbol{x}^2 - y_1^2) + d y_1 - u_0 \operatorname{Im}[\varphi(\boldsymbol{x} + i\boldsymbol{y})] = \alpha\gamma, \\ y_2 = y_3 = 0, \end{cases} \tag{63}$$

where

$$\varphi(\boldsymbol{q}) = ((\boldsymbol{P} + \boldsymbol{q})^2 + m^2)^{1/2} - P_0 - \frac{\boldsymbol{P}}{P_0}\boldsymbol{q}.$$

If we use the new variables $(\boldsymbol{x}, z_1 = y_1/\gamma)$, (63) becomes

$$\boldsymbol{x}^2 - \gamma^2 z_1^2 + d z_1 - u_0 z_1 \psi(\boldsymbol{x}, \gamma z_1) = \alpha, \tag{64}$$

where

$$\psi(\boldsymbol{x}, \gamma z_1) = \frac{P_1 + x_1}{((\boldsymbol{P} + \boldsymbol{x})^2 + m^2)^{1/2}} - \frac{P_1}{P_0} + \mathcal{O}(\gamma z_1),$$

the function $\mathcal{O}(\gamma z_1)$ being of the first order in γz_1.

Choose α sufficiently small so that $|u_0 \psi(\boldsymbol{x}, 0)| < d$ for $|\boldsymbol{x}| \leqq \sqrt{\alpha}$. [This is possible since $\psi(0, 0) = 0$.]

For $\gamma = 0$, (64) defines explicitly z_1 as a (positive) function of $\boldsymbol{x}$ in the region $|\boldsymbol{x}| \leqq \sqrt{\alpha}$. By virtue of the implicit function theorem, this equation also defines z implicitly as a function of $\boldsymbol{x}$ in that region, provided γ is sufficiently small[22]. We denote by γ_0 any strictly positive number such that $z_1(\boldsymbol{x})$ is unique and continuous in $|\boldsymbol{x}| \leqq \sqrt{\alpha}$ for $\gamma \leqq \gamma_0$.

The surface L_α in the region $|\boldsymbol{x}| \leqq \sqrt{\alpha}$ is thus closed and bounded. If α is chosen sufficiently small, then all the surfaces L_α for $\gamma \leqq \gamma_0$ also lie in the domain of analyticity of χ (and of $((\boldsymbol{P} + \boldsymbol{q})^2 + m^2)^{1/2}$), and $\hat{f}(u\tau)$ has thus the exponential fall-off $e^{-\alpha\gamma\tau}$ for $\gamma \leqq \gamma_0$.

Consider finally a set of χ that have a common support in the region $|\boldsymbol{q}| \leqq 1$ and a common upper bound K for $\boldsymbol{q}$ either real or in the complex domain lying between the surfaces $\{\boldsymbol{q}$ real, $|\boldsymbol{q}| \leqq \sqrt{\alpha}\}$ and $L_{\alpha\gamma_0}$.

[21] In nonrelativistic quantum mechanics, where the factor $((\boldsymbol{P} + \boldsymbol{q})^2 + m^2)^{1/2}$ is replaced by $(\boldsymbol{P} + \boldsymbol{q})^2/2m$, this maximal value is

$$\alpha = \frac{(\boldsymbol{u} - (\boldsymbol{P}/m)\, u_0)^2}{\gamma^2 + (u_0/2m)^2}$$

(if the domain of analyticity of χ is sufficiently large).

[22] It is sufficient to show that $(\partial/\partial z_1) F_\gamma(\boldsymbol{x}, z_1)$ is different from zero for $|\boldsymbol{x}| \leqq \sqrt{\alpha}$ on $F = \alpha$, where F denotes the left-hand side of (64). It is different from zero for small γ because $(\partial/\partial z_1) F_\gamma$ is a continuous function of γ and z_1 and $(\partial/\partial z_1) F_0 = d - u_0 \psi \neq 0$.

The above argument then gives the bound

$$|\hat{f}(u\tau)| < C\, e^{-\alpha\gamma\tau}, \tag{65}$$

where $C = K(A_{\alpha\gamma_0} + A')$, where $A_{\alpha\gamma_0}$ is the area of $L_{\alpha\gamma_0}$ (in the region $|\boldsymbol{x}| \leqq \sqrt{\alpha}$), and A' is the area of $|\boldsymbol{q}| \leqq 1$.

The choice of coordinates $\boldsymbol{q} = \boldsymbol{p} - \boldsymbol{P}$ is not important. Consider, for instance, the special coordinates $\boldsymbol{q}'$ such that $q_1' = u(p - P)$, and q_2', q_3' are some two of the components of $\boldsymbol{q}$. For an appropriate choice of these two components, the Jacobian $\partial(\boldsymbol{q}')/\partial(\boldsymbol{q})$ is different from zero in a neighborhood $\mathcal{N}$ of the origin, since u is not parallel to P.

The methods of this appendix are easily adapted to the above coordinates; the internal part around $\boldsymbol{q}' = 0$ is chosen to lie in $\mathcal{N}$, and the surface Σ_α is now simply

$$\Sigma_\alpha : \gamma \operatorname{Re} \mu(\boldsymbol{q}') + \operatorname{Im} q_1' = \alpha\gamma\,,$$

where $\mu(\boldsymbol{q}') = \boldsymbol{q}^2$.

These coordinates $\boldsymbol{q}'$ are precisely those that can be used, on the other hand, to show the rapid fall-off of $f(u\tau)$ in the case $\gamma = 0$, if u does not belong to the velocity cone of χ and if χ belongs to $\mathcal{D}$ (Ruelle's lemma).

We conclude with some remarks useful in adapting the methods of this appendix to the problems of Chapter IV 3.

(i) The choice of y_1 along the direction of $\boldsymbol{d} = \boldsymbol{u} - (\boldsymbol{P}/\mathrm{P}_0)\mathrm{u}_0$ is not important. It can be chosen along any direction $\boldsymbol{d}'$ such that $\boldsymbol{d}\boldsymbol{d}' \neq 0$. If $\boldsymbol{d}\boldsymbol{d}' > 0$, then y_1 is still a positive function of $\boldsymbol{x}$ for small γ.

(ii) in the proof given above, we placed the contour along $L_\alpha(\gamma)$ [see (63)]. We may, instead, keep the same $L_\alpha(\gamma_0)$ for all $\gamma \leqq \gamma_0$. The relevant quantity

$$\gamma(\boldsymbol{x}^2 - y_1^2) + d y_1 - u_0 \operatorname{Im} \varphi$$

is equal on $L_\alpha(\gamma_0)$ to $\alpha\gamma$ *plus* the term

$$\left(1 - \frac{\gamma}{\gamma_0}\right)(d y_1 - u_0 \operatorname{Im} \varphi)\,,$$

which is positive or equal to zero, for small γ. Consider now the set of surfaces $L_\alpha(\gamma_0)$, where $\alpha_1 \leqq \alpha \leqq \alpha_2$. These surfaces do not intersect each other for γ_0 sufficiently small, and their real bases are the spheres $|\boldsymbol{q}^2| = \alpha$. Consider a real domain $\mathcal{D}$ that lies between the domains $|\boldsymbol{q}|^2 < \alpha_1$ and $|\boldsymbol{q}|^2 < \alpha_2$. If we divide the domain of integration over real $\boldsymbol{q}$ in (61) into an internal part in $\mathcal{D}$ and an external part outside $\mathcal{D}$, then we may use any distortion $\mathcal{D}'$ of $\mathcal{D}$ that lies between L_{α_1} and L_{α_2}, and the methods of this appendix will yield a fall-off $e^{-\alpha_1\gamma\tau}$ for $\gamma \leqq \gamma_0$. The domain $\mathcal{N}$ of Chapter IV, Part 3, can be taken to be a domain $\mathcal{D}$ of this type, and $\mathcal{N}'$ can be taken to be of the type $\mathcal{D}'$.

Appendix IV

1. Contribution of the Surface $|v| = R$ in the $R \to \infty$ Limit

From (24), the term associated with the surface $|v| = R$ is

$$F_2^R(q) = -\lim_{R\to\infty}\left[R^{l-1}\int_{0<t<\gamma_0 R} dt\, d\Omega_v \hat{v}\cdot [\boldsymbol{H}(q,v,t)\, e^{iqv}\, e^{t\mu(q)}]_{|v|=R}\right], \quad (66)$$

where l is the dimension of the q-space ($l = 3(m+n) - 4$).

This immediately entails that

$$|F_2^R(q)| \leqq \lim_{R\to\infty}\left[R^{l-1}\int_{0<t<\gamma_0 R} dt\, e^{t\mu(q)} \operatorname*{Max}_{j=1,2,\ldots,l} |H_j(q,v,t)|_{|v|=R}\right] \quad (67)$$

up to a constant which is the product of l times the surface of the unit $(l-1)$-dimensional sphere.

To show that the limit of the right-hand side of (67) is zero, we use the following fall-off properties of the components of $\boldsymbol{H}(q, v, t)$:

(i) Eq. (11′) and the remark at the end of Chapter III 2 [together with the definitions (13) to (15) and (23)] yield the bound:

$$|H_j(q, v, \gamma|v|)| < C' e^{-\alpha\gamma|v|} \quad (68)$$

for $0 \leqq \gamma \leqq \gamma_0$, where C' can be chosen independent of j, v, γ and q for q in $\mathscr{E}$ [see (20)].

(ii) The function $F(q')\,\varrho_j(q, q')\,\mathrm{e}^{-t\mu(q')}$ belongs to $\mathscr{D}$ as a function of q'. This yields a uniform rapid fall-off of the components H_j of $\boldsymbol{H}$ in the $|v| \to \infty$ limit, for fixed t:

$$|H_j(q, v, t)| \leqq \frac{C_N(q, t, j)}{|v|^N} \quad (69)$$

for any positive integer N.

The dependence of C_N on t can be exhibited explicitly, as we now show in the case $N = 0$ and $N = 1$.

For $N = 0$, one immediately obtains

$$|H_j| < \int |F(q')\,\varrho_j(q, q')|\, dq' = C_0(q, j)\,. \quad (70)$$

For $N = 1$, we use the equalities

$$\begin{aligned}\int e^{-iq'v}\frac{\partial}{\partial q_i'}[F(q')\,e^{-t\mu(q')}\,\varrho_j(q,q')]\,dq' \\ = -\int \frac{\partial}{\partial q_i'}[e^{-iq'v}]\,[F(q')\,e^{-t\mu(q')}\,\varrho_j(q,q')]\,dq' \quad (71) \\ = -v_i H_j(q, v, t)\,,\end{aligned}$$

which holds because F is infinitely differentiable with compact support, and

$$\frac{\partial}{\partial q_i'}[F(q')\,e^{-t\mu(q')}\varrho_j(q,q')] = e^{-t\mu(q')}\frac{\partial}{\partial q_i'}[F(q')\,\varrho_j(q,q')] - t\,e^{-t\mu(q')}\frac{\partial}{\partial q_i}[\mu(q')]\,F(q')\,\varrho_j(q,q') \tag{72}$$

Since (71) is valid for all i, it is easy to see that

$$|H_j(q,v,t)| < \frac{C_0^{(1)}(q,j) + t\,C_1^{(1)}(q,j)}{|v|}. \tag{73}$$

Repeated application of this argument yields the bound

$$|H_j(q,v,t)| < \frac{C_0^{(N)} + C_N\,t^N}{|v|^N}, \tag{74}$$

where $C_0^{(N)}$ and C_N are independent of t and v and can also be chosen independent of j and q for q in $\mathscr{E}$.

Consider any fixed q inside the domain $\mathscr{E}$ and denote the *positive* quantity $\alpha - \operatorname{Re}\mu(q)$ by β. We define $t(R)$ by

$$t(R) = l\,\beta^{-1}\ln R, \tag{75}$$

and divide the domain of integration over t in the integral of (67) into the parts $0 < t < t(R)$ and $t(R) < t < \gamma_0 R$.

The contribution of the region $t(R) < t < \gamma_0 R$ is bounded [using (68)] by: $(t \equiv \gamma\,|v|)$

$$C'\,R^{l-1}\int_{t(R)}^{\gamma_0 R} e^{-\beta t}\,dt = C'\,R^{l-1}\frac{1}{\beta}[e^{-\beta t(R)} - e^{-\beta\gamma_0 R}].$$

Since $e^{-\beta t(R)} = R^l$, this contribution vanishes in the $R\to\infty$ limit. The contribution of the region $0 < t < t(R)$ is bounded [using (74)] by

$$\left|R^{l-1}\int_0^{t(R)} dt\;e^{t\mu(q)}\frac{C_0^{(N)} + C_N t^N}{R^N}\right|.$$

If we choose $N \geqq l\{1 + [\mu(q)/\beta]\} - 1$, this contribution also vanishes in the $R\to\infty$ limit. Hence $F_2^R \to 0$.

[This argument is only slightly changed if (11′) is replaced by the weaker (12): l is replaced in (75) by l plus the degree of $\mathscr{P}(\tau)$.]

2. Contribution of the Boundary of C′

This contribution is

$$\bar{F}(q) = -\lim_{R\to\infty}\left[\int_{0<t<\gamma_0 R} dt \int_{(t/\gamma_0)<|v|<R} d\Sigma_{\bar{c}}(v)\cdot [H(q,v,t)\, e^{t\mu(q)}\, e^{iqv}]_{v\in\bar{c}}\right], \quad (76)$$

where $\bar{c}$ is the boundary of C' (or C^+) and $d\Sigma_{\bar{c}}$ is the corresponding surface element.

We show that the integral on the right-hand side of (76) is absolutely convergent for q in the intersection of $\mathscr{E}$ and the polar cone C of C^+. This will imply the analyticity of $\bar{F}(q)$ in that domain. Eq. (68) is valid for q in $\mathscr{E}$ and v in $\bar{c}$, since the directions of v in $\bar{c}$ are by construction not causal. Thus the integral of the absolute value of the integrand is bounded up to a constant by

$$\int_{t>0} e^{-\beta t}\, dt \int_{v\in\bar{c}} e^{-\mathrm{Im}\, q\cdot v}\, |v|^{l-2}\, d|v| = \frac{(2\pi)^{1/2}}{\beta}\int_{v\in\bar{c}} e^{-\mathrm{Im}\, q\cdot v}\, |v|^{l-2}\, d|v|\,.$$

The polar cone C is defined by the condition that $\mathrm{Im}\, q\cdot v>0$ for all q in C for all v in C^+. The inequality holds also for v on the boundary $\bar{c}$ of C^+ for q inside C. The integral is therefore absolutely convergent for any q in C, and uniformly so for q in any compact set lying in the interior of C.

We must finally show that the infinitely differentiable function $\bar{F}(q)$ defined for real q by (76) is the limit of the analytic function defined by the same integral but with q in C. To show this, we divide the domain of integration into the regions $0<t<t(|v|)$ and $t(|v|)<t<\gamma_0(|v|)$, where $t(|v|)=l\beta^{-1}\ln|v|$. Using the same methods as in the preceding section one then shows that the integral is absolutely and uniformly convergent in the *closed* cone $\bar{C}$. The limit of the integral is therefore equal to the integral of the limit.

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