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A Remark on a Paper of J. F. Aarnes

ROBERT R. KALLMAN*

Mass. Institute of Technology, Cambridge

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Abstract. Let $\mathscr{A}$ be a C^*-algebra on the separable Hilbert space $\mathscr{H}$, and let $\mathscr{R}$ be the von Neumann generated by $\mathscr{A}$. Let G be a topological group and $a \to \varphi(a)$ a representation of G into the group of $*$-automorphisms of $\mathscr{A}$. Suppose that each $\varphi(a)$ extends to a $*$-automorphism of $\mathscr{R}$, and suppose that $a \to \langle \varphi(a)(T)x, y\rangle$ is continuous for each T in $\mathscr{A}$ and x, y in $\mathscr{H}$. Then, for a large class of groups G, one has automatically that $a \to \langle \varphi(a)(T)x, y\rangle$ is continuous for all T in $\mathscr{R}$ and x, y in $\mathscr{H}$.

Consider the following setup. Let $\mathscr{A}$ be a C^*-algebra (with identity) of operators on the Hilbert space $\mathscr{H}$, and let $\mathscr{R}$ be the von Neumann algebra generated by $\mathscr{A}$. Let G be a topological group, and let $a \to \phi(a)$ be a representation of G into the group of $*$-automorphisms of $\mathscr{A}$, denoted by $\mathrm{Aut}^*(\mathscr{A})$. Suppose that $a \to \langle \phi(a)(T)\, x, y\rangle$ is continuous on G for all T in $\mathscr{A}$ and x, y in $\mathscr{H}$, and suppose that each $\phi(a)$ extends to a $*$-automorphism of $\mathscr{R}$ (which we also denote by $\phi(a)$). One may easily check that the group property $\phi(ab)(T) = \phi(a)(\phi(b)(T))$ still holds. Question: Do we have that $a \to \langle \phi(a)(T)\, x, y\rangle$ is continuous on G for all T in $\mathscr{R}$ and x, y in $\mathscr{H}$?

This question was elegantly answered in the affirmative by J. F. Aarnes in [1] under the assumption that $(a, T) \to \phi(a)(T)$ (a in G, T in $\mathscr{A}$) satisfies a mild joint continuity condition. The purpose of this paper is to prove the following theorem, which roughly states that the continuity of the representation of G into $\mathrm{Aut}^*(\mathscr{R})$ is automatic under certain circumstances.

Theorem. *Let G, $\mathscr{A}$, $\mathscr{R}$, $a \to \phi(a)$, and $\mathscr{H}$ be as above. Suppose $\mathscr{H}$ is separable and the topology of G is given by a complete metric. Then $a \to \langle \phi(a)(T)\, x, y\rangle$ is continuous for all T in $\mathscr{R}$ and all x, y in $\mathscr{H}$.*

We remark that this theorem has wide applicability since the topology of every separable locally compact group is given by a complete metric. See Dixmier [2] for the elementary facts about von Neumann algebras used in the following proof.

Proof of the Theorem. Let T be in the unit ball of $\mathscr{R}$. Choose a sequence T_n ($n \geqq 1$) such that T_n converges strongly to T as $n \uparrow +\infty$. The Kaplansky

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density theorem implies that such a sequence exists since $\mathscr{H}$ is separable. Let $[x_i \mid i \geqq 1]$ be a dense sequence in the unit ball of $\mathscr{H}$. Since each $\phi(a)$ is in Aut*($\mathscr{R}$), each $\phi(a)$ is continuous in the weak operator topology on the unit ball of $\mathscr{R}$. Hence $\langle \phi(a)(T_n) x_i, x_j \rangle \to \langle \phi(a)(T) x_i, x_j \rangle$ as $n \uparrow +\infty$, for all a in G and $i, j \geqq 1$. Now, for each $n \geqq 1$, $a \to \langle \phi(a)(T_n) x_i, x_j \rangle$ is continuous on G. Hence, Osgood's theorem (see Kelley, Namioka, et al., [3], p. 86) implies that $a \to \langle \phi(a)(T) x_i, x_j \rangle$ is continuous on a set of second category, say S_{ij}. Since the topology of G is given by a complete metric, $S = \bigcap_{i,j \geqq 1} S_{ij}$ is also a set of second category. Since the topology of G is given by a complete metric, S is nonempty. Choose an element b of S. Let w and z be in the unit ball $\mathscr{H}$. Let $\varepsilon > 0$. Choose x_i and x_j such that $\|w - x_i\| \leqq \varepsilon$ *and* $\|z - x_j\| \leqq \varepsilon$. Easy estimates show that $|\langle \phi(a)(T) w, z \rangle - \langle \phi(b)(T) w, z \rangle| \leqq 4\varepsilon + |\langle \phi(a)(T) x_i, x_j \rangle - \langle \phi(b)(T) x_i, x_j \rangle|$. Hence, $\langle \phi(a)(T) w, z \rangle \to \langle \phi(b)(T) w, z \rangle$ as $a \to b$, for all w, z in $\mathscr{H}$. Let c be arbitrary in G and let $a \to c$. Then $bc^{-1}a \to b$. Hence, $\phi(bc^{-1}a)(T) \to \phi(b)(T)$ in the weak operator topology. But $\phi(cb^{-1})$ is an element of Aut*($\mathscr{R}$), and any $*$-automorphism of $\mathscr{R}$ is continuous in the weak operator topology on the unit ball of $\mathscr{R}$. Hence, $\phi(a)(T) = \phi(cb^{-1})(\phi(bc^{-1}a)(T)) \to \phi(cb^{-1})(\phi(b)(T)) = \phi(c)(T)$ in the weak operator topology as $a \to c$.

Q.E.D.

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Robert. R. Kallman
Department of Mathematics
Mass. Institute of Technology
Cambridge, Mass. 02139, USA