Commun. math. Phys. 13, 317—334 (1969)

On a Class of Equilibrium States under the Kubo-Martin-Schwinger Boundary Condition

I. Fermions

F. Rocca*,**, M. Sirugue*, and D. Testard
University of Aix-Marseille

Received April 2, 1969; revised form June 10, 1969

Abstract. Using the Kubo-Martin-Schwinger boundary condition for equilibrium states of quantum statistical mechanics of fermion gas, we prove that for $T \neq 0$ a one-particle evolution (corresponding essentially to bilinear hamiltonians) generally defines a unique equilibrium state, which is quasi-free. Conversely any quasi-free state is the equilibrium state for a single one-particle evolution if it has no Fock part in its product decomposition. Limiting cases where $T \to 0$ and $T \to \infty$ are studied. In the case where $T \to 0$ one shows that the state generally converges to a Fock state linked to the evolution.

Introduction

Quasi-free states have been recently studied in an extensive way [1, 2] as possible states for statistical systems of fermions. It is our goal to study the possible dynamics associated with such states.

Our main tool in that study will be the so-called Kubo-Martin-Schwinger boundary condition, in a form given for instance in [3]. This condition has been used in order to derive general properties of equilibrium states under general evolutions. In this paper, we shall restrict ourselves to more specific evolutions, namely the quasi-free evolutions, which are defined in the first section. Nevertheless, some important results of this paper are extended to more sophisticated evolutions.

The second section is devoted to "complexification" of such evolutions; in this way, we are able to define more general automorphisms of a dense subalgebra of the Clifford algebra, which are not *-automorphisms.

We can then prove the main theorem which states that, for a given non-zero temperature, the Kubo-Martin-Schwinger boundary condition establishes a unique correspondence between quasi-free evolutions and quasi-free states which are not of the Fock type. This restriction is quite clear since these states certainly do not satisfy the second condition of Kubo-Martin-Schwinger [3].

---

* Attaché de Recherche au C.N.R.S.

** This work is a part of a "Thèse de Doctorat d'Etat" presented to the "Faculté des Sciences de Marseille", April 23, 1969, under the number A.O. 3073.

This theorem by no means spoils the possibility of more sophisticated models, as the Bardeen-Cooper-Schrieffer model, since we describe actually infinite systems.

The last section is devoted to the study of limiting cases where $T \to 0$ and $T \to \infty$. In the case where $T \to 0$, we show that, for a given quasi-free evolution, the corresponding quasi-free state converges to a Fock state which is linked to this evolution.

I. Notations and Definitions

Let $\mathscr{H}$ be a real Hilbert space (the one-particle space) with the symmetric positive definite form s. $\mathfrak{A}$ will be the C^*-algebra built over $\mathscr{H}$ (Clifford algebra), generated by the hermitian elements $B(\psi)$, $\psi \in \mathscr{H}$, which satisfy the canonical anticommutation relations:

$$B(\psi)\, B(\varphi) + B(\varphi)\, B(\psi) = 2s(\psi, \varphi)\,,$$

A quasi-free state over $\mathfrak{A}$ is a state the truncated functions of which are zero except:

$$\omega(B(\psi)\, B(\varphi))^T = \omega(B(\psi)\, B(\varphi))\,.$$

As is well known, it appears as a Slater determinant (or a Pfaffian):

$$\omega\left(\prod_{i=1}^{2n} B(\varphi_i)\right) = \sum_{\substack{i_1 < i_2 < \cdots < i_n \\ i_k < j_k}} \chi_\sigma \prod_{k=1}^{n} \omega(B(\varphi_{i_k})\, B(\varphi_{j_k})) \tag{1.1}$$

χ_σ being the parity of the permutation σ:

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots\ldots & 2n-1 & 2n \\ i_1 & j_1 & \ldots\ldots & i_n & j_n \end{pmatrix}.$$

A quasi-free state ω defines an operator A on $\mathscr{H}$ according to [1]

$$\omega(B(\varphi)\, B(\psi)) = s(\varphi, \psi) + i s(A\varphi, \psi) \tag{1.2}$$

satisfying both

$$A^+ = -A \quad \text{and} \quad \|A\| \leqq 1 \tag{1.3}$$

where A^+ is the adjoint of A with respect to s.

Conversely an operator satisfying (1.3) defines a unique quasi-free state ω_A.

An orthogonal operator T (i.e. $T^+T = TT^+ = 1$) of $\mathscr{H}$ defines a $*$-automorphism of $\mathfrak{A}$ according to

$$\alpha_T\, B(\psi) = B(T\psi)\,. \tag{1.4}$$

These automorphisms are called "one-particle automorphisms".

Definition[1]. *A quasi-free evolution is a homorphic mapping $t \to \alpha_t$ of the additive group of the real line into the group of one-particle automorphisms $t \to \alpha_t X$ ($X \in \mathfrak{A}$) which are strongly continuous.*

[1] $t \to \alpha_t$ is not necessarily the time translation since for instance in the grand canonical formalism it contains already the chemical potential (see [3]).

According to formula (1.4), we get a homorphic mapping $t \to T_t$ of the additive group of the line into the orthogonal group of $\mathscr{H}$ such that $t \to T_t \psi (\psi \in \mathscr{H})$ is strongly continuous, since $\|B(\psi)\| = \|\psi\|$.

Definition. *The state ω is an equilibrium state for the evolution α_t if it satisfies the Kubo-Martin-Schwinger (K.M.S.) conditions* [3, 4] *with respect to this evolution: for every X and $Y \in \mathfrak{A}$, $t \to \omega(X \alpha_t Y)$ can be extended to an analytic function of t in the strip $0 < \mathrm{Im}\, t < \beta$ continuous on the boundary and such that:*

$$\omega(X \alpha_t Y)_{t = i\beta} = \omega(YX) \tag{1.5}$$

and

$$\omega(X^* X) = 0 \Rightarrow X = 0 \,. \tag{1.6}$$

It is interesting to remark that an equilibrium state is an invariant state. Indeed we have the following result:

Let $t \to \alpha_t$ an evolution over an arbitrary C^*-algebra $\mathfrak{A}$ and let ω be a state satisfying the K.M.S. conditions.

Then $\omega \circ \alpha_t = \omega$ for every $t \in R$.

The proof can be found for instance in [9].

II. Complexification of Quasi-Free Evolutions

It is proved in [3] and in [4] that, given a time evolution $t \to \alpha_t$ of the C*-algebra $\mathfrak{A}$, one can define an automorphism $\alpha_{i\beta}$ of $\mathfrak{A}$ which is not a *-automorphism. This automorphism is of central importance in the algebraic formulation of the K.M.S. boundary condition.

In this paper, we shall consider only quasi-free evolutions and, in this section, we want to derive more specific results pertinent to this case. Actually, one can use the results and proofs in [4] adapted to quasi-free evolutions, but we prefer to give a slighty different approach, which uses explicitly the analiticity properties of $t \to \alpha_t B(\psi)$ for a dense set of ψ; let be more precise:

Proposition 1. *Let $t \to \alpha_t$ be a quasi-free evolution; there exists a dense subspace $\tilde{\mathscr{H}}$ of $\mathscr{H}$ for which*

$$t \to \alpha_t B(\tilde{\psi}) = B(T_t \tilde{\psi}), \qquad \tilde{\psi} \in \tilde{\mathscr{H}}$$

can be extended to an entire function $z \in C \to \alpha_z B(\tilde{\psi})$; the extension is of course unique.

Proof. Let $\mathscr{H}_0$ be the set of vectors $\tilde{\varphi}$ such that

$$\tilde{\varphi} = T(\hat{f})\, \varphi = \int \hat{f}(t)\, T_t \varphi\, dt \tag{2.1}$$

for some $\varphi \in \mathscr{H}$ and for $\hat{f}$ a real function the Fourier transform of which f belongs to $\mathscr{D}$. For the sake of brevity, we shall call $\widehat{\mathscr{D}_R}$ this subset of $\mathscr{S}$, i.e. the set of real functions the Fourier transform of which belongs to $\mathscr{D}$, and $\mathscr{D}_R$ the corresponding subset in $\mathscr{D}$. $\tilde{\mathscr{H}}$ is the linear closure of $\mathscr{H}_0$.

The right hand side in (2.1) is to be understood as a Bochner integral just as

$$B(\tilde{\varphi}) = \int B(T_t\varphi)\,\hat{f}(t)\,dt \tag{2.2}$$

where the integration has been intertwined with the continuous real linear application $\tilde{\varphi} \to B(\tilde{\varphi})$.

Let e_z be the function $w \to \exp(-izw)$. The application

$$z \to \int B(T_t\varphi)\,\widehat{fe_z}(t)\,dt \tag{2.3}$$

realizes the extension. Actually, one can easily see that

$$\int B(T_t\varphi)\,\widehat{fe_z}(t)\,dt = \sum_{n=0}^{\infty} z^n a_n \tag{2.4}$$

where

$$a_n = (-1)^n (n!)^{-1} \int B(T_t\varphi)\,\hat{f}^{(n)}(t)\,dt \tag{2.5}$$

and for $z = u$ real

$$\int B(T_t\varphi)\,\widehat{fe_u}(t)\,dt = \int B(T_t\varphi)\,\hat{f}(t-u)\,dt = B(T_u\tilde{\varphi}) = \alpha_u B(\tilde{\varphi})\,.$$

We need now to prove that $\tilde{\mathscr{H}}$ is dense in $\mathscr{H}$; this can be achieved by proving that $\mathscr{H}_0$ is a total set in $\mathscr{H}$. Let $\chi \in \mathscr{H}$ such that

$$s(\chi, \tilde{\varphi}) = 0 \qquad \forall\, \tilde{\varphi} \in \mathscr{H}_0$$

$\tilde{\varphi}$ in $\mathscr{H}_0$ is of the form (2.1), so that by the continuity of s:

$$\int \hat{f}(t)\, s(\chi, T_t\varphi)\,dt = 0$$

for all $\hat{f} \in \mathscr{S}$ and hence $s(\chi, T_t\varphi) = 0$ (continuity of $t \to s(\chi, T_t\varphi)$) for all $t \in R$; then the regularity of s implies that $\chi = 0$. Q.E.D.

Moreover, it is clear that the linear subspace (resp. algebra $\tilde{\mathfrak{A}}$) generated by $B(\tilde{\varphi})$, $\tilde{\varphi} \in \mathscr{H}_0$, is dense in the linear space (resp. algebra $\mathfrak{A}$) generated by $B(\varphi)$, $\varphi \in \mathscr{H}$.

In the sequel we shall need the notation:

$$\mathfrak{H} = \{\varphi \in \mathscr{H};\, T_t\varphi = \varphi, \forall t \in R\}\,. \tag{2.6}$$

$\mathfrak{H}$ at least contains the null vector and is contained in $\mathscr{H}_0$. We shall consider the case where z is purely imaginary $z = i\beta$; in that case we introduce the notations, for a $f \in \mathscr{D}_R$

$$\begin{aligned} f^{u_\beta}(w) &= f(w)\,\mathrm{ch}(\beta w)\,,\\ f^{v_\beta}(w) &= -\,\mathrm{i}f(w)\,\mathrm{sh}(\beta w)\,. \end{aligned} \tag{2.7}$$

Let us remark that f^{u_β} and f^{v_β} belong to $\mathscr{D}_R$.

We give now an important lemma:

Lemma 1. *If* $\sum_{k=1}^{n} T(\hat{f_k})\,\varphi_k = 0$, $f_k \in \mathscr{D}_R$, $\varphi_k \in \mathscr{H}$, *then*

$$\sum_{k=1}^{n} T(\widehat{f_k^{u_\beta}})\,\varphi_k = \sum_{k=1}^{n} T(\widehat{f_k^{v_\beta}})\,\varphi_k = 0\,.$$

Indeed, for any $\psi \in \mathscr{H}$ and u real, one has:

$$s\left(\psi, T_u \sum_{k=1}^{n} T(\widehat{f_k})\, \varphi_k\right) = 0$$

or more explicitly

$$\sum_{k=1}^{n} \int s(\psi, T_t \varphi_k)\, \widehat{f_k e_u}(t)\, dt = 0$$

actually the left hand side is the restriction to u real of an entire function (cf. proposition 1), so that the function is zero everywhere. Specially

$$\sum_{k=1}^{n} \int s(\psi, T_t \varphi_k)\, \widehat{f_k e_{i\beta}}(t)\, dt = 0 \tag{2.8}$$

if now we remark that

$$\widehat{f e_{i\beta}} = \widehat{f^{u_\beta}} + i \widehat{f^{v_\beta}}$$

and rewrite (2.8) as

$$s\left(\psi, \sum_{k=1}^{n} T(\widehat{f_k^{u_\beta}})\, \varphi_k\right) + i\, s\left(\psi, \sum_{k=1}^{n} T(\widehat{f_k^{v_\beta}})\, \varphi_k\right) = 0$$

the result follows.

This lemma justifies the following definition:

Definition. *U_β and V_β are linear operators defined in $\tilde{\mathscr{H}}$ as follows:*

$$U_\beta\left(\sum_{k=1}^{n} T(\widehat{f_k})\, \varphi_k\right) = \sum_{k=1}^{n} T(\widehat{f_k^{u_\beta}})\, \varphi_k\,, \tag{2.9}$$

$$V_\beta\left(\sum_{k=1}^{n} T(\widehat{f_k})\, \varphi_k\right) = \sum_{k=1}^{n} T(\widehat{f_k^{v_\beta}})\, \varphi_k\,. \tag{2.10}$$

One has, using notations of the proposition 1:

$$\alpha_{i\beta} B(\tilde{\psi}) = B(U_\beta \tilde{\psi}) + i B(V_\beta \tilde{\psi}) \qquad \tilde{\psi} \in \tilde{\mathscr{H}}\,. \tag{2.11}$$

We gather in three propositions the properties of U_β and V_β which will be important in the sequel.

Proposition 2.

(U 1) *U_β is symmetric.*

(U 2) *$U_\beta \geqq 1$ and consequently $(a + U_\beta)$ is injective for every $a > -1$.*

(U 3) *$U_\beta \tilde{\mathscr{H}} = \tilde{\mathscr{H}}$ and $(1 + U_\beta)\, \tilde{\mathscr{H}} = \tilde{\mathscr{H}}$; consequently $(1 + U_\beta)^{-1} (1 - U_\beta)$ is defined on $\tilde{\mathscr{H}}$.*

(U 4) *$0 \leqq (1 + U_\beta)^{-1} (-1 + U_\beta) \leqq 1$.*

(U 5) *$U_\beta T_t - T_t U_\beta = 0$ on $\tilde{\mathscr{H}}$.*

Proposition 3.

(V 1) *V_β is antisymmetric.*

(V 2) $V_\beta T_t - T_t V_\beta = 0$ on $\tilde{\mathscr{H}}$.
(V 3) Ker $V_\beta = \mathfrak{H}$.

Proposition 4.

(UV 1) $U_\beta^2 + V_\beta^2 = 1$ *on* $\tilde{\mathscr{H}}$.
(UV 2) $U_\beta V_\beta - V_\beta U_\beta = 0$ *on* $\tilde{\mathscr{H}}$.
(UV 3) Ker $V_\beta = \mathrm{Ker}(U_\beta - 1)$.

Proofs. First let us remark that both

$$U_\beta \tilde{\mathscr{H}} \subset \tilde{\mathscr{H}}, \tag{2.12}$$

$$V_\beta \tilde{\mathscr{H}} \subset \tilde{\mathscr{H}}. \tag{2.13}$$

The proofs of U 1 and V 1 are essentially the same; so that we restrict to U 1. Let us consider the following elements of $\tilde{\mathscr{H}}$:

$$\tilde{\varphi} = \sum_{k=1}^{n} T(\widehat{f_k})\,\varphi_k \qquad f_k, g_k \in \mathscr{D}_R; \quad \varphi_k, \psi_k \in \mathscr{H}$$

$$\tilde{\psi} = \sum_{k=1}^{m} T(\widehat{g_k})\,\psi_k$$

such that

$$s(\tilde{\varphi}, U_\beta \tilde{\psi}) = \sum_{k=1}^{n} \sum_{j=1}^{m} \int s(T_t \varphi_k, T_s \psi_j)\, \widehat{f_k}(t)\, \widehat{g_j^{u_\beta}}(s)\, dt\, ds$$

using the orthogonality of T_t and the real character of $\hat{f}$, we get after a change of variables

$$s(\tilde{\varphi}, U_\beta \tilde{\psi}) = \sum_{k=1}^{n} \sum_{j=1}^{m} \int s(\varphi_k, T_u \psi_j)\, \widehat{\overline{f_k} \cdot g_j^{u_\beta}}(u)\, du = s(U_\beta \tilde{\varphi}, \tilde{\psi})$$

using the obvious relation $\overline{f} \cdot g^{u_\beta} = \overline{f^{u_\beta}} \cdot g$.

Let us now prove U 2. For any $\tilde{\varphi} \in \tilde{\mathscr{H}}$, one has:

$$s(\tilde{\varphi}, \tilde{\varphi}) = \sum_{k=1}^{n} \sum_{j=1}^{n} \int s(\varphi_k, T_u \varphi_j)\, \widehat{\overline{f_k} \cdot f_j}(u)\, du \geqq 0 \tag{2.14}$$

this relation is true for any finite family of vectors $\varphi_k \in \mathscr{H}$ and any finite family of functions $f_k \in \mathscr{D}_R$; so we can choose

$$f_k(w) = i\sqrt{2}\,\mathrm{sh}\left(\frac{\beta w}{2}\right) g_k(w), \qquad g_k \in \mathscr{D}_R \tag{2.15}$$

so that

$$\overline{f_k}(w)\, f_j(w) = 2\,\mathrm{sh}^2\left(\frac{\beta w}{2}\right) g_k(w)\, g_j(w) = g_k(w)\, g_j^{u_\beta}(w) - \overline{g_k}(w)\, g_j(w)$$

we introduce this result into (2.14) and we get:

$$s(\tilde{\psi}, U_\beta \tilde{\psi}) \geqq s(\tilde{\psi}, \tilde{\psi})$$

where

$$\tilde{\psi} = \sum_{k=1}^{n} T(\widehat{g_k})\,\varphi_k.$$

Hence U 2.

We prove now U 3; it suffices to prove that the general element of $\tilde{\mathscr{H}}$, $\tilde{\varphi} = \sum_{k=1}^{n} T(\widehat{f_k})\, \varphi_k$, $f_k \in \mathscr{D}_R$, $\varphi_k \in \mathscr{H}$, is of the form $U_\beta \tilde{\varphi}_1$ or $(1 + U_\beta)\, \tilde{\varphi}_2$.

This is achieved if one chooses

$$\tilde{\varphi}_1 = \sum_{k=1}^{n} T(\widehat{g_k})\, \varphi_k \quad \text{with} \quad g_k(w) = \frac{1}{\operatorname{ch} \beta w} f_k(w) \in \mathscr{D}_R\,, \tag{2.16}$$

$$\tilde{\varphi}_2 = \sum_{k=1}^{n} T(\widehat{h_k})\, \varphi_k \quad \text{with} \quad h_k(w) = \frac{1}{2 \operatorname{ch}^2(\beta w)} f_k(w) \in \mathscr{D}_R\,. \tag{2.17}$$

In order to prove U 4 one has to consider the relation (2.14), but instead of the choice (2.15) we take successively

$$f_k(w) = i \operatorname{th}\left(\frac{\beta w}{2}\right) g_k(w) \tag{2.18}$$

$$f_k(w) = \frac{1}{\operatorname{ch}\left(\frac{\beta w}{2}\right)} g_k(w) \tag{2.19}$$

and the proof goes along the same line of reasoning as for U 2. Property U 5 is obvious, just as V 2, UV 1 and UV 2. Let us now come to V 3. Consider $\tilde{\psi} = T(\hat{f})\, \psi \in \mathscr{H}$ such that $V_\beta \tilde{\psi} = 0$. For every $\varphi \in \mathscr{H}$, $s(\varphi, V_\beta \tilde{\psi}) = 0$ or

$$-i \int \widehat{f^{V_\beta}}(t)\, s(\varphi, \psi_t)\, dt = 0,\ f \in \mathscr{D}_R\,. \tag{2.20}$$

By linearity, we extend (2.20) to every $f \in \mathscr{D}$. Moreover $t \to s(\varphi, \psi_t)$ is bounded and continuous; so it is the Fourier transform of a distribution which, due to (2.20), has its support reduced to $\{0\}$. So that $s(\varphi, \psi_t)$ is at most a polynomial in t and since it is bounded it is a constant:

$$s(T_t \tilde{\psi}, \varphi) = s(\tilde{\psi}, \varphi) \qquad \forall\, \varphi \in \mathscr{H}$$

therefore

$$T_t \tilde{\psi} = \tilde{\psi} \qquad \text{and} \qquad \tilde{\psi} \in \mathfrak{H}\,.$$

Conversely let $\psi \in \mathscr{H}$ such that $T_t \psi = \psi$. One has seen that $\psi \in \tilde{\mathscr{H}}$; let $\tilde{\varphi} \in \mathscr{H}_0$ i.e. $\tilde{\varphi} = T(\hat{f})\, \varphi$, $f \in \mathscr{D}_R$. We can write:

$$\begin{aligned} s(\psi, U_\beta \tilde{\varphi}) + i s(\psi, V_\beta \tilde{\varphi}) &= s(\psi, \varphi) \int \widehat{f e_{i\beta}}(t)\, dt \\ &= s(\psi, \varphi)\, (f e_{i\beta})\, (0) \\ &= s(\psi, \varphi) \int \hat{f}(t)\, dt\,. \end{aligned}$$

We conclude then that

$$s(\psi, (U_\beta - 1)\, \tilde{\varphi}) = 0$$

$$s(\psi, V_\beta \tilde{\varphi}) = 0$$

and the proof of $V\,3$ is achieved with the help of $V\,1$. Finally, $UV\,3$ is proved by using $UV\,1$, $V\,1$ and $V\,3$.

III. States Satisfying the K.M.S. Condition with Respect to a Quasi-Free Evolution

Definition. *Let $\alpha_{i\beta}$ be an automorphism of $\mathfrak{A}$ defined by*

$$\alpha_{i\beta} B(\psi) = B(U\psi) + i B(V\psi) \qquad \psi \in \tilde{\mathscr{H}} \tag{3.1}$$

U and V having the properties of U_β and V_β in § 2; a state ω over $\mathfrak{A}$ will satisfy the K.M.S. condition with respect to $\alpha_{i\beta}$ if:

$$\omega(X\alpha_{i\beta} Y) = \omega(YX) \qquad X \in \mathfrak{A},\ Y \in \tilde{\mathfrak{A}}\,. \tag{3.2}$$

Definition. *Let $\mathfrak{A}_2$ the linear subspace generated by products of the type $B(\psi)\,B(\varphi)$; two states are equivalent with respect to $\mathfrak{A}_2$ if they coincide on $\mathfrak{A}_2$. We shall call $\hat{\omega}$ the class of $\omega \in \mathfrak{A}_1^{*+}$ with respect to this equivalence relation.*

Remark that any class contains a unique quasi-free state so that it is completely determined by one operator A defined in (1.2) with properties (1.3).

Before proving the main theorem, let us give a lemma:

Lemma 2. *Given an automorphism $\alpha_{i\beta}$ of the type* (3.1), *there exists a unique class $\hat{\omega}$ of states such that:*

$$\omega(B(\varphi)\,\alpha_{i\beta} B(\psi)) = \omega(B(\psi)\,B(\varphi)) \tag{3.3}$$

for any $\varphi \in \mathscr{H}$, $\psi \in \tilde{\mathscr{H}}$ and $\omega \in \hat{\omega}$.

Proof. Suppose such a class exists. Let A be the corresponding operator on $\mathscr{H}$; then (3.3) implies relations (use (1.2)):

$$1 = U + AV \quad \text{on} \quad \tilde{\mathscr{H}} \tag{3.4}$$

$$A = V - AU \quad \text{on} \quad \tilde{\mathscr{H}} \tag{3.5}$$

the relation (3.5) suggests the solution

$$A = (U+1)^{-1} V\,. \tag{3.6}$$

We see, using $UV\,1$, that (3.4) is also satisfied by the expression (3.6) for A.

To show that (3.6) is an actual solution, we prove relations (1.3). A is antisymmetric since U is symmetric and V antisymmetric. Furthermore

$$A^+ A = (U+1)^{-1}(U-1)$$

which is bounded by 1 (cf. $U\,4$). So $\|A\| \leqq 1$.

Finally let us remark the important relations deduced from (3.6) and $V\,3$:

$$\operatorname{Ker} A = \operatorname{Ker} V = \mathfrak{H}\,, \tag{3.7}$$

$$AU - UA = AV - VA = 0\,. \tag{3.8}$$

As V is antisymmetric, $\tilde{\mathscr{H}}$ split into two orthogonal subspaces:

$$\tilde{\mathscr{H}} = \mathfrak{H} \oplus V \tilde{\mathscr{H}}$$

and for technical reasons we shall prove the main theorem first in the case where $\mathfrak{H} = \{0\}$.

Lemma 3. *Let $\alpha_{i\beta}$ be an automorphism of the type* (3.1) *such that $\mathfrak{H} = \{0\}$; there is a unique state satisfying* (3.2) *and this state is a quasi-free state.*

Proof. Using the fact, deduced from (3.1), that

$$(\alpha_{i\beta} Y)^* = \alpha_{-i\beta}(Y^*)$$

and relation (3.2), one can verify that for any Y in the dense subalgebra $\tilde{\mathfrak{A}}$, any $X \in \mathfrak{A}$ and ω satisfying (3.2):

$$\omega(X(\alpha_{i\beta} Y - \alpha_{-i\beta} Y)) = \omega([X, (Y - \alpha_{-i\beta} Y)]_+), \tag{3.9}$$

$$\omega(X(\alpha_{i\beta} Y - \alpha_{-i\beta} Y)) = \omega([(Y + \alpha_{-i\beta} Y), X]_-). \tag{3.10}$$

Our goal is to show that ω is actually the quasi-free state which, by the lemma 2, corresponds to the operator A given by (3.6). Our first task is to show that ω is zero over the odd monomials. Let us specialize to

$$\begin{aligned} X &= \prod_{i=1}^{2n} B(\varphi_i) \qquad \varphi_i \in \mathscr{H} \\ Y &= B(\varphi_{2n+1}) \qquad \varphi_{2n+1} \in \tilde{\mathscr{H}} \end{aligned} \tag{3.11}$$

one can verify, using (3.1), that (3.10) rewrite:

$$\begin{aligned} 2i\omega(X B(V\varphi_{2n+1})) &= \omega([B((U+1)\varphi_{2n+1}) - iB(V\varphi_{2n+1}), X]) \\ &= 2\sum_{i=1}^{2n} (-1)^{i+1} [s((U+1)\varphi_{2n+1}, \varphi_i) \\ &\quad - is(V\varphi_{2n+1}, \varphi_i)]\, \omega\left(\prod_{j \neq i}^{2n} B(\varphi_j)\right). \end{aligned} \tag{3.12}$$

If $n = 0$ one explicitly finds that:

$$\omega(B(V\varphi)) = 0 \qquad \varphi \in \tilde{\mathscr{H}}.$$

Since V has a dense range in $\mathscr{H}$ (recall that $\mathfrak{H} = \{0\}$) we can conclude first that

$$\omega(B(\varphi)) = 0 \qquad \varphi \in \mathscr{H} \tag{3.13}$$

and secondly by the recursion formula (3.12) that

$$\omega\left(\prod_{i=1}^{2n+1} B(\varphi_i)\right) = 0 \qquad \varphi_i \in \mathscr{H}. \tag{3.14}$$

Let us now come back to the value of ω over even products and take

$$\begin{aligned} X &= \prod_{i=1}^{2n-1} B(\varphi_i) \qquad \varphi_i \in \mathscr{H} \\ Y &= B(\varphi_{2n}) \qquad \varphi_{2n} \in \tilde{\mathscr{H}} \end{aligned} \tag{3.15}$$

in the same way as before, we rewrite (3.9)

$$\omega(XB(V\varphi_{2n})) = \sum_{i=1}^{2n-1} (-1)^{i+1}[s(V\varphi_{2n}, \varphi_i) - is((1-U)\varphi_{2n}, \varphi_i)] \cdot \omega\left(\prod_{i \neq j}^{2n-1} B(\varphi_j)\right). \tag{3.16}$$

Using the lemma 2, we can see that

$$\begin{aligned} s(V\varphi_{2n}, \varphi_i) - is((1-U)\varphi_{2n}, \varphi_i) &= s(V\varphi_{2n}, \varphi_i) - is(AV\varphi_{2n}, \varphi_i) \\ &= \omega(B(\varphi_i)\, B(V\varphi_{2n}))\,. \end{aligned} \tag{3.17}$$

Since V has a dense range in $\mathscr{H}$, we obtain for a dense set of elements:

$$\omega\left(\prod_{i=1}^{2n} B(\varphi_i)\right) = \sum_{i=1}^{2n-1} (-1)^{i+1}\, \omega(B(\varphi_i)\, B(\varphi_{2n}))\, \omega\left(\prod_{j \neq i}^{2n-1} B(\varphi_j)\right) \tag{3.18}$$

so that the truncated functions of order higher of two are zero (see Appendix); the continuity of ω ensures then that it is true for every element in $\mathscr{H}$. Hence ω is quasi-free, and unique since it is determined by the operator A given by (3.6).

Now, we go on the general case where $\mathfrak{H}$ is not reduced to zero, and, unless stated otherwise, we assume that its dimension is even or infinite. First we prove a rather obvious lemma.

Lemma 4. *Let $\alpha_{i\beta}$ an automorphism of the type* (3.1) *and $\mathfrak{A}(\mathfrak{H})$ the Clifford algebra built over $\mathfrak{H}$. Then there exists a unique state over $\mathfrak{A}(\mathfrak{H})$ which satisfies* (3.2); *it is the central state of $\mathfrak{A}(\mathfrak{H})$.*

The proof is immediate if one uses $V\,3$ and $UV\,3$ to see that:

$$\alpha_{i\beta} X = X, \qquad X \in \mathfrak{A}(\mathfrak{H})$$

so

$$\omega(XY) = \omega(YX) \quad, X, Y \in \mathfrak{A}(\mathfrak{H})\,.$$

It is well known that the central state of a Clifford algebra is unique and quasi-free if the dimension of the one-particle space is even or infinite. So given an automorphism of the type (3.1), we know that the restrictions of ω to the sub-C*-algebras $\mathfrak{A}(\overline{V\tilde{\mathscr{H}}})$ and $\mathfrak{A}(\mathfrak{H})$ are uniquely determined by the condition (3.2); we gather these results to prove the following theorem:

Theorem 1. *Let $t \to \alpha_t$ a quasi-free evolution of the algebra $\mathfrak{A}$, such that the dimension of $\mathfrak{H}$ is even or infinite. Then there exists a unique state satisfying the K.M.S. conditions with respect to α_t. This state is a product state (à la* POWERS [5]*) of the state described in lemma* 3 *on $\mathfrak{A}(\overline{V\tilde{\mathscr{H}}})$ with the central state of $\mathfrak{A}(\mathfrak{H})$; hence it is quasi-free.*

Proof. One has previously noticed that

$$\mathscr{H} = \mathfrak{H} \oplus \overline{V\tilde{\mathscr{H}}}$$

is a decomposition of $\mathscr{H}$ into mutually orthogonal subspaces. We have also (cf. UV 3 and U 1)

$$\mathscr{H} = \mathfrak{H} \oplus \overline{(U-1)\tilde{\mathscr{H}}} .$$

Suppose there exists a state ω satisfying (3.2) with respect to $\alpha_{i\beta}$. Let us take

$$X = \prod_{i=1}^{n} B(\varphi_i),\ \varphi_i \in \mathfrak{H}, \text{ and } Y = \prod_{i=1}^{m} B(\psi_i),\ \psi_i \in \overline{V\tilde{\mathscr{H}}} = \overline{(U-1)\tilde{\mathscr{H}}} .$$

Remark that if $m - n$ is odd formula (3.12) holds and induction shows that $\omega(X\,Y) = 0$.

Similarly if both m and n are odd:

$$\omega(\alpha_{i\beta} X\,Y) = \omega(Y X) = (-1)^{mn}\,\omega(X\,Y) = 0 .$$

If now both m and n are even:

$$\omega\left((\alpha_{i\beta}\,B(\varphi_1))\prod_{i=2}^{n} B(\varphi_i)\prod_{j=1}^{m} B(\psi_j)\right) = \omega\left(\prod_{i=2}^{n} B(\varphi_i)\prod_{j=1}^{m} B(\psi_j)\,B(\varphi_1)\right)$$

so that simple algebra shows:

$$\omega(X\,Y) = \sum_{j=2}^{n} (-1)^{j-1}\,s(\varphi_1,\varphi_j)\,\omega\left(\prod_{\substack{i=2\\ i\neq j}}^{n} B(\varphi_i)\prod_{k=1}^{m} B(\psi_k)\right)$$

induction on m allows to write:

$$\omega(X\,Y) = \omega_0(X)\cdot\omega_1(Y) \tag{3.19}$$

where ω_0 is the central state of $\mathfrak{A}(\mathfrak{H})$ and ω_1 the state defined by lemma 3 on $\mathfrak{A}(\overline{V\tilde{\mathscr{H}}})$. The formula (3.19) ensures that ω is the product state, in the sense of POWERS [5], of ω_0 and ω_1.

In general a quasi-free state ω_A is a product state in the sense of Powers of the form $\omega_{A_1} \otimes \omega_{A_2} \otimes \omega_{A_3}$ (see [6]), where A_i is the restriction of A to $\mathscr{H}_i$ and

$$\mathscr{H}_1 = \operatorname{Ker}(|A| - 1)$$
$$\mathscr{H}_2 = \operatorname{Ker} A$$
$$\mathscr{H}_3 = \mathscr{H} \ominus (\mathscr{H}_1 \oplus \mathscr{H}_2)$$

ω_{A_1} is of the Fock type. In our case, the Fock type component is eliminated by the second condition of K.M.S. (cf. (1.6)), a condition which we have not used up to now. As we shall see below this corresponds to the fact that β is strictly finite. Then the quasi-free state ω is written, by (3.7) and (3.19):

$$\omega = \omega_A = \omega_{A_2} \otimes \omega_{A_3} \quad \text{with} \quad A = (1+U)^{-1}V .$$

Finally we consider the case where the dimension of $\mathfrak{H}$ is odd. There is no unicity of the central state on $\mathfrak{A}(\mathfrak{H})$ and only one of these states is quasi-free. Then it is possible to prove that the theorem 1 is true, unless in the case where $\dim \mathfrak{H} = \dim \mathscr{H} = 2n + 1$. Let us remark that the situation where the kernel of A is of odd dimensionality is really unphysical; indeed it is impossible to define on this subspace a complex structure so that the evolution cannot be considered as a unitary group of transformations in a complexification of $\mathscr{H}$.

IV. Quasi-Free Evolution Linked to a Quasi-Free State through the K.M.S. Condition

The theorem we shall prove in this section is something like the converse theorem of the previous one. More precisely, we shall find, using earlier results and for every quasi-free state, the unique evolution for which this state is an equilibrium state: it is a quasi-free evolution.

In order to reach this goal, we shall need a result which is the analogue of the Stone theorem, in a real space.

Proposition 5. *Let T_t, $t \in R$, be a one parameter abelian group of orthogonal operators over a real Hilbert space $\mathscr{H}$, such that $t \to T_t$ is strongly continuous. Then, there exists an antisymmetric operator Z (possibly unbounded) such that:*

$$T_t = \exp(Zt)\,. \tag{4.1}$$

We prove first a lemma using the notation (2.1) (see also [10]):

Lemma 5. *Let $\tilde{\varphi} = \sum_{k=1}^{n} T(\hat{f}_k)\,\varphi_k = 0\, f_k \in \mathscr{D}_R, \varphi_k \in \mathscr{H}$. Then $\sum_{k=1}^{n} T(\hat{f}'_k)\,\varphi_k = 0$, where the prime denotes the derivative of $\hat{f}_k$.*

Proof. If $\sum_{k=1}^{n} T(\hat{f}_k)\,\varphi_k = 0$, then for every $\varphi \in \mathscr{H}$ and $u \in R$

$$\sum_{k=1}^{n} \int s(\varphi, T_t\varphi_k)\,[\hat{f}_k(t-u) - \hat{f}_k(t)]\,dt = 0$$

so that if we let $u \to 0$ and taking into account that $\widehat{f_k} \in \mathscr{S}$, we get

$$s\left(\varphi, \sum_{k=1}^{n} T(\hat{f}'_k)\,\varphi_k\right) = 0$$

so the lemma.

Let us define now the operator Z on the elements $\tilde{\varphi}$ of $\tilde{\mathscr{H}}$:

$$Z\left(\sum_{k=1}^{n} T(\hat{f}_k)\,\varphi_k\right) = -\sum_{k=1}^{n} T(\widehat{i f_k \cdot \mathfrak{I}})\,\varphi_k \tag{4.2}$$

where

$$\mathfrak{I} \cdot f_k(w) = w f_k(w)\,.$$

Furthermore the series

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} Z^m \tilde{\varphi} = \sum_{m=0}^{\infty} \sum_{k=1}^{n} \int \frac{(-i)^m t^m}{m!} \widehat{\mathfrak{J}^m \cdot f_k}(u)\, T_u \varphi_k du \tag{4.3}$$

converges absolutely (standart proof) and on $\tilde{\mathscr{H}}$

$$T_t \tilde{\varphi} = \exp(Zt)\, \tilde{\varphi}$$

the antisymmetry of Z is obvious and $\|\exp(Zt)\| = 1$, so that

$$T_t = \exp(Zt)\,.$$

Now we can give the main result of this section:

Theorem 2. *Let ω_A be a quasi-free state such that its decomposition as product state (see previous section) contains no Fock part. Then there exists a unique quasi-free evolution for which ω_A is an equilibrium state.*

Proof. Suppose that there exists a quasi-free evolution for which ω_A is an equilibrium state. We have previously seen that in this case A and T_t are linked by the relation, deduced from (3.6), (2.9) and (2.10) and true on $\tilde{\mathscr{H}}$:

$$A\tilde{\varphi} = A\left(\sum_{k=1}^{n} T(f_k)\varphi_k\right) = -\sum_{k=1}^{n} T\left(\widehat{i f_k \cdot \operatorname{th}\frac{\beta\mathfrak{J}}{2}}\right)\varphi_k\,. \tag{4.4}$$

Let us now calculate

$$\frac{1}{\beta}\sum_{n=0}^{\infty}(-1)^n \frac{A^{2n+1}}{2n+1}\tilde{\varphi} = -\sum_{k=1}^{n} T(\widehat{\mathrm{i}f_k \cdot \mathfrak{J}})\varphi_k \tag{4.5}$$

$$= Z\tilde{\varphi}$$

where we used the proposition 5 to ensure the existence, for our quasi-free evolution, of the antisymmetric operator Z given by (4.2) and such that $T_t = \exp(Zt)$. The convergence of the series on the dense set $\tilde{\mathscr{H}}$ can be proved using standard procedures.

Conversely the previous definition of Z on the dense domain $(1 + A^2)\tilde{\mathscr{H}}$ i.e.:

$$Z = \frac{1}{\beta}\sum_{n=0}^{\infty}(-1)^n \frac{A^{2n+1}}{2n+1} \tag{4.6}$$

clearly determines through (4.1) a unique quasi-free evolution for which ω_A is an invariant state and satisfies the K.M.S. conditions.

It is to be remarked that:

i) the formula (4.5) is meaningless over the kernel of $|A| - 1$; it is the reason for which we excluded the Fock part of the quasi-free state.

ii) if A is of norm strictly less than 1, the operator Z is bounded; this is of course an unphysical situation.

iii) on the kernel of A, $Z = 0$ and the evolution is trivial.

The cases i) and iii) are clearly linked to $\beta = \infty$ and $\beta = 0$, i.e. respectively to the systems with temperature zero and to the systems with temperature infinite $\left(\text{since one defines } \frac{1}{kT} = \beta\right)$.

At this stage we want to present a rather obvious lemma which extends the generality of our previous results:

Lemma 6. *Let ω be a state satisfying the K.M.S. boundary conditions with respect to some evolution $t \to \alpha_t$. Then, if β is a $*$-automorphism of $\mathfrak{A}$, the state $\omega \circ \beta$ satisfies the K.M.S. conditions with respect to $t \to \beta^{-1} \alpha_t \beta$.*

Hence our results extend to the whole set of evolutions which are conjugated to a quasi-free evolution within the group of $*$-automorphisms and to the whole set of states which are of the form

$$\omega = \omega_1 \circ \beta$$

where ω_1 is quasi-free and β any $*$-automorphism of $\mathfrak{A}$.

V. Limiting Cases

In this section we want to investigate the cases where $\beta \to 0$ or $\beta \to \infty$; the first one has been previously studied in [7] for more general cases.

We have proved that, given a finite non-zero temperature and given a quasi-free evolution, we can define a quasi-free state in a unique way by the K.M.S. conditions. In the explicit expression for this state, we let $\beta \to 0$ and $\beta \to \infty$. Then we prove that the state goes to the central state ($\beta \to 0$) in some sense and to a Fock state ($\beta \to \infty$) uniquely defined by the evolution. We shall need in the sequel a lemma which essentially gives the polar decomposition of the operator Z previously defined.

Lemma 7. *Let Z be the operator defined by* (4.2). *Then*

$$Z = I\Omega \quad \textit{on} \quad (U-1)\tilde{\mathscr{H}} \tag{5.1}$$

where

$$\Omega\tilde{\varphi} = \sum_{k=1}^{n} T(\widehat{f_k^w})\varphi_k \qquad \tilde{\varphi} \in (U-1)\tilde{\mathscr{H}} \tag{5.2}$$

$$I\tilde{\varphi} = -\sum_{k=1}^{n} T(\widehat{\mathrm{i}f_k^\varepsilon})\varphi_k \tag{5.3}$$

with

$$f_k^w(w) = f_k(w)\,|w| \tag{5.4}$$

$$f_k^\varepsilon(w) = f_k(w)\,\varepsilon(w) \qquad \varepsilon(w) = \text{sign of } w\,. \tag{5.5}$$

Moreover i) $\Omega \geqq 0$ and it is defined on $(U-1)\tilde{\mathscr{H}}$

ii) I is a complexification of $\overline{(U-1)\tilde{\mathscr{H}}}$ i.e.

$I^+ = -I$ and $I^2 = -I$ on $\overline{(U-1)\tilde{\mathscr{H}}}$.

The proof is obvious.

Let us come to the first result:

Theorem 3. *Let $t \to \alpha_t$ a quasi-free evolution and A_β the operator built from α_t as in section III. Then*:

$$\lim_{\beta \to \infty} A_\beta = I \quad on \quad \overline{(U-1)\tilde{\mathscr{H}}}$$

in the sense of strong convergence of operators on $\mathscr{H}$; I is the complexification linked to α_t, given by (5.3).

Proof. Let us for sake of simplicity restrict ourselves to $(U-1)\mathscr{H}_0$; for $\tilde{\varphi} = T(\hat{f})\varphi$, $f \in \mathscr{D}_R$, $\varphi \in \mathscr{H}$, one has:

$$\|(A_{n\beta} - I)\varphi\|^2 = \int s(\varphi, \varphi_u)\, \widehat{|f|^2 \left(\operatorname{th} \frac{n\beta\mathfrak{J}}{2} - \varepsilon\right)^2}(u)\, du$$

since $\left(\operatorname{th}\left(\frac{n\beta w}{2}\right) - \varepsilon(w)\right)^2 \leqq 1$, we can exchange twice the limit $n \to \infty$ with integrations. On the other hand we have pointwise:

$$\operatorname{th}\left(\frac{n\beta w}{2}\right) - \varepsilon(w) \xrightarrow[n\to\infty]{} 0$$

so the result.

Quite similar techniques can be used to show the next theorem:

Theorem 4. *With the same notations of the theorem* 3, *we have*:

$$\lim_{\beta \to 0} A_\beta = 0$$

in the sense of strong convergence of operators on $\mathscr{H}$.

It is interesting to note that the convergence of A_β (actually only the weak convergence is needed) implies the weak convergence of ω_{A_β}. Indeed one has, firstly on $\mathfrak{A}_2$:

$$|\omega_{A_1}(B(\varphi)\,B(\psi)) - \omega_{A_2}(B(\varphi)\,B(\psi))| = |s((A_1 - A_2)\varphi, \psi)|\,.$$

Moreover the proof goes by induction using formula (3.25):

$$\left|\omega_{A_1}\left(\prod_{i=1}^{2n} B(\varphi_i)\right) - \omega_{A_2}\left(\prod_{i=1}^{2n} B(\varphi_i)\right)\right| = \left|\sum_{i=1}^{2n-1} \left\{\omega_{A_1}(B(\varphi_i)\,B(\varphi_{2n}))\right.\right.$$

$$\left.\left.\cdot\, \omega_{A_1}\left(\prod_{j\neq i}^{2n-1} B(\varphi_j)\right) - \omega_{A_2}(B(\varphi_i)\,B(\varphi_{2n}))\,\omega_{A_2}\left(\prod_{j\neq i}^{2n-1} B(\varphi_j)\right)\right\}\right|$$

$$\leqq \sum_{i=1}^{2n-1} |\omega_{A_1}(B(\varphi_i)\,B(\varphi_{2n}))|\, \omega_{A_1}\left(\prod_{j\neq i}^{2n-1} B(\varphi_j)\right) - \omega_{A_2}\left(\prod_{j\neq i}^{2n-1} B(\varphi_j)\right)$$

$$+ \sum_{i=1}^{2n-1} |\omega_{A_1}(B(\varphi_i)\,B(\varphi_{2n})) - \omega_{A_2}(B(\varphi_i)\,B(\varphi_{2n}))| \left|\omega_{A_2}\left(\prod_{j\neq i}^{2n-1} B(\varphi_j)\right)\right|.$$

So that for any $X \in \mathfrak{A}$:

$$|\omega_{A_1}(X) - \omega_{A_2}(X)| \to 0$$

f $A_1 \to A_2$ (weak convergence).

So by the theorems 3 and 4 one has:

$$\lim_{\beta \to 0} \omega_{A_\beta} = \omega_0 \quad \text{(the central state)} \tag{5.6}$$

$$\lim_{\beta \to \infty} \omega_{A_\beta} = \omega_I \quad \text{(Fock state)} \tag{5.7}$$

in the weak convergence sense.

These results are intuitively already known for finite volume from the well known formula (see for instance [8]):

$$\langle A \rangle = \frac{1}{\operatorname{Tr}(e^{-\beta H + \mu H})} \operatorname{Tr}(A e^{-\beta H + \mu H})$$

Conclusion

The theorems we have proved and which establish for a fermion system a correspondence between quasi-free states and quasi-free evolutions, deliver a host of rather simple models of fermion gas. The majority of these models are deprived of any physical sense, mainly due to their lack of any invariance. Indeed it will in general be impossible to define the simplest thermodynamical quantities as density, density of energy or entropy.

We have already shown that the state defined by a quasi-free evolution retains the whole invariance of the evolution and conversely. So we postpone to a next paper the study of the case where some invariance is present, specifically the translation invariance.

Appendix

In this appendix, we give another writing of quasi-free states over the Clifford algebra.

Lemma. *Let ω be an even state such that*

$$\omega\left(\prod_{i=1}^{2n} B(\varphi_i)\right) = \sum_{i=1}^{2n-1} (-1)^{i+1} \omega(B(\varphi_i)\, B(\varphi_{2n}))\, \omega\left(\prod_{\substack{j=1 \\ j \neq i}}^{2n-1} B(\varphi_j)\right)$$

then ω is a quasi-free state.

The proof goes by induction; suppose that for $(n-2)$:

$$\omega\left(\prod_{\substack{j=1\\ j\neq i}}^{2n-1} B(\varphi_j)\right) = \sum_{\sigma} \chi_0\omega(B(\varphi_{i_1})\, B\varphi_{j_1})) \ldots \omega(B(\varphi_{i_{n-1}})\, B(\varphi_{j_{n-1}}))$$

the sum being extended to all permutations of $1,2,\ldots,2n-1$ except i, such that $i_1 < i_2 \ldots < i_{n-1}$ and $i_k < j_k$, and χ_0 being the parity of the permutation $(1,2,\ldots,i-1,i+1,\ldots,2n-1) \to (i_1,j_1,\ldots,i_{n-1},j_{n-1})$. Then, according to the hypothesis of the lemma:

$$\omega\left(\prod_{i=1}^{2n} B(\varphi_i)\right) = \sum_{i=1}^{2n-1}\sum_{\sigma} \chi_0(-1)^{i+1}\omega(B(\varphi_{i_1})\, B(\varphi_{j_1}))$$

$$\cdots\omega(B(\varphi_i)\, B(\varphi_{2n}))\cdots\omega(B(\varphi_{i_{n-1}})\, B(\varphi_{j_{n-1}}))$$

but the permutation

$$\sigma_1 : (1,2\ldots,i,\ldots,2n) \to (i_1,j_1,\ldots,i,2n,\ldots,i_{n-1},j_{n-1})$$

has the same parity of the permutation

$$\sigma_2 : (1,2,\ldots,i,\ldots,2n) \to (i,i_1,j_1,\ldots,i_{n-1},j_{n-1},2n)\,.$$

So let

$$\sigma_3 : (i,1,2,\ldots,i-1,i+1,\ldots,2n) \to (i,i_1,j_1,\ldots,i_{n-1},j_{n-1},2n)\,.$$

One has

$$\chi_3 = \chi_0, \quad \text{but} \quad \chi_3 = (-1)^{i-1}\chi_2 = (-1)^{i-1}\chi_1\,,$$

so that

$$\chi_1 = (-1)^{i+1}\chi_0\,. \qquad \text{Q.E.D.}$$

Acknowledgments. The authors would like to thank Prof. D. KASTLER for his interest in this work, and Prof. N. M. HUGENHOLTZ and Doctor M. WINNINK for illuminating discussions and advices.

References

1. BALSLEV, E., J. MANUCEAU, and A. VERBEURE: Comm. math. Phys. 8, 315 (1968).
2. ROCCA, F., M. SIRUGUE, and D. TESTARD: Ann. Inst. H. Poincaré **3**, 247 (1969).
3. HAAG, R., N. M. HUGENHOLTZ, and M. WINNINK: Comm. math. Phys. **5**, 215 (1967).
4. KASTLER, D., J. C. T. POOL, and E. THUE POULSEN: Comm. Math. Phys. **12**, 175 (1969).
5. POWERS, R. T.: Thesis, Princeton University 1967.
6. MANUCEAU, J., F. ROCCA, and D. TESTARD: Comm. math. Phys. **12**, 43 (1969).
7. HUGENHOLTZ, N. M.: Comm. math. Phys. **6**, 189 (1967).

8. Kadanoff, L. P., and G. Bayn: Quantum Statistical Mechanics. New York: Benjamin 1962.
9. Winnink, M.: Thesis, Groningen 1968.
10. Nelson, E.: Ann. math. **70**, 572 (1959).

F. Rocca
M. Sirugue
D. Testard
Centre de Physique
Théorique C.N.R.S.
31 Chemin J. Aiguier
F 13 Marseille 9ème